Takagi-Sugeno fuzzy PID controllers: mathematical models and stability analysis with multiple fuzzy sets

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Abstract: This paper deals with nonlinear Takagi-Sugeno (TS) fuzzy PID controllers with multiple fuzzy sets. Two models of fuzzy PID controllers are proposed using algebraic product (AP) triangular norm, bounded sum (BS)/maximum (Max) triangular co-norm and centre of gravity (CoG) defuzzifier. The inputs are fuzzified by three or more fuzzy sets with trapezoidal/triangular type membership functions. A new rule base is proposed consisting of four rules which reduce the number of tunable parameters. The models of the fuzzy PID controllers reveal that they are (nonlinear) variable gain/structure controllers, i.e., the gains are a function of input variables and the structure of the controller changes in the input space. The variations of gain and the properties of the controllers are investigated. The bounded-input bounded-output (BIBO) stability of the closed loop system with one of the proposed models in the loop is studied. The applicability of the controllers is demonstrated with the help of two examples.

Keywords: fuzzy control; Takagi-Sugeno controller; variable gain controller; mathematical model; BIBO stability.

Reference to this paper should be made as follows: Raj, R. and Mohan, B.M. (2020) 'Takagi-Sugeno fuzzy PID controllers: mathematical models and stability analysis with multiple fuzzy sets', *Int. J. Fuzzy Computation and Modelling*, Vol. 3, No. 1, pp.33–60.

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This paper is a revised and expanded version of a paper entitled 'Analytical structure of a general Takagi-Sugeno fuzzy PID controller' presented at TENCON, Penang, Malaysia, 5–9 November 2017.

1 Introduction

With the introduction of fuzzy sets (Zadeh, 1965), fuzzy control attempts to design controllers for complex and ill-defined systems. The algorithm to deal with these complex systems was introduced by Zadeh (1973) where the control strategy is in the form of IF-THEN statements. Human-like thinking and qualitative knowledge of the processes are translated into the form of IF-THEN statements which decide the control strategy. Two types of Fuzzy Logic Control (FLC) exist in the literature: Mamdani type (Mamdani, 1974) and Takagi-Sugeno (TS) type (Takagi and Sugeno, 1985). Mamdani type FLC employs fuzzy sets whereas Takagi-Sugeno type FLC uses linear functions of the input variables in the rule consequent part. FLC is an active area of research and has many practical applications in industrial processes. For example, control of a cryogenic process (Santos and Dexter, 2002), unbalance compensation in an active magnetic bearing (Chen et al., 2009), and antilock braking systems (Sharkawy, 2010).

Conventional PID controllers are still prevalent in the industry due to its ease of operation and low cost of implementation. The linear controllers operate well on linear processes and lower order plants. The problem arises when complex nonlinear processes with higher order need to be controlled. FLC provides an alternative solution to this. The analysis and design of FLC requires the knowledge of its structure which is generally not available. Also, the fuzzy controllers do not have a single fixed model as it depends on various factors such as rule base, triangular norms and co-norms and fuzzification and defuzzification strategies. Stability analysis and design become convenient once the model of the controller is known.

Some historical developments in the area of fuzzy control are presented here. Modelling, identification and control of complex systems using fuzzy logic was introduced by Zadeh (1994). In the literature, several papers dealing with Mamdani type fuzzy PID controllers (Mizumoto, 1995; Misir et al., 1996; Kim and Oh, 2000; Mohan and Sinha, 2006, 2008a, 2008b) were reported whereas a limited number of papers was available for TS type fuzzy PID controllers (Ying, 2000; Mohan, 2011; Raj and Mohan, 2017). PID controllers (Mizumoto, 1995) were realised by 'product-sum-gravity' method and 'simplified fuzzy reasoning' method. The design, tracking performance and stability analysis of fuzzy PI+D controller was studied in Misir et al. (1996). The fuzzy PI and fuzzy D controllers were designed separately and the overall control law was obtained by algebraically combining the individual control laws. In Kim and Oh (2000), a nonlinear fuzzy PID control method was proposed to improve the transient responses of nonlinear systems. Thirteen different classes of fuzzy controllers of Mamdani type were derived in Mohan and Sinha (2006) and five different classes of fuzzy controllers of Mamdani type were derived and proved (Mohan and Sinha, 2008b) to be unsuitable for control application. Analytical structure of the simplest fuzzy PID controller was derived (Mohan and Sinha, 2008a) using Mamdani minimum inference method and centre of sums defuzzification method. The explicit structure of TS type fuzzy PID controller was derived (Ying, 2000) using simplified linear control rules. A modified rule base was proposed (Mohan, 2011) for the TS fuzzy PID controller and models were developed using AP triangular norm and BS triangular co-norm. Very recently, two models of the simplest TS fuzzy PID controller were derived (Raj and Mohan, 2017) using only two fuzzy sets and the modified rule base, and it was shown that the controller models in Mohan (2011) are a special case of the derived models. In the literature several design methods (Tanaka and Wang, 2001) for TS fuzzy controllers have been reported. A function-based evaluation approach (Hu et al., 2001) has been proposed while addressing the issues of simplicity and nonlinearity of fuzzy PID controllers. In Foulloy and Galichet (2003), two types of fuzzy inputs were presented and their use in FLCs was discussed. An implementation of fuzzy predictive functional control on an open-loop unstable process was presented in Lepetic et al. (2003). A magnetic suspension system was considered and a lead compensator was used to stabilise it. An adaptive PID controller design based on fuzzy models was proposed (Savran and Kahraman, 2014) for uncertain systems. The fuzzy models of the processes were constructed from the measured input-output data.

Stability is one of the major concerns while designing a controller. The analytical structures of controllers help in establishing stability criteria which guarantee the stability of the closed-loop system. Several attempts have been made in this direction so far. Lyapunov's direct method was applied to establish a sufficient condition (Tanaka and Sugeno, 1992; Chen et al., 1993) for stability of a fuzzy system. The stability result was then employed for the design of fuzzy controllers. Stability region of PID parameters was derived and hence a sufficient condition was obtained for a stable fuzzy controller using the passivity theorem (Sio and Lee, 1998). Sufficient conditions were established using circle criterion (Cao et al., 2011) for global asymptotic stability of the simplest TS fuzzy control systems. Sufficient conditions for BIBO stability of nonlinear feedback systems were derived using the small gain theorem (Chou et al., 2015).

It seems from the literature that generalised structure of the fuzzy PID controller is not investigated. So an attempt has been made to generalise TS type fuzzy PID controllers using multiple fuzzy sets, i.e., the number of fuzzy sets is user defined whereas in Raj and Mohan (2017) there is no such flexibility as the number of fuzzy sets is fixed. We have considered fuzzy sets with trapezoidal membership functions which can be modified to triangular form if required. A new TS rule base is introduced and applied to develop mathematical models of TS type fuzzy PID controllers by considering different universes of discourse (UoD) for all three input variables and using AP triangular norm and BS/Max triangular co-norm. The rule base consists of four general rules. The rule base is constructed in such a manner to reduce the number of tunable parameters. Two classes of fuzzy PID controllers of TS type are proposed.

Explicit structures of the fuzzy controllers are derived. It is shown that the fuzzy PID controllers are (nonlinear) variable gain/structure controllers. Also, the issue of stability is addressed using the small gain theorem (Khalil, 2015). A sufficient condition for the BIBO stability of a closed loop system with one of the proposed controllers in the loop is established. Two examples of nonlinear processes are considered for simulation study which demonstrates the applicability of the proposed fuzzy PID controllers.

Rest of the paper is organised as follows: The configuration of TS fuzzy PID controllers is discussed in Section 2. In Section 3, the analytical structures and properties of TS fuzzy PID controllers are presented. The stability of fuzzy control systems is investigated in Section 4. Simulations have been carried out in Section 5. The final section concludes the paper.

2 Principle components of fuzzy PID controllers

A typical block diagram of a closed-loop system with a continuous-time (CT) plant is shown in Figure 1. The incremental output of a discrete-time linear PID controller is given by

$$\Delta u(k) = K_P \Delta e(k) + K_I e(k) + K_D \Delta^2 e(k) \tag{1}$$

where $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$ are error, change of error and double change of error, respectively, and are defined as $e(k) = r(k) - y(k)$, $\Delta e(k) = e(k) - e(k-1)$ and $\Delta^2 e(k) = \Delta e(k) - \Delta e(k-1)$. $r(k)$ and $y(k)$ are the reference command and process output at k^{th} instant, respectively. The inputs of the PID controller are shown in Figure 2. The overall control effort is given by $u(k) = u(k-1) + \Delta u(k)$.

Figure 1 A typical closed-loop control system

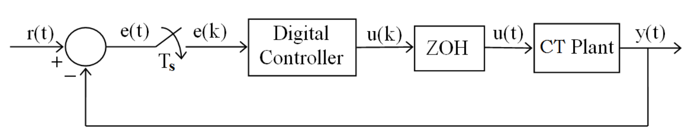
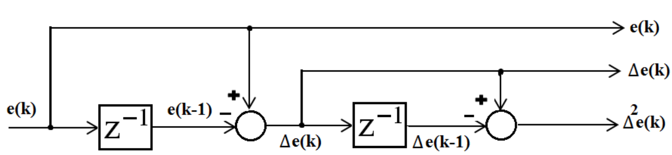
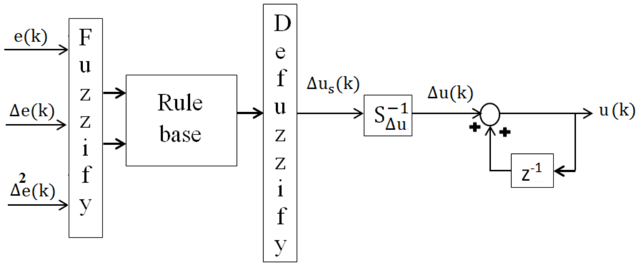


Figure 2 Inputs of PID controller



The principle structure of fuzzy PID controller is shown in Figure 3. $\Delta u(k)$, $\Delta u_s(k)$ and $S_{\Delta u}$ are the incremental controller output, scaled version of incremental controller output and the output scaling factor, respectively.

Figure 3 Fuzzy PID controller



2.1 *Fuzzification*

The inputs $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$ are fuzzified by N_1, N_2 and N_3 fuzzy sets with trapezoidal membership functions as shown in Figure 4. We consider J_1, J_2 and J_3 number of fuzzy sets on negative $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$ respectively, one fuzzy set for near zero on each of $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$, and J_1, J_2 and J_3 number of fuzzy sets on positive $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$ respectively. So we have $N_1 = 2J_1 + 1$, $N_2 = 2J_2 + 1$ and $N_3 = 2J_3 + 1$ with $J_1, J_2, J_3 \geq 1$. $2B_1$, $2B_2$, $2B_3$ and $2A_1$, $2A_2$, $2A_3$ are the lower and upper sides of trapezoidal membership functions for $e(k)$, $\Delta e(k)$, $\Delta^2 e(k)$, respectively. The trapezoidal membership functions reduce to triangular ones if $A_1 = A_2 = A_3 = 0$. Uniform distribution of fuzzy sets is considered over $[-L_1, L_1]$ for $e(k)$, over $[-L_2, L_2]$ for $\Delta e(k)$, and over $[-L_3, L_3]$ for $\Delta^2 e(k)$, where L_1, L_2, and L_3 are the design parameters. The central value $W_1(W_2, W_3)$ of the first fuzzy set on $e(k)(\Delta e(k), \Delta^2 e(k))$ is given by $W_1 = A_1 + B_1(W_2 = A_2 + B_2, W_3 = A_3 + B_3)$. The mathematical description of membership functions of i^{th}, j^{th}, and k^{th} fuzzy sets on $e(k)$, $\Delta e(k)$, and $\Delta^2 e(k)$, respectively, is given by

$$\mu_i(e) = \begin{cases} \dfrac{e(k) - iW_1 + B_1}{B_1 - A_1}, & iW_1 - B_1 \leq e(k) \leq iW_1 - A_1 \\ 1, & iW_1 - A_1 \leq e(k) \leq iW_1 + A_1 \\ \dfrac{-e(k) + iW_1 + B_1}{B_1 - A_1}, & iW_1 + A_1 \leq e(k) \leq iW_1 + B_1 \end{cases}$$

$$\mu_j(\Delta e) = \begin{cases} \dfrac{\Delta e(k) - jW_2 + B_2}{B_2 - A_2}, & jW_2 - B_2 \leq \Delta e(k) \leq jW_2 - A_2 \\ 1, & jW_2 - A_2 \leq \Delta e(k) \leq jW_2 + A_2 \\ \dfrac{-\Delta e(k) + jW_2 + B_2}{B_2 - A_2}, & jW_2 + A_2 \leq \Delta e(k) \leq jW_2 + B_2 \end{cases}$$

$$\mu_k(\Delta^2 e) = \begin{cases} \dfrac{\Delta^2 e(k) - kW_3 + B_3}{B_3 - A_3}, & kW_3 - B_3 \leq \Delta^2 e(k) \leq kW_3 - A_3 \\ 1, & kW_3 - A_3 \leq \Delta^2 e(k) \leq kW_3 + A_3 \\ \dfrac{-\Delta^2 e(k) + kW_3 + B_3}{B_3 - A_3}, & kW_3 + A_3 \leq \Delta^2 e(k) \leq kW_3 + B_3 \end{cases}$$

The description of membership functions of negative and positive fuzzy sets J_1, J_2 and J_3 on $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$ is as follows:

$$\mu_{-J_1}(e) = \begin{cases} 1, & -L_1 \le e(k) \le -l_1 + A_1 \\ \dfrac{-e(k) - l_1 + B_1}{B_1 - A_1}, & -l_1 + A_1 \le e(k) \le -l_1 + B_1 \end{cases}$$

$$\mu_{J_1}(e) = \begin{cases} \dfrac{e(k) - l_1 + B_1}{B_1 - A_1} & 1 - B_1 \le e(k) \le l_1 - A_1 \\ 1, & l_1 - A_1 \le e(k) \le L_1 \end{cases}$$

$$\mu_{-J_2}(\Delta e) = \begin{cases} 1, & -L_2 \le \Delta e(k) \le -l_2 + A_2 \\ \dfrac{-\Delta e(k) - l_2 + B_2}{B_2 - A_2}, & -l_2 + A_2 \le \Delta e(k) \le -l_2 + B_2 \end{cases}$$

$$\mu_{J_2}(\Delta e) = \begin{cases} \dfrac{\Delta e(k) - l_2 + B_2}{B_2 - A_2}, & l_2 - B_2 \le \Delta e(k) \le l_2 - A_2 \\ 1, & l_2 - A_2 \le \Delta e(k) \le L_2 \end{cases}$$

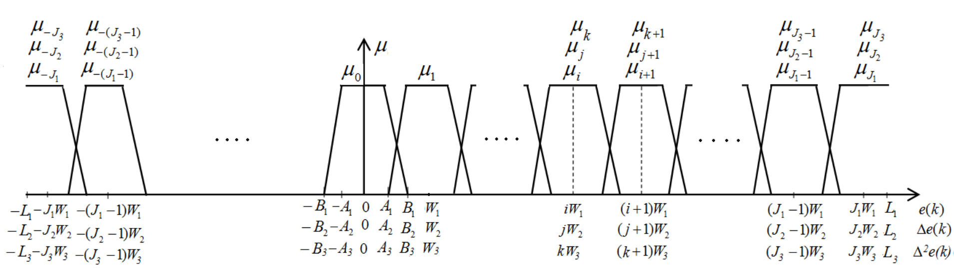
$$\mu_{-J_3}(\Delta^2 e) = \begin{cases} 1, & -L_3 \le \Delta^2 e(k) \le -l_3 + A_3 \\ \dfrac{-\Delta^2 e(k) - l_3 + B_3}{B_3 - A_3}, & -l_3 + A_3 \le \Delta^2 e(k) \le -l_3 + B_3 \end{cases}$$

$$\mu_{J_3}(\Delta^2 e) = \begin{cases} \dfrac{\Delta^2 e(k) - l_3 + B_3}{B_3 - A_3}, & l_3 - B_3 \le \Delta^2 e(k) \le l_3 - A_3 \\ 1, & l_3 - A_3 \le \Delta^2 e(k) \le L_3 \end{cases}$$

where $l_1 = J_1 W_1$, $l_2 = J_2 W_2$, and $l_3 = J_3 W_3$. Also,

$$\mu_i(e) + \mu_{i+1}(e) = 1 \text{ for } -L_1 \le e(k) \le L_1,$$
$$\mu_j(\Delta e) + \mu_{j+1}(\Delta e) = 1 \text{ for } -L_2 \le \Delta e(k) \le L_2 \text{ and}$$
$$\mu_k(\Delta^2 e) + \mu_{k+1}(\Delta^2 e) = 1 \text{ for } -L_3 \le \Delta^2 e(k) \le L_3.$$

Figure 4 Fuzzy sets with their membership functions on input variables



2.2 *Rule base*

$N_1.N_2.N_3$ control rules are required to cover $N_1 \times N_2 \times N_3$ possible combinations of the input fuzzy sets. The rule base consists of four rules as follows:

r_1 IF $e(k)$ is E_i AND $\Delta e(k)$ is ΔE_j AND $\Delta^2 e(k)$ is $\Delta^2 E_k$ THEN
$\Delta u_1(k) = a_{|i+j+k|} e(k) + b_{|i+j+k|} \Delta e(k) + c_{|i+j+k|} \Delta^2 e(k)$

r_2 IF ($e(k)$ is E_{i+1} AND $\Delta e(k)$ is ΔE_j AND $\Delta^2 e(k)$ is $\Delta^2 E_k$) OR ($e(k)$ is E_i AND $\Delta e(k)$ is ΔE_{j+1} AND $\Delta^2 e(k)$ is $\Delta^2 E_k$) OR ($e(k)$ is E_i AND $\Delta e(k)$ is ΔE_j AND $\Delta^2 e(k)$ is $\Delta^2 E_{k+1}$) THEN
$\Delta u_2(k) = a_{|i+j+k+1|} e(k) + b_{|i+j+k+1|} \Delta e(k) + c_{|i+j+k+1|} \Delta^2 e(k)$

r_3 IF ($e(k)$ is E_{i+1} AND $\Delta e(k)$ is ΔE_{j+1} AND $\Delta^2 e(k)$ is $\Delta^2 E_k$) OR ($e(k)$ is E_{i+1} AND $\Delta e(k)$ is ΔE_j AND $\Delta^2 e(k)$ is $\Delta^2 E_{k+1}$) OR ($e(k)$ is E_i AND $\Delta e(k)$ is ΔE_{j+1} AND $\Delta^2 e(k)$ is $\Delta^2 E_{k+1}$) THEN
$\Delta u_3(k) = a_{|i+j+k+2|} e(k) + b_{|i+j+k+2|} \Delta e(k) + c_{|i+j+k+2|} \Delta^2 e(k)$

r_4 IF $e(k)$ is E_{i+1} AND $\Delta e(k)$ is ΔE_{j+1} AND $\Delta^2 e(k)$ is $\Delta^2 E_{k+1}$ THEN
$\Delta u_4(k) = a_{|i+j+k+3|} e(k) + b_{|i+j+k+3|} \Delta e(k) + c_{|i+j+k+3|} \Delta^2 e(k)$

where $\Delta u_1, \Delta u_2$, Δu_3 and Δu_4 represent the incremental control outputs of PID controller for rules r_1, r_2, r_3 and r_4, respectively, and $a_{i,j,k}$, $b_{i,j,k}$ and $c_{i,j,k}$ are design parameters for $i = -J_1, ..., -1, 0, 1, ..., J_1 - 1$, $j = -J_2, ..., -1, 0, 1, ..., J_2 - 1$ and $k = -J_3, ..., -1, 0, 1, ..., J_3 - 1$.

Justification: In the literature, the rule base for fuzzy PID controller consists of eight rules (Ying, 2000; Mohan and Sinha, 2006, 2008a, 2008b). Now, considering the Mamdani rule base in Mohan and Sinha (2008a), we have

R_1 IF d_N is $n.d$ AND v_N is $n.v$ AND a_N is $n.a$ THEN Δu_s is O_{-2}

R_2 IF d_N is $p.d$ AND v_N is $n.v$ AND a_N is $n.a$ THEN Δu_s is O_{-1}

R_3 IF d_N is $n.d$ AND v_N is $p.v$ AND a_N is $n.a$ THEN Δu_s is O_{-1}

R_4 IF d_N is $n.d$ AND v_N is $n.v$ AND a_N is $p.a$ THEN Δu_s is O_{-1}

R_5 IF d_N is $p.d$ AND v_N is $p.v$ AND a_N is $n.a$ THEN Δu_s is O_{+1}

R_6 IF d_N is $p.d$ AND v_N is $n.v$ AND a_N is $p.a$ THEN Δu_s is O_{+1}

R_7 IF d_N is $n.d$ AND v_N is $p.v$ AND a_N is $p.a$ THEN Δu_s is O_{+1}

R_8 IF d_N is $p.d$ AND v_N is $p.v$ AND a_N is $p.a$ THEN Δu_s is O_{+2}.

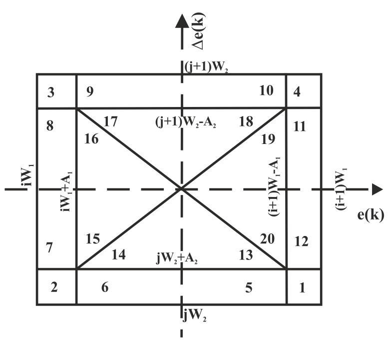
Here, we can see that rules R_2, R_3 and R_4 have the same consequent part. Similarly, R_5, R_6 and R_7 also have the same consequent. So we merge rules R_2, R_3 and R_4 into r_2 and rules R_5, R_6 and R_7 into r_3 of the proposed rule base. So the proposed rule base consists of only four rules. From Figure 4, one can find that the fuzzy sets on the negative x-axis are the mirror image of the fuzzy sets on the positive x-axis. So based on this symmetric arrangement of the fuzzy sets, we introduce the modulus operator in the subscript of the rule consequent parameters. The modified rule base having four rules helps in reducing the number of tunable parameters and thereby the complexity of the controller. The total number of tunable parameters is given by $3(J_1 + J_2 + J_3 + 1)$.

Triangular norms and co-norms are mathematical operations that provide logical conjunction (AND) and disjunction (OR), respectively. T-norm is a mapping $t : [0, 1] \;\; X \;\; [0, 1] \to [0, 1] \;\; \forall \;\; x, y, x', y', z \in [0, 1]$ that satisfies boundary conditions: $t(x, 0) = 0$, $t(x, 1) = x$, commutativity property: $t(x, y) = t(y, x)$, monotonicity property: $(x \leq x', y \leq y') \to t(x, y) \leq t(x', y')$, and associativity property: $t(t(x, y), z) = t(x, t(y, z))$. T-conorm (or s-norm) is a mapping that satisfies boundary conditions: $s(x, 0) = x$, $s(x, 1) = 1$, and commutativity, monotonicity and associativity properties. In this work, we have considered AP t-norm and BS/Max t-conorm as the AND and OR operators. They are defined as follows:

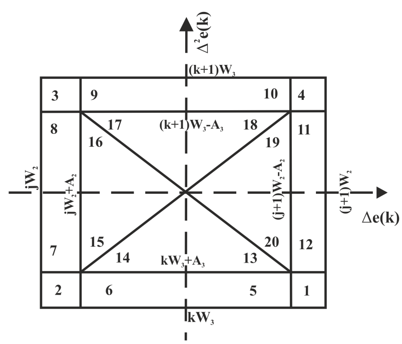
Algebraic product (AP):	$t(\mu_A(x), \mu_B(y)) = \mu_A(x)\mu_B(y)$
Bounded sum (BS):	$s_1(\mu_A(x), \mu_B(y)) = min(1, \mu_A(x) + \mu_B(y))$
Maximum (Max):	$s_2(\mu_A(x), \mu_B(y)) = max(\mu_A(x), \mu_B(y))$

Note that $s_2 \leq s_1$.

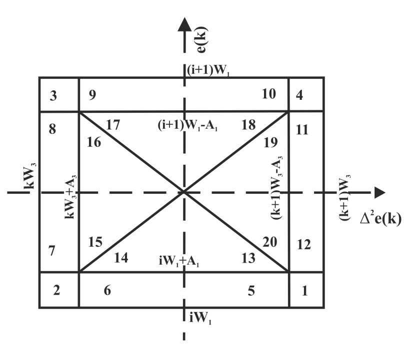
Figure 5 Representation of 3-dimensional input space into 2-dimensional planes, (a) $e(k) - \Delta e(k)$ plane (b) $\Delta e(k) - \Delta^2 e(k)$ plane (c) $\Delta^2 e(k) - e(k)$ plane (inner regions)



(a)



(b)



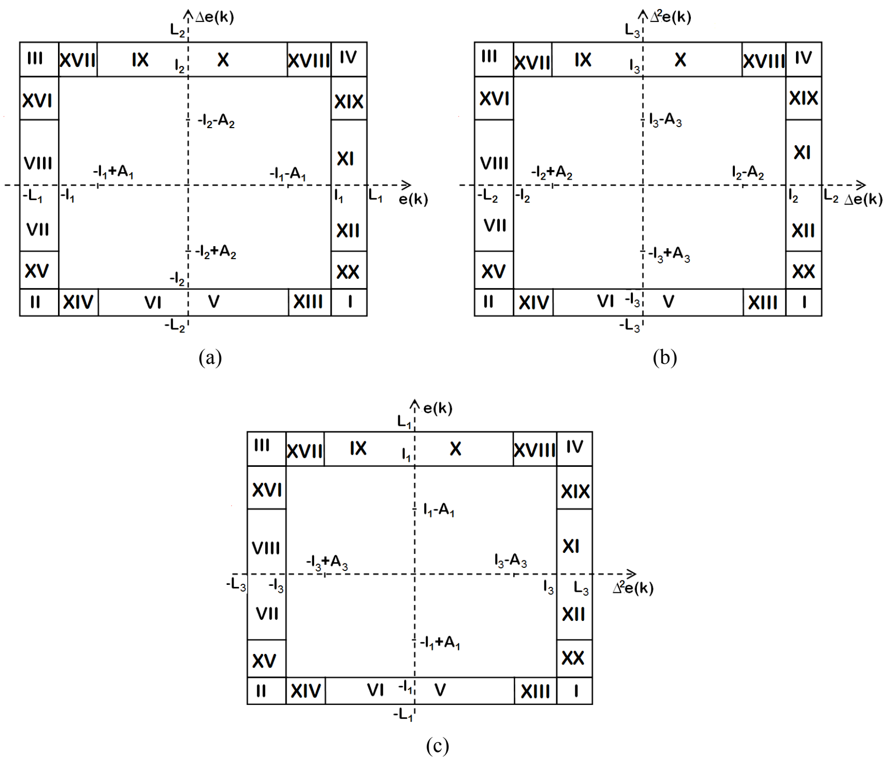
(c)

Two classes of controllers are defined using the combinations of t-norm and t-co-norms:

Class 1 with t and s_1

Class 2 with t and s_2.

Figure 6 Representation of 3-dimensional input space into 2-dimensional planes, (a) $e(k) - \Delta e(k)$ plane (b) $\Delta e(k) - \Delta^2 e(k)$ plane (c) $\Delta^2 e(k) - e(k)$ plane (outer regions)



The inputs form a 3-dimensional space. To visualise the 3D space clearly, we have drawn 2D planes as shown in Figures 5 and 6. Now any point in this 3D space can be located by taking its projections on $e - \Delta e$, $\Delta e - \Delta^2 e$ and $\Delta^2 e - e$ planes. So a point, say $(e^*, \Delta e^*, \Delta^2 e^*)$, can be located uniquely in the 3D space by a triplet (n_1, n_2, n_3), called a cell, where n_1, n_2, n_3 = 1, 2, ..., 20, I, II, ..., XX as shown in Figures 5 and 6. A cell is said to be valid if the relation between e and Δe in Figure 5(a) [or Figure 6(a)] and the relation between Δe and $\Delta^2 e$ in Figure 5(b) [or Figure 6(b)] produce the relation between $\Delta^2 e$ and e in Figure 5(c) [or Figure 6(c)]. For example, the triplet (13, 14, 16) denotes a valid cell with 13 taken from Figure 5(a), 14 taken from Figure 5(b) and 16 taken from Figure 5(c). Similarly, a triplet (II, XV, XIV) denotes a valid cell with II taken from Figure 6(a), XV taken from Figure 6(b) and XIV taken from Figure 6(c). A cell defines the space where the input variables lie. The control rules are applied in each cell to obtain the corresponding control law. The rule base consists of two parts – antecedent part and consequent part. The antecedent part of the rule base is evaluated and tabulated in Tables 1–3. Some cells are clubbed together in the form of regions as their resultant membership functions are the same.

Table 1 Resultant membership functions in outer regions for both the classes of controllers

Regions	*CELLS*	μ_1	μ_2	μ_3	μ_4
1	(I,II,III), (I,XV,XVII), (II,III,I), (II,XVI,XIII), (III,I,II), (III,XX,XIV), (XIII,II,XVI), (XIV,III,XX), (XV,XVII,I), (XVI,XIII,II), (XVII,I,XV), (XX,XIV,III)	0	1	0	0
2	(I,III,IV), (I,XVI,XVIII), (III,IV,I), (III,XIX,XIII), (IV,I,III), (IV,XX,XVII), (XIII,III,XIX), (XIV,XVIII,I), (XVII,IV,XX), (XVIII,I,XVI), (XIX,XIII,III), (XX,XVII,IV)	0	0	1	0
3	(II,II,II), (II,XV,XIV), (XIV,II,XX), (XV,XIV,II)	1	0	0	0
4	(IV,IV,IV), (IV,XIX,XIV), (XVIII,IV,XIX), (XIX,XVIII,IV)	0	0	0	1
5	(I,VII,IX), (I,VIII,X), (III,XI,V), (III,XII,VI)	0	μ_{-J_3}	μ_{J_3}	0
6	(II,VII,VI), (II,VIII,V)	μ_{-J_3}	μ_{J_3}	0	0
7	(IV,XI,X), (IV,XII,IX)	0	0	μ_{-J_3}	μ_{J_3}
8	(V,II,VIII), (VI,II,VII)	μ_{-J_1}	μ_{J_1}	0	0
9	(V,III,XI), (VI,III,XII), (IX,I,VII), (X,I,VIII)	0	μ_{-J_1}	μ_{J_1}	0
10	(VII,VI,II), (VIII,V,II)	μ_{-J_2}	μ_{J_2}	0	0
11	(VII,IX,I), (VIII,X,I), (XI,V,III), (XII,VI,III)	0	μ_{-J_2}	μ_{J_2}	0
12	(IX,IV,XII), (X,IV,XI)	0	0	μ_{-J_1}	μ_{J_1}
13	(XI,X,IV), (XII,IX,IV)	0	0	μ_{-J_2}	μ_{J_2}

2.3 Defuzzification

A defuzzification interface converts the conclusions of the inference mechanism into actual inputs to the process. It provides a means to choose a single output from the overall output generated by the control rules.

A typical fuzzy rule in a first-order TS fuzzy model has the form

IF x is A_i AND y is B_i THEN $z_i = p_i x + q_i y$

where A and B are the fuzzy sets in the antecedent, while $z_i = p_i x + q_i y$ is a crisp function in the consequent. Since each rule has a crisp output, the final output z inferred from n implications is given as the average of all z_i with the weights μ_i:

$$z = \frac{\sum_{i=1}^{n} \mu_i \cdot z_i}{\sum_{i=1}^{n} \mu_i}$$

where $\mu_i = \mu_{A_i}(x).\mu_{B_i}(y)$. Notice that z_i is a linear mapping. We see that the TS fuzzy system performs a nonlinear interpolation between linear mappings. As an example, suppose that $n = 2$, and that we have the rules

IF x is A_1 THEN $z_1 = p_1 x$
IF x is A_2 THEN $z_2 = p_2 x$

with the UoD for x given in Figure 7 so that μ_1 and μ_2 represent A_1 and A_2. We have

$$z = \frac{\mu_1 z_1 + \mu_2 z_2}{\mu_1 + \mu_2} = \mu_1 z_1 + \mu_2 z_2$$

as $\mu_1 + \mu_2 = 1$. We see that for $x > 1$, $\mu_1 = 0$ and $\mu_2 = 1$, so $z = z_2 = p_2 x$ which is a straight line. If $x < -1$, $\mu_1 = 1$ and $\mu_2 = 0$, so $z = z_1 = p_1 x$ which is another straight line as $p_1 \neq p_2$. For $-1 \leq x \leq 1$, the output z is an interpolation between the two straight lines. Figure 8 shows how this interpolation is achieved. Thus, in general, the TS fuzzy system provides an intuitive representation of a nonlinear system as a nonlinear interpolation between n linear mappings.

Figure 7 Membership functions for TS fuzzy system example

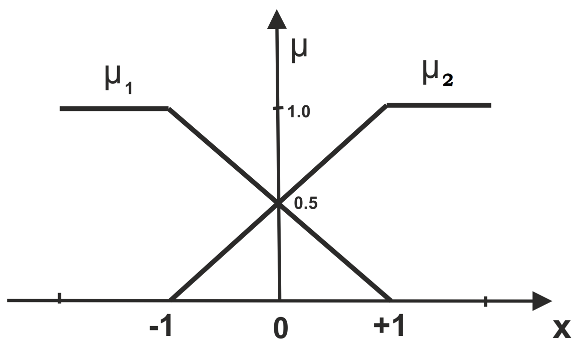


Figure 8 Interpolator between linear mappings (see online version for colours)

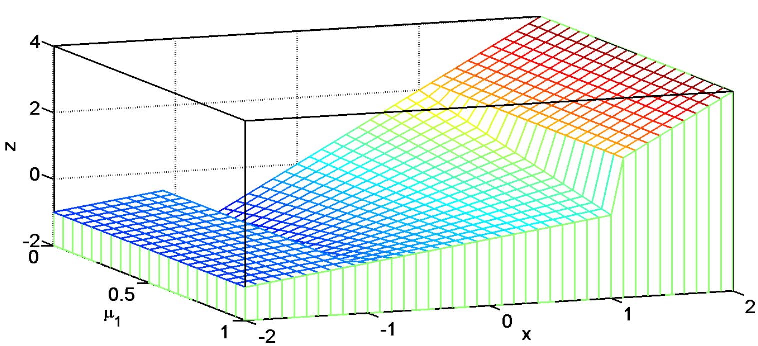


Table 2 Resultant membership functions in inner regions for class 1 controller

Regions	*CELLS*	μ_1	μ_2	μ_3	μ_4
1	(1,2,3), (2,3,1), (3,1,2)	0	1	0	0
2	(1,3,4), (3,4,1), (4,1,3)	0	0	1	0
3	(2,2,2)	1	0	0	0
4	(4,4,4)	0	0	0	1
5	(1,7,9), (1,8,10), (3,11,5), (3,12,6)	0	μ_k	μ_{k+1}	0
6	(2,7,6), (2,8,5)	μ_k	μ_{k+1}	0	0
7	(4,11,10), (4,12,9)	0	0	μ_k	μ_{k+1}
8	(5,2,8), (6,2,7)	μ_i	μ_{i+1}	0	0
9	(5,3,11), (6,3,12), (9,1,7), (10,1,8)	0	μ_i	μ_{i+1}	0
10	(7,6,2), (8,5,2)	μ_j	μ_{j+1}	0	0
11	(7,9,1), (8,10,1), (11,5,3), (12,6,3)	0	μ_j	μ_{j+1}	0
12	(9,4,12), (10,4,11)	0	0	μ_i	μ_{i+1}
13	(11,10,4), (12,9,4)	0	0	μ_j	μ_{j+1}
14	(5,7,16), (5,7,17), (5,8,18), (5,8,19), (6,7,14), (6,7,15), (6,8,13), (6,8,20)	$\mu_i\mu_k$	$\mu_{i+1}\mu_k + \mu_i\mu_{k+1}$	$\mu_{i+1}\mu_{k+1}$	0
15	(7,14,6), (7,15,6), (7,16,5), (7,17,5), (8,13,6), (8,18,5), (8,19,5), (8,20,6)	$\mu_j\mu_k$	$\mu_j\mu_{k+1} + \mu_{j+1}\mu_k$	$\mu_{j+1}\mu_{k+1}$	0
16	(9,11,13), (9,11,20), (9,12,14), (9,12,15), (10,11,18), (10,11,19), (10,12,16), (10,12,17)	0	$\mu_i\mu_k$	$\mu_i\mu_{k+1} + \mu_{i+1}\mu_k$	$\mu_{i+1}\mu_{k+1}$
17	(11,13,9), (11,18,10), (11,19,10), (11,20,9), (12,14,9), (12,15,9), (12,16,10), (12,17,10)	0	$\mu_j\mu_k$	$\mu_j\mu_{k+1} + \mu_{j+1}\mu_k$	$\mu_{j+1}\mu_{k+1}$
18	(13,6,8), (14,6,7), (15,6,7), (16,5,7), (17,5,7), (18,5,8), (19,5,8), (20,6,8)	$\mu_i\mu_j$	$\mu_i\mu_{j+1} + \mu_{i+1}\mu_j$	$\mu_{i+1}\mu_{j+1}$	0
19	(13,9,11), (14,9,12), (15,9,12), (16,10,12), (17,10,12), (18,10,11), (19,10,11), (20,9,11)	0	$\mu_i\mu_j$	$\mu_i\mu_{j+1} + \mu_{i+1}\mu_j$	$\mu_{i+1}\mu_{j+1}$
20	$\sharp$	$\mu_i\mu_j\mu_k$	$\mu_i\mu_j\mu_{k+1}$ $+\mu_i\mu_{j+1}\mu_k$ $+\mu_{i+1}\mu_j\mu_k$	$\mu_i\mu_{j+1}\mu_{k+1}$ $+\mu_{i+1}\mu_j\mu_{k+1}$ $+\mu_{i+1}\mu_{j+1}\mu_k$	$\mu_{i+1}\mu_{j+1}\mu_{k+1}$

$\sharp \Longrightarrow$ (13,14,16), (13,15,16), (13,15,17), (13,16,18), (13,16,19), (13,17,19), (14,14,15), (14,15,14), (14,15,15), (14,16,13), (14,16,20), (14,17,20), (15,14,14), (15,14,15), (15,15,14), (15,16,13), (15,17,13), (15,17,20), (16,13,14), (16,13,15), (16,18,13), (16,18,20), (16,19,13), (16,20,14), (17,13,15), (17,18,20), (17,19,13), (17,19,20), (17,20,14), (17,20,15), (18,13,16), (18,18,19), (18,19,18), (18,19,19), (18,20,16), (18,20,17), (19,13,16), (19,13,17), (19,18,18), (19,18,19), (19,19,18), (19,20,17), (20,14,16), (20,14,17), (20,15,17), (20,16,18), (20,17,18), (20,17,19)

Table 3 Resultant membership functions in inner regions for class 2 controller

Regions	*CELLS*	μ_1	μ_2	μ_3	μ_4
1–13	Same as in Table 2				
14	(5,7,16), (5,7,17), (5,8,18), (6,7,15)	$\mu_i\mu_k$	$\mu_{i+1}\mu_k$	$\mu_{i+1}\mu_{k+1}$	0
15	(5,8,19), (6,7,14), (6,8,13), (6,8,20)	$\mu_i\mu_k$	$\mu_i\mu_{k+1}$	$\mu_{i+1}\mu_{k+1}$	0
16	(7,14,6), (8,13,6), (8,19,5), (8,20,6)	$\mu_j\mu_k$	$\mu_{j+1}\mu_k$	$\mu_{j+1}\mu_{k+1}$	0
17	(7,15,6), (7,16,5), (7,17,5), (8,18,5)	$\mu_j\mu_k$	$\mu_j\mu_{k+1}$	$\mu_{j+1}\mu_{k+1}$	0
18	(9,11,13), (9,11,20), (9,12,14), (10,11,19)	0	$\mu_i\mu_k$	$\mu_i\mu_{k+1}$	$\mu_{i+1}\mu_{k+1}$
19	(9,12,15), (10,11,18), (10,12,16), (10,12,17)	0	$\mu_i\mu_k$	$\mu_{i+1}\mu_k$	$\mu_{i+1}\mu_{k+1}$
20	(11,13,9), (11,19,10), (11,20,9), (12,14,9)	0	$\mu_j\mu_k$	$\mu_{j+1}\mu_k$	$\mu_{j+1}\mu_{k+1}$
21	(11,18,10), (12,15,9), (12,16,10), (12,17,10)	0	$\mu_j\mu_k$	$\mu_j\mu_{k+1}$	$\mu_{j+1}\mu_{k+1}$
22	(13,6,8), (14,6,7), (19,5,8), (20,6,8)	$\mu_i\mu_j$	$\mu_{i+1}\mu_j$	$\mu_{i+1}\mu_{j+1}$	0
23	(13,9,11), (14,9,12), (19,10,11), (20,9,11)	0	$\mu_i\mu_j$	$\mu_{i+1}\mu_j$	$\mu_{i+1}\mu_{j+1}$
24	(15,6,7), (16,5,7), (17,5,7), (18,5,8)	$\mu_i\mu_j$	$\mu_i\mu_{j+1}$	$\mu_{i+1}\mu_{j+1}$	0
25	(15,9,12), (16,10,12), (17,10,12), (18,10,11)	0	$\mu_i\mu_j$	$\mu_i\mu_{j+1}$	$\mu_{i+1}\mu_{j+1}$
26	(13,14,16), (14,14,15), (19,13,16), (19,13,17), (19,19,18), (19,20,17), (20,14,16), (20,14,17)	$\mu_i\mu_j\mu_k$	$\mu_{i+1}\mu_j\mu_k$	$\mu_{i+1}\mu_{j+1}\mu_k$	$\mu_{i+1}\mu_{j+1}\mu_{k+1}$
27	(13,15,16), (13,15,17), (13,16,18), (14,15,15), (19,18,18), (20,15,17), (20,16,18), (20,17,18)	$\mu_i\mu_j\mu_k$	$\mu_{i+1}\mu_j\mu_k$	$\mu_{i+1}\mu_j\mu_{k+1}$	$\mu_{i+1}\mu_{j+1}\mu_{k+1}$
28	(13,16,19), (13,17,19), (14,15,14), (14,16,13), (14,16,20), (14,17,20), (19,18,19), (20,17,19)	$\mu_i\mu_j\mu_k$	$\mu_i\mu_j\mu_{k+1}$	$\mu_{i+1}\mu_j\mu_{k+1}$	$\mu_{i+1}\mu_{j+1}\mu_{k+1}$
29	(15,14,14), (16,13,14), (16,19,13), (16,20,14), (17,19,13), (17,19,20), (17,20,14), (18,19,19)	$\mu_i\mu_j\mu_k$	$\mu_i\mu_{j+1}\mu_k$	$\mu_i\mu_{j+1}\mu_{k+1}$	$\mu_{i+1}\mu_{j+1}\mu_{k+1}$
30	(15,14,15), (16,13,15), (17,13,15), (17,20,15), (18,13,16), (18,19,18), (18,20,16), (18,20,17)	$\mu_i\mu_j\mu_k$	$\mu_i\mu_{j+1}\mu_k$	$\mu_{i+1}\mu_{j+1}\mu_k$	$\mu_{i+1}\mu_{j+1}\mu_{k+1}$
31	(15,15,14), (15,16,13), (15,17,13), (15,17,20), (16,18,13), (16,18,20), (17,18,20), (18,18,19)	$\mu_i\mu_j\mu_k$	$\mu_i\mu_j\mu_{k+1}$	$\mu_i\mu_{j+1}\mu_{k+1}$	$\mu_{i+1}\mu_{j+1}\mu_{k+1}$

We apply the above weighted average defuzzification method to obtain the expression for crisp incremental control output, given by

$$\Delta u_s(k) = \frac{\sum_{l=1}^{4} \mu_l(\Delta u) \cdot \Delta u_l(k)}{\sum_{l=1}^{4} \mu_l(\Delta u)} = \frac{\sum_{l=1}^{4} \mu_l(\Delta u) \left[a_{|i+j+k+l-1|} e(k) + b_{|i+j+k+l-1|} \Delta e(k) + c_{|i+j+k+l-1|} \Delta^2 e(k)\right]}{\sum_{l=1}^{4} \mu_l(\Delta u)} \tag{2}$$

The output of the TS fuzzy PID controller is obtained by $u(k) = u(k-1) + \Delta u(k)$.

3 Analytical structures and properties of TS fuzzy PID controllers

The models of the general TS fuzzy PID controller are derived in this section. The input-output structural relationship of the TS fuzzy PID controller takes the following form:

$$\begin{aligned} \Delta u_s(k) = {} & \gamma_1(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k)) \cdot e(k) + \gamma_2(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k)) \cdot \Delta e(k) \\ & + \gamma_3(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k)) \cdot \Delta^2 e(k) \end{aligned} \tag{3}$$

where $\tilde{e}(k) = e(k) - (i+0.5)W_1$, $\Delta\tilde{e}(k) = \Delta e(k) - (j+0.5)W_2$, $\Delta^2\tilde{e}(k) = \Delta^2 e(k) - (k+0.5)W_3$, and $\gamma_1(.)$, $\gamma_2(.)$, and $\gamma_3(.)$ are the variable gains of the fuzzy controller. On comparing with the expression of linear PID controller given in equation (1), we can say that $\gamma_1(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$ is variable integral gain, $\gamma_2(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$ is variable proportional gain, and $\gamma_3(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$ is variable derivative gain. The exact expressions for these gains have been derived for the general TS fuzzy PID controllers.

When the inputs $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$ lie within $[-l_1, l_1]$, $[-l_2, l_2]$, and $[-l_3, l_3]$, respectively, at any sampling instant k, the input variables must satisfy $iW_1 \le e(k) \le (i+1)W_1$, $jW_2 \le \Delta e(k) \le (j+1)W_2$, and $kW_3 \le \Delta e(k) \le (k+1)W_3$. This is shown in Figure 5. The analytical structure has been computed using the outcomes of the rule premise parts and is tabulated in Tables 4 and 5. The expressions of θ_1, θ_2 and θ_3 in Tables 4 and 5 are given by

$$\begin{aligned} \theta_1 &= 0.5(2B_1 - W_1) = 0.5(B_1 - A_1) \\ \theta_2 &= 0.5(2B_2 - W_2) = 0.5(B_2 - A_2) \\ \theta_3 &= 0.5(2B_3 - W_3) = 0.5(B_3 - A_3) \end{aligned}$$

Table 4 Expressions of gains in inner regions for class 1 controller

Regions	$\gamma_1(\tilde{e}, \Delta\tilde{e}, \Delta^2\tilde{e})$, $\gamma_2(\tilde{e}, \Delta\tilde{e}, \Delta^2\tilde{e})$, $\gamma_3(\tilde{e}, \Delta\tilde{e}, \Delta^2\tilde{e})$
1, 2	C_2
3, 4	C_1
5, 6, 7	$\frac{(\theta_3-\Delta^2\tilde{e})C_2+(\theta_3+\Delta^2\tilde{e})C_3}{2\theta_3}$
8, 9, 12	$\frac{(\theta_1-\tilde{e})C_2+(\theta_1+\tilde{e})C_3}{2\theta_1}$
10, 11, 13	$\frac{(\theta_2-\Delta\tilde{e})C_2+(\theta_2+\Delta\tilde{e})C_3}{2\theta_2}$
14, 16	$\frac{\tilde{e}\Delta^2\tilde{e}(C_1-2C_2+C_3)+(\theta_1\Delta^2\tilde{e}+\theta_3\tilde{e})(-C_1+C_3)+\theta_1\theta_3(C_1+2C_2+C_3)}{4\theta_1\theta_3}$
15, 17	$\frac{\Delta\tilde{e}\Delta^2\tilde{e}(C_1-2C_2+C_3)+(\theta_2\Delta^2\tilde{e}+\theta_3\Delta\tilde{e})(-C_1+C_3)+\theta_2\theta_3(C_1+2C_2+C_3)}{4\theta_2\theta_3}$
18, 19	$\frac{\tilde{e}\Delta\tilde{e}(C_1-2C_2+C_3)+(\theta_1\Delta\tilde{e}+\theta_2\tilde{e})(-C_1+C_3)+\theta_1\theta_2(C_1+2C_2+C_3)}{4\theta_1\theta_2}$
20	$\frac{[\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e}(-C_1+3C_2-3C_3+C_4)+(\theta_1\Delta\tilde{e}\Delta^2\tilde{e}+\theta_2\tilde{e}\Delta^2\tilde{e}+\theta_3\tilde{e}\Delta\tilde{e})(C_1-C_2-C_3+C_4)+(\theta_1\theta_2\Delta^2\tilde{e}+\theta_2\theta_3\tilde{e}+\theta_1\theta_3\Delta\tilde{e})(-C_1-C_2+C_3+C_4)+\theta_1\theta_2\theta_3(C_1+3C_2+3C_3+C_4)]}{8\theta_1\theta_2\theta_3}$

Table 5 Expressions of gains in inner regions for class 2 controller

Regions	$\gamma_1(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$, $\gamma_2(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$, $\gamma_3(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$
1–13	Same as in Table 4
14; 15, 19; 18	$\frac{\tilde{e}\Delta^2\tilde{e}(C_1-C_2+C_3)+\theta_1\Delta^2\tilde{e}(-C_1\mp C_2+C_3)+\theta_3\tilde{e}(-C_1\pm C_2+C_3)+\theta_1\theta_3(C_1+C_2+C_3)}{\tilde{e}\Delta^2\tilde{e}\mp\theta_1\Delta^2\tilde{e}\pm\theta_3\tilde{e}+3\theta_1\theta_3}$
16; 17, 20; 21	$\frac{\Delta\tilde{e}\Delta^2\tilde{e}(C_1-C_2+C_3)+\theta_2\Delta^2\tilde{e}(-C_1\mp C_2+C_3)+\theta_3\Delta\tilde{e}(-C_1\pm C_2+C_3)+\theta_2\theta_3(C_1+C_2+C_3)}{\Delta\tilde{e}\Delta^2\tilde{e}\mp\theta_2\Delta^2\tilde{e}\pm\theta_3\Delta\tilde{e}+3\theta_2\theta_3}$
22; 24, 23; 25	$\frac{\tilde{e}\Delta\tilde{e}(C_1-C_2+C_3)+\theta_1\Delta\tilde{e}(-C_1\mp C_2+C_3)+\theta_2\tilde{e}(-C_1\pm C_2+C_3)+\theta_1\theta_2(C_1+C_2+C_3)}{\tilde{e}\Delta\tilde{e}\mp\theta_1\Delta\tilde{e}\pm\theta_2\tilde{e}+3\theta_1\theta_2}$
26; 31	$\frac{\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e}(-C_1+C_2-C_3+C_4)+\theta_1\Delta\tilde{e}\Delta^2\tilde{e}(C_1\pm C_2\mp C_3+C_4)+\theta_2\tilde{e}\Delta^2\tilde{e}(C_1-C_2-C_3+C_4)+\theta_3\tilde{e}\Delta\tilde{e}(C_1\mp C_2\pm C_3+C_4)+\theta_1\theta_2\Delta^2\tilde{e}(-C_1\mp C_2\mp C_3+C_4)+\theta_2\theta_3\tilde{e}(-C_1\pm C_2\pm C_3+C_4)+\theta_1\theta_3\Delta\tilde{e}(-C_1-C_2+C_3+C_4)+\theta_1\theta_2\theta_3(C_1+C_2+C_3+C_4)}{2(\theta_1\Delta\tilde{e}\Delta^2\tilde{e}+\theta_3\tilde{e}\Delta\tilde{e}\mp\theta_1\theta_2\Delta^2\tilde{e}\pm\theta_2\theta_3\tilde{e}+2\theta_1\theta_2\theta_3)}$
27; 29	$\frac{\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e}(-C_1+C_2-C_3+C_4)+\theta_1\Delta\tilde{e}\Delta^2\tilde{e}(C_1\pm C_2\mp C_3+C_4)+\theta_2\tilde{e}\Delta^2\tilde{e}(C_1\mp C_2\pm C_3+C_4)+\theta_3\tilde{e}\Delta\tilde{e}(C_1-C_2-C_3+C_4)+\theta_1\theta_2\Delta^2\tilde{e}(-C_1-C_2+C_3+C_4)+\theta_2\theta_3\tilde{e}(-C_1\pm C_2\pm C_3+C_4)+\theta_1\theta_3\Delta\tilde{e}(-C_1\mp C_2\mp C_3+C_4)+\theta_1\theta_2\theta_3(C_1+C_2+C_3+C_4)}{2(\theta_1\Delta\tilde{e}\Delta^2\tilde{e}+\theta_2\tilde{e}\Delta^2\tilde{e}\pm\theta_2\theta_3\tilde{e}\mp\theta_1\theta_3\Delta\tilde{e}+2\theta_1\theta_2\theta_3)}$
28; 30	$\frac{\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e}(-C_1+C_2-C_3+C_4)+\theta_1\Delta\tilde{e}\Delta^2\tilde{e}(C_1-C_2-C_3+C_4)+\theta_2\tilde{e}\Delta^2\tilde{e}(C_1\mp C_2\pm C_3+C_4)+\theta_3\tilde{e}\Delta\tilde{e}(C_1\pm C_2\mp C_3+C_4)+\theta_1\theta_2\Delta^2\tilde{e}(-C_1\pm C_2\pm C_3+C_4)+\theta_2\theta_3\tilde{e}(-C_1-C_2+C_3+C_4)+\theta_1\theta_3\Delta\tilde{e}(-C_1\mp C_2\mp C_3+C_4)+\theta_1\theta_2\theta_3(C_1+C_2+C_3+C_4)}{2(\theta_2\tilde{e}\Delta^2\tilde{e}+\theta_3\tilde{e}\Delta\tilde{e}\pm\theta_1\theta_2\Delta^2\tilde{e}\mp\theta_1\theta_3\Delta\tilde{e}+2\theta_1\theta_2\theta_3)}$

Note: In Table 5, the meaning of $\pm$ and $\mp$ in regions r_1; r_2 – the top sign is for region r_1 and the bottom sign is for region r_2.

Similarly, the case when the inputs $e(k)$, $\Delta e(k)$, $\Delta^2 e(k)$ lie outside $[-l_1, l_1]$, $[-l_2, l_2]$, $[-l_3, l_3]$, respectively, is depicted in Figure 6. The analytical structure has been computed in the outer regions using the outcomes of the rule premise parts. The expressions of gains $\gamma_1(.)$, $\gamma_2(.)$, and $\gamma_3(.)$ in outer regions 1–13 are computed and are tabulated in Table 6 for both the classes of controllers. The values of coefficients

C_1, C_2, C_3, and C_4 for class 1 and class 2 PID controllers are given in Table 7. It is evident from Tables 4–6 that the gains $\gamma_1(.)$, $\gamma_2(.)$, and $\gamma_3(.)$ are nonlinear functions of the inputs. Also, one can notice that the structure of the controller varies from one region to the other. So the general TS fuzzy PID controllers are variable gain/structure controllers.

Note: For triangular membership functions $A_1 = A_2 = A_3 = 0$. Hence the expressions of gains in Tables 4 and 5 can be modified accordingly.

Table 6 Expressions of gains in outer regions for both classes of controllers

Region	$\gamma_1(\tilde{e}, \Delta\tilde{e}, \Delta^2\tilde{e})$, $\gamma_2(\tilde{e}, \Delta\tilde{e}, \Delta^2\tilde{e})$, $\gamma_3(\tilde{e}, \Delta\tilde{e}, \Delta^2\tilde{e})$
1, 2	C_2
3, 4	C_1
5, 6, 7	$\dfrac{(B_3 - l_3 - \Delta^2 e)C_2 + (B_3 - l_3 + \Delta^2 e)C_3}{2(B_3 - l_3)}$
8, 9, 12	$\dfrac{(B_1 - l_1 - e)C_2 + (B_1 - l_1 + e)C_3}{2(B_1 - l_1)}$
10, 11, 13	$\dfrac{(B_2 - l_2 - \Delta e)C_2 + (B_2 - l_2 + \Delta e)C_3}{2(B_2 - l_2)}$

Table 7 Coefficients for class 1 and class 2 controllers

Regions class 1 controller	*Regions class 2 controller*	$\gamma_1(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$, $\gamma_2(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$, $\gamma_3(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$			
		C_1	C_2	C_3	C_4
1, 3, 5, 9, 11, 14, 15, 18, 20	1, 3, 5, 9, 11, 14–17, 22, 24, 26–31	$X_{\lvert i+j+k \rvert}$	$X_{\lvert i+j+k+1 \rvert}$	$X_{\lvert i+j+k+2 \rvert}$	$X_{\lvert i+j+k+3 \rvert}$
2, 7, 12, 13, 16, 17, 19	2, 7, 12, 13, 18–21, 23, 25	$X_{\lvert i+j+k+1 \rvert}$	$X_{\lvert i+j+k+2 \rvert}$	$X_{\lvert i+j+k+3 \rvert}$	$X_{\lvert i+j+k \rvert}$
4, 6, 8, 10	4, 6, 8, 10	$X_{\lvert i+j+k+3 \rvert}$	$X_{\lvert i+j+k \rvert}$	$X_{\lvert i+j+k+1 \rvert}$	$X_{\lvert i+j+k+2 \rvert}$
where $X = a$ for $\gamma_1(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$; $X = b$ for $\gamma_2(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$ and $X = c$ for $\gamma_3(\tilde{e}\Delta\tilde{e}\Delta^2\tilde{e})$					

Notes: A region is represented by a number whereas the cell is represented by a triplet (n_1, n_2, n_3). A region contains one or more cells.

3.1 Properties

Upon investigating the analytical structures of fuzzy controllers the following points can be enunciated:

- When all the inputs are zero, i.e., $[\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k)]$ = [0,0,0], the gains become

$$\begin{aligned}
\gamma_1(.) &= \frac{a_{|i+j+k|} + 3a_{|i+j+k+1|} + 3a_{|i+j+k+2|} + a_{|i+j+k+3|}}{8} \\
\gamma_2(.) &= \frac{b_{|i+j+k|} + 3b_{|i+j+k+1|} + 3b_{|i+j+k+2|} + b_{|i+j+k+3|}}{8} \qquad (4) \\
\gamma_3(.) &= \frac{c_{|i+j+k|} + 3c_{|i+j+k+1|} + 3c_{|i+j+k+2|} + c_{|i+j+k+3|}}{8}
\end{aligned}$$

for class 1 controller, and

$$
\begin{aligned}
\gamma_1(.) &= \frac{a_{|i+j+k|} + a_{|i+j+k+1|} + a_{|i+j+k+2|} + a_{|i+j+k+3|}}{4} \\
\gamma_2(.) &= \frac{b_{|i+j+k|} + b_{|i+j+k+1|} + b_{|i+j+k+2|} + b_{|i+j+k+3|}}{4} \\
\gamma_3(.) &= \frac{c_{|i+j+k|} + c_{|i+j+k+1|} + c_{|i+j+k+2|} + c_{|i+j+k+3|}}{4}
\end{aligned}
\tag{5}
$$

for class 2 controller. These gains are called static gains.

- The gains $\gamma_1(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$, $\gamma_2(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$, and $\gamma_3(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$ are nonlinear functions of the inputs $e(k)$, $\Delta e(k)$, and $\Delta^2 e(k)$.
- The necessary and sufficient condition for the general TS fuzzy PID controller to become a linear controller is $a_{|i+j+k|} = a_{|i+j+k+1|} = a_{|i+j+k+2|} = a_{|i+j+k+3|}$, $b_{|i+j+k|} = b_{|i+j+k+1|} = b_{|i+j+k+2|} = b_{|i+j+k+3|}$ and $c_{|i+j+k|} = c_{|i+j+k+1|} = c_{|i+j+k+2|} = c_{|i+j+k+3|}$.
- The gains $\gamma_1(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$, $\gamma_2(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$, and $\gamma_3(\tilde{e}(k), \Delta\tilde{e}(k), \Delta^2\tilde{e}(k))$ are bounded by the smallest value and the largest value of the rule consequent parameters. The gains satisfy the following inequalities:

$$
\begin{aligned}
&min(a_{|i+j+k|}, a_{|i+j+k+1|}, a_{|i+j+k+2|}, a_{|i+j+k+3|}) \\
&\qquad \le \gamma_1(.) \le max(a_{|i+j+k|}, a_{|i+j+k+1|}, a_{|i+j+k+2|}, a_{|i+j+k+3|}) \\
&min(b_{|i+j+k|}, b_{|i+j+k+1|}, b_{|i+j+k+2|}, b_{|i+j+k+3|}) \\
&\qquad \le \gamma_2(.) \le max(b_{|i+j+k|}, b_{|i+j+k+1|}, b_{|i+j+k+2|}, b_{|i+j+k+3|}) \\
&min(c_{|i+j+k|}, c_{|i+j+k+1|}, c_{|i+j+k+2|}, c_{|i+j+k+3|}) \\
&\qquad \le \gamma_3(.) \le max(c_{|i+j+k|}, c_{|i+j+k+1|}, c_{|i+j+k+2|}, c_{|i+j+k+3|})
\end{aligned}
\tag{6}
$$

The inequalities in equation (6) show that the gains $\gamma_1(.)$, $\gamma_2(.)$, and $\gamma_3(.)$ are bounded by the smallest value and the largest value of the rule consequent parameters.

4 Stability analysis

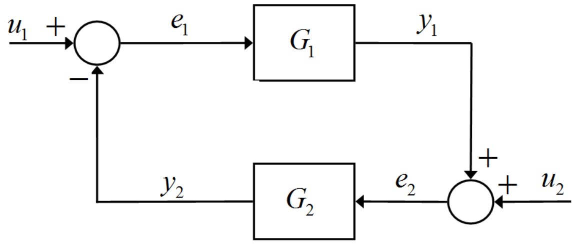
In this section, BIBO stability of the closed loop control system having one of the proposed fuzzy controllers in the loop has been investigated. The small gain theorem is used to establish the stability result. The result is presented in the following theorem:

Theorem: The sufficient condition for the BIBO stability of the feedback control system in the whole input space, defined by $[iW_1, (i+1)W_1] \times [jW_2, (j+1)W_2] \times [kW_3, (k+1)W_3]$ can be stated as

$$
max(a_{ijk} + b_{ijk} + c_{ijk}).||G_2|| < 1 \tag{7}
$$

for $i = -J_1, ..., 0, ..., J_1 - 1$; $j = -J_2, ..., 0, ..., J_2 - 1$ and $k = -J_3, ..., 0, ..., J_3 - 1$. where $max()$ operator chooses the maximum value from a set of parameters values for all i, j and k.

Figure 9 A feedback control system



Proof: Consider a closed loop control system as shown in Figure 9. The following equations govern the closed loop system.

$$e_1 = u_1 - y_2, e_2 = u_2 + y_1, y_1 = G_1 e_1, y_2 = G_2 e_2$$

Considering G_1 as TS fuzzy controller and G_2 as any nonlinear plant, and using the small gain theorem, we can state that $||G_1||.||G_2|| < 1$, i.e., the product of gains of G_1 and G_2 must be less than 1. Also for any bounded input pair (u_1, u_2), the output pair (y_1, y_2) must be bounded. The norm of plant G_2 is defined as

$$||G_2|| := \sup_{\substack{\tilde{u}_1 \neq \tilde{u}_2 \\ k \geq 0}} \frac{|\tilde{y}_1(k) - \tilde{y}_2(k)|}{|\tilde{u}_1(k) - \tilde{u}_2(k)|}$$

where $\tilde{u}_1(k)$ and $\tilde{u}_2(k)$ are any two control signals in the set $\{\tilde{u}(k)\}$. $\tilde{y}_1(k)$ and $\tilde{y}_2(k)$ are outputs of plant G_2 for control signals $\tilde{u}_1(k)$ and $\tilde{u}_2(k)$, respectively. The norm of G_2 is the gain of the nonlinear plant under consideration.

From equation (3) we have,

$$\begin{aligned}
||\Delta u_s(k)|| &= ||\gamma_1(.)e(k) + \gamma_2(.)\Delta ek) + \gamma_3(.)\Delta^2 e(k)|| \\
&= ||\gamma_1(.)e(k) + \gamma_2(.)(e(k) - e(k-1)) + \gamma_3(.)(e(k) - 2e(k-1) \\
&\quad + e(k-2))|| \\
&\leq ||\gamma_1(.) + \gamma_2(.) + \gamma_3(.)||.|e(k)| + ||\gamma_2(.) + 2\gamma_3(.)||.|e(k-1)| \\
&\quad + ||\gamma_3(.)||.|e(k-2)| \\
&\leq ||\gamma_1(.) + \gamma_2(.) + \gamma_3(.)||.|e(k)| + ||\gamma_2(.) + 2\gamma_3(.)||.e_{max1} + ||\gamma_3(.)||.e_{max2}
\end{aligned}$$

where e_{max1} and e_{max2} are the suprema of the error signal, defined as

$$e_{max1} := \sup_{k \geq 1} |e(k-1)|$$

$$e_{max2} := \sup_{k \geq 2} |e(k-2)|$$

From equation (6) we have

$$||\gamma_1(.) + \gamma_2(.) + \gamma_3(.)|| \leq a_{ijk} + b_{ijk} + c_{ijk};$$

$$||\gamma_2(.) + 2\gamma_3(.)|| \leq b_{ijk} + 2c_{ijk}; \text{ and } ||\gamma_3(.)|| \leq c_{ijk}$$

where

$$a_{ijk} = max(a_{|i+j+k|}, a_{|i+j+k+1|}, a_{|i+j+k+2|}, a_{|i+j+k+3|}),$$
$$b_{ijk} = max(b_{|i+j+k|}, b_{|i+j+k+1|}, b_{|i+j+k+2|}, b_{|i+j+k+3|}) \text{ and}$$
$$c_{ijk} = max(c_{|i+j+k|}, c_{|i+j+k+1|}, c_{|i+j+k+2|}, c_{|i+j+k+3|})$$

Therefore,

$$||\Delta u_s(k)|| \leq (a_{ijk} + b_{ijk} + c_{ijk})||e(k)|| + \delta$$

where $\delta = (b_{ijk} + 2c_{ijk})e_{max1} + c_{ijk}e_{max2}$.

Hence, by applying the small gain theorem we obtain

$$max(a_{ijk} + b_{ijk} + c_{ijk}).||G_2|| < 1. \qquad \square$$

5 Simulation studies

Two examples of nonlinear dynamical systems have been considered to investigate the behaviour of the proposed fuzzy PID controllers. We perform simulation studies on nonlinear systems considering trapezoidal membership functions ($A_1 \neq 0$, $A_2 \neq 0$ and $A_3 \neq 0$) and the least number of fuzzy sets for the inputs $e(k)$, $\Delta e(k)$ and $\Delta^2 e(k)$, i.e., $N_1 = N_2 = N_3 = 3$.

Example 1: We consider a nonlinear plant (Mohan, 2011) having the dynamics

$$\dot{y}(t) = y(t) + \sin^2(\sqrt{|y(t)|}) + u(t) \tag{8}$$

Table 8 Parameters of linear and fuzzy controllers in Example 1

		Linear PID	
	K_P=12.54	K_I=53.643	K_D=0.056
		Fuzzy PID class 1	
$S_{\Delta u}^{-1}$=2.55	a_0=0.8838	b_0=46.956	c_0=1.1264
	a_1=0.862	b_1=34.996	c_1=1.664
	a_2=0.6835	b_2=40.134	c_2=8.64
	a_3=0.258	b_3=15.756	c_3=0.272
	A_1=43.33	A_2=0.2365	A_3=215.44
	B_1=134.5	B_2=0.3987	B_3=350.66
		Fuzzy PID class 2	
$S_{\Delta u}^{-1}$=6.8342	a_0=0.9206	b_0=28.986	c_0=1.408
	a_1=0.8979	b_1=21.603	c_1=2.08
	a_2=0.712	b_2=24.775	c_2=10.8
	a_3=0.2688	b_3=9.726	c_3=0.34
	A_1=41.6	A_2=0.383	A_3=172.35
	B_1=129.12	B_2=0.646	B_3=280.525

A step input of magnitude 10 is applied as the reference signal and the performance of the proposed fuzzy controllers is evaluated in comparison with the corresponding linear PID controller.

Genetic algorithm (GA) is used to find the optimal parameters of linear and fuzzy PID controllers. The cost function is selected as

$$J_{PID} = \frac{1}{T}\int_0^T (e^2(t) + u^2(t))dt \tag{9}$$

The number of tuneable parameters of the linear PID controller is 3 while it is 19 for the proposed fuzzy PID controller. These parameters are optimised. The values of the tuneable parameters of the controllers are given in Table 8.

Figure 10 Step responses of nonlinear system in Example 1 (see online version for colours)

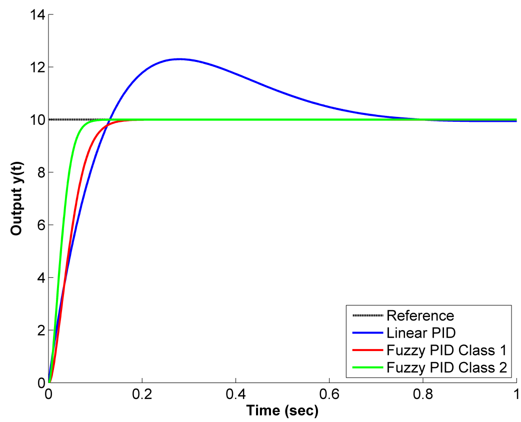
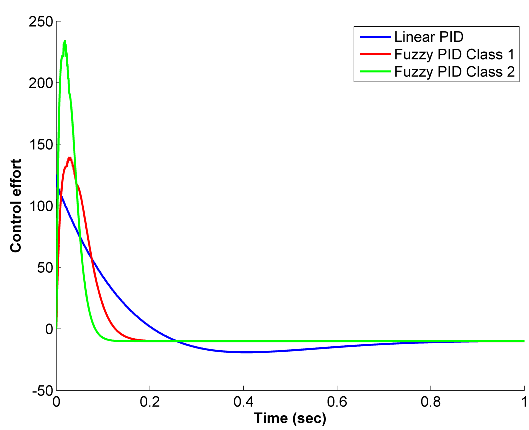


Figure 11 Control efforts for nonlinear system in Example 1 (see online version for colours)



The output step responses of the nonlinear plant and the control efforts of linear and fuzzy PID controllers are shown in Figures 10 and 11. Sampling time T_s = 1 ms is considered for simulation. The time-domain performance data related to the linear and fuzzy PID controllers is enlisted in Table 9. The performance of the proposed general TS fuzzy PID controllers is better compared to the linear PID controller performance.

Table 9 Time-domain performance data for Example 1

Controller	t_r *(in sec)*	t_s *(in sec)*	M_p *(in %)*
Linear PID	0.108	0.8	23
Fuzzy PID class 1	0.088	0.1275	0
Fuzzy PID class 2	0.055	0.078	0

Note: t_r – rise time; t_s – settling time; M_p – peak overshoot.

Example 2: We consider the two-tank system as shown in Figure 12. This system can be configured as:

1 single-input single-output (SISO) system, where the pump feeds into the upper tank and the lower tank is not used

2 state-coupled SISO system, where the pump feeds into the upper tank which in turn feeds into the lower tank.

Figure 12 A two-tank system (see online version for colours)

The dynamics of the two-tank system is given by

$$\dot{L_1}(t) = -\frac{a_1}{A_1}\sqrt{2gL_1(t)} + \frac{K_p}{A_1}V_p(t) \tag{10}$$

$$\dot{L_2}(t) = \frac{a_1}{A_2}\sqrt{2gL_1(t)} - \frac{a_2}{A_2}\sqrt{2gL_2(t)} \tag{11}$$

where L_1 and L_2 are the water levels in the tank 1 and tank 2, respectively. $A_1(A_2)$ is the cross-sectional area of the tank 1 (tank 2), and $a_1(a_2)$ is the cross-sectional area of the outflow orifice of tank 1 (tank 2). V_p and K_p are, respectively, the pump voltage and pump constant. g is the gravitational constant. The values of the parameters are given in Table 10.

Table 10 List of parameters of the two-tank system

A_1, A_2	15.379 cm^2
a_1, a_2	0.178 cm^2
K_p	4.6 cm^3/volts-sec
g	980 cm/sec^2

Figure 13 Block diagram of the closed-loop system for water level control

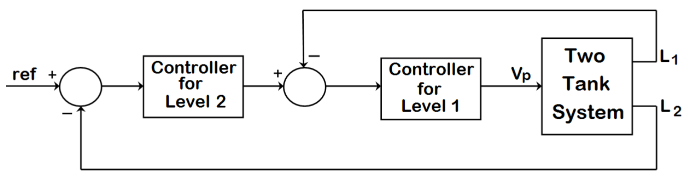


Figure 13 shows the block diagram of the closed loop system where two-tank is the plant under control and two controllers are employed to control the levels of water in tank 1 and tank 2. There are two loops; the inner loop regulates the water level in tank 1 and the outer loop maintains the water level in tank 2. Considering single tank operation, the inner loop will be active whereas the outer loop is not required. The reference signal will be given to the inner loop and hence the controller for water level in tank 2 is not required.

Case 1 SISO system, where the pump feeds into the upper tank and the lower tank is not used. The inner loop in Figure 13 is in operation. The objective is to maintain the level of water in tank 1 at a desired level (say 15 cm). We design the controllers (both linear and fuzzy) by obtaining optimal parameters using GA. The cost function is selected as in equation (9). The linear and fuzzy controller parameters are enlisted in Table 11.

The output responses (water level in tank 1) and the corresponding control efforts obtained with linear and fuzzy PID controllers are shown in Figures 14 and 15. Sampling time of 1 ms is considered for simulation. The time-domain performance data of water level control in tank 1 with linear and fuzzy PID controllers is enlisted in Table 12.

Case 2 State-coupled SISO system, where the pump feeds into the upper tank which in turn feeds into the lower tank. The control strategy is shown in Figure 13. The water level in tank 2 is to be maintained at 15 cm.

The parameters of linear and fuzzy PID controllers are tuned using GA and the cost function in equation (9). The tuned parameters of linear and fuzzy controllers are as in Table 13.

The output responses (water level in tank 2) and the control efforts obtained with linear and fuzzy PID controllers are shown in Figures 16 and 17. Sampling time of 1 ms is considered for simulation. The time-domain performance data of water level control in tank 2 with linear and fuzzy PID controllers is enlisted in Table 14.

Table 11 Parameters of linear and fuzzy controllers for water level control in tank 1

		Linear PID	
	$K_P = 8.3$	$K_I = 2.8$	$K_D = 0.3$
		Fuzzy PID class 1 (level 1)	
$S_{\Delta u}^{-1} = 0.8$	$a_0 = 0.0442$	$b_0 = 36.12$	$c_0 = 0.0704$
	$a_1 = 0.0431$	$b_1 = 26.92$	$c_1 = 0.104$
	$a_2 = 0.0342$	$b_2 = 30.87$	$c_2 = 0.54$
	$a_3 = 0.0129$	$b_3 = 12.12$	$c_3 = 0.017$
	$A_1 = 866.67$	$A_2 = 0.3075$	$A_3 = 3447$
	$B_1 = 2690$	$B_2 = 0.5183$	$B_3 = 5610.5$
		Fuzzy PID class 2 (level 1)	
$S_{\Delta u}^{-1} = 1.3$	$a_0 = 0.0589$	$b_0 = 45.15$	$c_0 = 0.0845$
	$a_1 = 0.0575$	$b_1 = 33.65$	$c_1 = 0.1248$
	$a_2 = 0.0456$	$b_2 = 38.59$	$c_2 = 0.648$
	$a_3 = 0.0172$	$b_3 = 15.15$	$c_3 = 0.0204$
	$A_1 = 650$	$A_2 = 0.246$	$A_3 = 2872.5$
	$B_1 = 2017.5$	$B_2 = 0.4146$	$B_3 = 4675.42$

Figure 14 Output responses with linear and fuzzy PID controllers for water level control in tank 1 (see online version for colours)

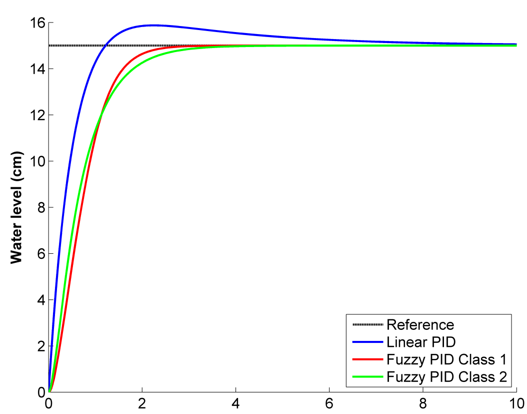


Figure 15 Control efforts for water level control in tank 1 (see online version for colours)

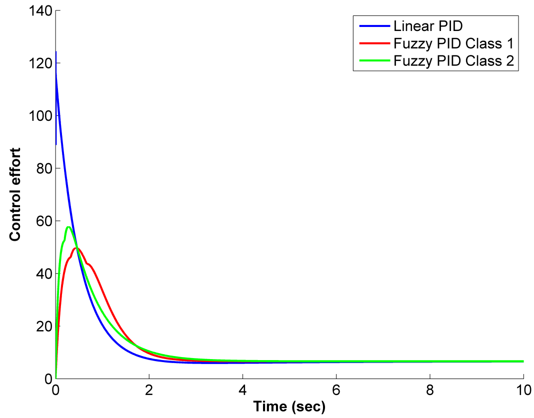
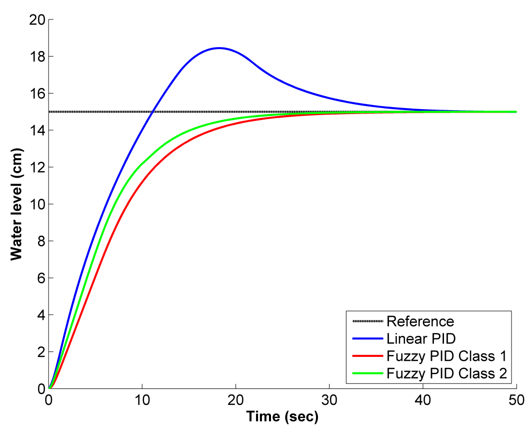


Table 12 Time-domain performance data for water level control in tank 1

Controller	t_r *(in sec)*	t_s *(in sec)*	M_p *(in %)*
Linear PID	0.845	9.2	5.83
Fuzzy PID class 1	1.4285	2.1	0
Fuzzy PID class 2	1.565	2.58	0

Figure 16 Output responses with linear and fuzzy PID controllers for water level control in tank 2 (see online version for colours)



It is quite evident from both the examples that the proposed fuzzy PID controllers outperform their linear counterparts. In general, the fuzzy controllers outperform the linear controllers, the challenge lies in tuning the parameters. In general, the fuzzy controllers have far more number of tuneable parameters compared to the number of parameters of linear controllers. Naturally tuning becomes a hectic task. The challenge lies in tuning the parameters of fuzzy controllers.

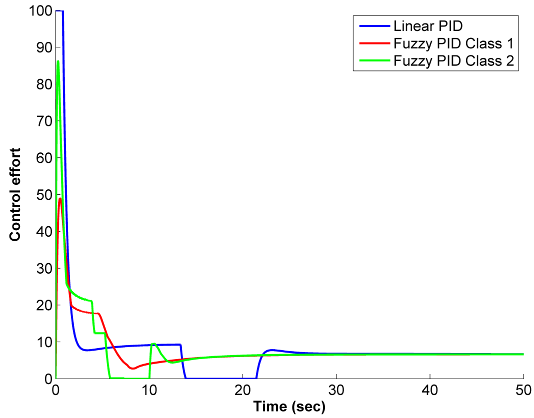
Table 13 Parameters of linear and fuzzy controllers for water level control in tank 2

	Linear PID (inner loop)		
	K_P = 8.51	K_I = 2.31	K_D = 0.04
	Linear PID (outer loop)		
	K_P = 3.62	K_I = 0.52	K_D = 0.123
	Fuzzy PID class 1 (inner loop)		
$S_{\Delta u}^{-1}$ = 0.14	a_0 = 0.0134	b_0 = 44.33	c_0 = 0.306
	a_1 = 0.0287	b_1 = 74.03	c_1 = 0.21
	a_2 = 0.0225	b_2 = 78.98	c_2 = 0.81
	a_3 = 0.0086	b_3 = 33.33	c_3 = 0.257
	A_1 = 130	A_2 = 14.66	A_3 = 29.8
	B_1 = 303.5	B_2 = 31.16	B_3 = 40.7
	Fuzzy PID class 1 (outer loop)		
$S_{\Delta u}^{-1}$ = 0.242	a_0 = 0.0048	b_0 = 35.26	c_0 = 1.971
	a_1 = 0.0093	b_1 = 58.89	c_1 = 2.912
	a_2 = 0.0074	b_2 = 67.53	c_2 = 15.12
	a_3 = 0.0028	b_3 = 28.03	c_3 = 0.476
	A_1 = 40	A_2 = 1.406	A_3 = 12.393
	B_1 = 124.15	B_2 = 2.369	B_3 = 18.232
	Fuzzy PID class 2 (inner loop)		
$S_{\Delta u}^{-1}$ = 0.183	a_0 = 0.0482	b_0 = 57.73	c_0 = 0.767
	a_1 = 0.0823	b_1 = 96.41	c_1 = 0.526
	a_2 = 0.0506	b_2 = 102.86	c_2 = 2.03
	a_3 = 0.0284	b_3 = 43.41	c_3 = 0.643
	A_1 = 577.78	A_2 = 11.26	A_3 = 9.23
	B_1 = 1348.89	B_2 = 23.92	B_3 = 21.56
	Fuzzy PID class 2 (outer loop)		
$S_{\Delta u}^{-1}$ = 0.53	a_0 = 0.0037	b_0 = 23.656	c_0 = 0.141
	a_1 = 0.0072	b_1 = 39.505	c_1 = 0.208
	a_2 = 0.0057	b_2 = 45.305	c_2 = 1.08
	a_3 = 0.0022	b_3 = 18.802	c_3 = 0.034
	A_1 = 52	A_2 = 2.095	A_3 = 173.5
	B_1 = 161.4	B_2 = 3.532	B_3 = 255.25

Table 14 Time-domain performance data for water level control in tank 2

Controller	t_r *(in sec)*	t_s *(in sec)*	M_p *(in %)*
Linear PID	9.5	39.2	23.33
Fuzzy PID class 1	15.5	25.2	0
Fuzzy PID class 2	13.3	22.1	0

Figure 17 Control efforts for water level control in tank 2 (see online version for colours)



6 Discussion and conclusions

Two classes of TS type fuzzy PID controllers have been proposed using AP triangular norm, BS/Max triangular co-norm, different UoDs and different number of fuzzy sets for the input variables. An attempt has been made to generalise the fuzzy PID controller using multiple fuzzy sets with trapezoidal membership functions. The trapezoidal membership functions can be modified to triangular ones by considering $A_1 = A_2 = A_3 = 0$. In this case the expressions of the controllers will be modified accordingly, i.e., by replacing θ_1 by $0.5B_1$, θ_2 by $0.5B_2$ and θ_3 by $0.5B_3$. The proposed controllers are equivalent to (nonlinear) variable gain/structure controllers as their gains are nonlinear functions of input variables, and their structures vary. Moreover, the controllers become linear if $a_{|i+j+k|} = a_{|i+j+k+1|} = a_{|i+j+k+2|} = a_{|i+j+k+3|}$, $b_{|i+j+k|} = b_{|i+j+k+1|} = b_{|i+j+k+2|} = b_{|i+j+k+3|}$ and $c_{|i+j+k|} = c_{|i+j+k+1|} = c_{|i+j+k+2|} = c_{|i+j+k+3|}$. The proposed rule base consists of only four rules compared to eight rules in the literature. This helps in reducing the number of tuneable parameters. Tuning of fuzzy controllers becomes relatively easier as the number of tuneable parameters reduces. The number of tuneable parameters for the proposed rule base is $3(J_1 + J_2 + J_3 + 1)$. A sufficient condition for BIBO stability of the closed loop system has been established using the small gain theorem. Simulation studies on two nonlinear processes show the applicability of the proposed controllers.

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