Strengthening preference ranking organisation method for enrichment evaluation with features of paraconsistent Pavelka style fuzzy logic

Abdulai Inusah*

Department of Mathematics,
Tampere University,
Hervanta Campus,
Tampere, Finland
Email: inusah.abdulai@tuni.fi
*Corresponding author

Esko Turunen

Department of Mathematics and Statistics,
Tampere University,
33101 Tampere, Finland
Email: esko.turunen@tuni.fi

Abstract: In this study, a new technique of complete ranking is proposed to augment the efficiency of the preference ranking organisation method for enrichment evaluation (PROMETHEE) methodology. The technique employs some ideas of the PROMETHEE method and paraconsistent Pavelka style fuzzy logic. To illustrate the effectiveness and efficiency of this novel technique, data on the performance of five mobile phone operators in Ghana is analysed and the results compared with the ranking of the conventional PROMETHEE I and II.

Keywords: paraconsistent logic; Pavelka logic; leaving and entering flows; evidence couple; evidence matrices; PROMETHEE; paraconsistent Pavelka style fuzzy logic; MV-algebra; complete ranking.

Reference to this paper should be made as follows: Inusah, A. and Turunen, E. (2021) 'Strengthening preference ranking organisation method for enrichment evaluation with features of paraconsistent Pavelka style fuzzy logic', *Int. J. Multicriteria Decision Making*, Vol. 8, No. 3, pp.211–232.

Biographical notes: Abdulai Inusah received his MPhil in Mathematics at the Kwame Nkrumah University of Science and Technology, Kumasi, Ghana. Currently, he is pursuing his doctoral studies at the Department of Mathematics, Tampere University of Technology, Tampere, Finland. His research interests include issues related to decision making in both fuzzy and non-fuzzy environments.

Esko Turunen earned his PhD (Mathematics) in 1994, and currently he is working at Tampere University, Finland. His research areas are various generalised logics, data mining and their applications. He has published more than 50 scientific papers including three books. He has worked several years in various universities including Charles University and Prague University of Technology in Czechia, University of Naples, Salerno and Pisa in Italy and

Technical University in Vienna, Austria. In addition, he has participated as a mathematical expert in several industrial research projects developing traffic signal systems, medical expert systems and other intelligent control systems.

1 Introduction

It is always an easy task for all outranking methods including the preference ranking organisation method for enrichment evaluation (PROMETHEE) to rank completely a finite set of comparable alternatives. However, the task becomes quite challenging for these methods when there are incomparable options among the alternatives. In the literature, experts have expressed a reservation as to the way complete ranking of alternatives is achieved under PROMETHEE II of the PROMETHEE methodology. According to the experts, obtaining complete ranking of choices by simply subtracting the value of the entering flow from the value of the corresponding leaving flow of an alternative leads to the loss of information (Brans and Vincke, 1985; Brans et al., 1986; Figueira et al., 2005). In this article, an improved technique of complete ranking that is unveiling and more information preserving is proposed for the PROMETHEE methodology. The technique combines some essential features of the PROMETHEE I, II methods, and paraconsistent Pavelka style fuzzy logic (Turunen et al., 2010). Essentially, we slightly modified Definition 4.1 in Inusah and Turunen (2021) for those parts where the matrices *M* and *N* are not comparable; then, the theorems that follow from them are very similar to Theorem 4.1 in Inusah and Turunen (2021).

The PROMETHEE methodology was developed by Brans in the early ‘80s as a formal approach for resolving decision problems characterised by several and sometimes conflicting criteria (Figueira et al., 2005). Further extensions were made to it by Brans and Mareschal in subsequent years (Figueira et al., 2005).

Presently, the PROMETHEE I and II methods are among the famous multi-criteria decision making methods in the literature owing mainly to their simplicity, user-friendliness and versatility. The basic objective of the method is not necessarily to find out the most upright decision, but instead to identify the option(s) among others that optimally matches the aspirations and perception of the decision-maker. It is also today the method adopted by most decision-makers and experts in various fields to resolve choice and ranking problems. Apart from helping individuals to address simple decision making challenges, the PROMETHEE technique is most functional in facilitating group decision making in situations where a heterogeneous group of people per their qualification, experience, background, specialisation, perception, and judgement work together to rectify a given complex decision task. Over the years, the PROMETHEE method has chalked up numerous successes in its application across a wide range of fields including banking, industrial location, planning, tourism, telecommunication, healthcare, water resource, investment, military and many more.

However, the method is not without some highs and lows. At the policy level, PROMETHEE assists decision makers to assess the strengths and weaknesses of various policy options and then provide a ranking of the options (De Keyser and Peeters, 1996). The assessment process allows for individual or group participation. In other words, it supports individual or group level decision making via dialogue and consensus building. This ranking makes it easier for decision makers to choose the most suitable policy option for implementation. However, it is observed that after PROMETHEE has helped to identify the optimal policy option it lacks the capacity to implement and evaluate that policy. In terms of sustainability, PROMETHEE can compare the impacts of sustainable development dimensions, namely economic, social, environmental, financial and technological.

Apart from being applicable to spatial data, PROMETHEE can be used to compare long-term impacts independently of the year under consideration. It can also compare impacts independently of global dimension (De Keyser and Peeters, 1996). Thus, PROMETHEE becomes a very important tool for the comparison of sustainable impacts since some of these dimensions cannot be expressed quantitatively. This is also to say that, PROMETHEE can handle at the same time both qualitative and quantitative criteria. However, the operational aspect of the method is quite involving and expensive. The human resources, time, data and data availability as well as the cost of applying the method cannot be easily determined and vary from one area of application to another. Moreover, high expert judgement is most often required to explain the findings.

Although the processes involved in analysing preferences under PROMETHEE are tedious, complex and difficult to explain to laymen, the method has a high degree of transparency. Apart from the fact that the application of PROMETHEE is not limited by time and space, it can be applied to resolve decision problems associated with uncertainty and fuzzy input data. Nonetheless, ranking inconsistencies (rank reversal problem) may occur when new alternatives are introduced into the process (De Keyser and Peeters, 1996).

As another strength, PROMETHEE can also work in combination with other tools such as the geographic information system (GIS) and spatial models for assessment of the suitability of land use. The same way, it can be used to compare the impact of policy options induced by various tools that include physical assessment tools, modelling tools and environmental assessment tools. However, PROMETHEE falls short of strength to handle efficiently large decision-making problems: decision problems with numerous alternatives and criteria and sometimes with sub-criteria. In such cases, decision-makers tend to lose track of their line of analysis as the view of the whole problem gets blurred. Furthermore, PROMETHEE method does not proffer any formal way to determine weights of criteria, but instead assumes that decision-makers can fix the weights on their own. Another setback of the PROMETHEE method which is the focus of our article is the establishment of complete ranking (PROMETHEE II). It is observed that obtaining a complete ranking by subtracting the value of the global weakness of an alternative course of action from the value of its global strength gives rise to the loss of information (Brans and Vincke, 1985; Brans et al., 1986).

Therefore, in this study, we introduce an improved version of the complete ranking (PROMETHEE II) that is more revealing and more informative than the traditional PROMETHEE II.

Pavelka style logic is a broad logical system that covers different formal systems and embraces the assignment of graded truth-values to formulas (Belohlavek, 2015). This logic allows for, among other things, the drawing of approximate conclusions from collections of approximate premises. Pavelka logic, thus, constitutes a way by which the standard logic (classical logic) has been generalised. This logic admits and prepares, primarily, for the reality of having to work with exact, vague, and inexact circumstances of daily life. To this end, Pavelka logic provides a wider set of truth values by relaxing the restriction on the two truth values 0 and 1 in order to incorporate assumptions, theorems, inference rules and derived formulas with different degrees of truth other than the two truth degrees associated with classical logic.

A paraconsistent logic on the other hand is a logic in which contradictory items of information do not imply every single thing (Gabbay and Woods, 2007). This logic is built on the core belief that not all contradictions are false – some inconsistencies are true. In the literature, there are many different paraconsistent logics all designed to serve diverse purposes. The principle of explosion as found in Gabbay and Woods (2007), thus, creates a stark contrast between paraconsistent logic and classical logic. In fact, in classical logic, contradictions entail everything. The paraconsistent logic of interest in this article is the one developed by Belnap (1977), extended by Perny and Tsoukias (1998) and further advanced by Turunen et al. (2010) into what is called *paraconsistent Pavelka style fuzzy logic*.

According to Belnap (1977), based on available proof (evidence) any proposition α can take one of four possible states or values: *false*, *contradictory*, *unknown* and *true* but not just the normal yes and no (or completely true and completely false values) we usually search for. This means:

- Statement α is *false* if there is no proof in support of α but there is proof against α. So denoting falsehood by F, 'there is proof' by 1 and 'there is no proof' by 0, we represent this quantity as $F = (0, 1)$.
- α is *contradictory* (*inconsistent*) if there is proof in support of α and there is also proof against α. Denoting contradictory by C, we have $C = (1, 1)$.
- α is *unknown* if there is no proof in support of α and there is no proof against α. Representing unknown by U, $U = (0, 0)$.
- α is *true* if there is proof in support of α and there is no proof against α. We denote true here by T and so $T = (1, 0)$.

Later, this four valued logic was extended to cover the whole interval [0, 1] by Perny and Tsoukias (1998) to measure the magnitude of truth, falsehood, contradiction and unknown in every proposition (α). In other words, this fuzzy extension was made to create room for the above states to also assume intermediate values between 0 and 1 to depict partially true, partially contradictory, partially falsehood, and partially unknown in every logical formula. This partiality is largely caused by perhaps lack of information, incomplete information or the availability of excess information. The authors defined these four values on the interval [0, 1] as follows:

$$F(\alpha) = \min(b, 1-a) \tag{1}$$

$$C(\alpha) = \max(0, a+b-1) \tag{2}$$

$$U(\alpha) = \max(0, 1-a-b) \tag{3}$$

$$T(\alpha) = \min(1-b, a) \tag{4}$$

From this point, Turunen et al. (2010) further developed this logic into a logical system known as paraconsistent Pavelka style fuzzy logic whose algebraic structure is the Łukasiewicz structure (an injective MV-algebra). The binary operations $\odot$, $\wedge$, $\oplus$ and the unary operation * are the operations of the Łukasiewicz structure so that for all $a, b \in [0, 1]$, $a \odot b = \max(0, a+b-1)$, $a \wedge b = \min(a, b)$, $a^* = 1-a$ and $a \oplus b = \min(1, a+b)$. Moreover, on this structure according to the authors, the $F(\alpha)$, $C(\alpha)$, $U(\alpha)$, and $T(\alpha)$ as defined by equations (1)–(4) are re-expressed as $F(\alpha) = a^* \wedge b, C(\alpha) = a \odot b, U(\alpha) = a^* \odot b^*$ and $T(\alpha) = a \wedge b^*$. Hence, in terms of a 2-by-2 matrix, for every $a, b \in [0, 1]$, we have

$$\begin{bmatrix} F(\alpha) & C(\alpha) \\ U(\alpha) & T(\alpha) \end{bmatrix} = \begin{bmatrix} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{bmatrix}$$

The pair of values (a, b) in $[0, 1]$ is called an *evidence couple*. Thus, the evidence couples: $F = (0, 1)$, $C = (1, 1)$, $U = (0, 0)$, and $T = (1, 0)$ in terms of evidence matrices are:

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In this article, we seek to link paraconsistent Pavelka style fuzzy logic to the PROMETHEE I and II methodology and then demonstrate how such a synergy can be used to augment the performance of the PROMETHEE method.

Many multi-attribute decision making (MADM) methods such as PROMETHEE, TOPSIS, ELECTRE and others draw their truth values from the real unit interval [0, 1]. In particular, the PROMETHEE method derives the values of the leaving flow and the entering flow from the real unit interval [0, 1]; the TOPSIS acquire its values of relative closeness to the ideal solution from within the interval [0, 1] and in the ELECTRE method, both the concordance and the discordance thresholds are well within this same interval [0, 1]. This implies that the values of the quantities referred to in these three methods are not always 0 or 1 but could also be other values between 0 and 1. Similarly, Pavelka's generalisation of the two-valued logic is based on the same notion of the real unit interval [0, 1] as the truth value set albeit equipped with a complete MV-algebraic structure. This means that in the field of MADM, particularly in the PROMETHEE method we can equip the truth value set [0, 1] with a complete MV-algebraic structure so as to make the decision making method benefit immensely from the strengths of the Pavelka style logic. For instance, by endowing the PROMETHEE approach with some features of the paraconsistent Pavelka style fuzzy logic as done in this work, we obtain the right algebraic framework to advance our definitions, theorems and proofs in a way that adds meaning and value to the outranking method.

In this study, we equip the truth value set L (the interval [0, 1]) for the leaving and entering flows with the Łukasiewicz structure which is an injective MV-algebra to have a Pavelka style fuzzy logic. Furthermore, via the idea of evidence couples, every alternative course of action in the PROMETHEE method is viewed from the perspective of the four possible values (false, contradictory, unknown and truth) and by so doing a paraconsistent dimension as introduced by Turunen et al. (2010) is added to the Pavelka style fuzzy logic to produce what is called the *paraconsistent Pavelka style fuzzy logic*. Based on this framework, we introduce our definitions, theorems and proofs. Moreover, essential concepts in the paraconsistent Pavelka style fuzzy logic to be brought to bear on the PROMETHEE approach are well and correctly interpreted in the parlance of the PROMETHEE methodology. The interpretation is as follows:

1 Formula: Every formula α is regarded in PROMETHEE as a preference statement about an alternative course of action in relation to others, and such a statement is evaluated through a given set of criteria under the PROMETHEE method. For example, α can represent the statement: alternative A_k is preferable to alternative A_l.

2 The evidence in favour of α and the evidence against α which we denoted in the paraconsistent Pavelka style fuzzy logic by *a* and *b* respectively are synonymous with what we call the *leaving flow* ($\phi^+(\alpha)$) and the *entering flow* ($\phi^-(\alpha)$) respectively under the PROMETHEE method.

3 A given alternative α can be assigned one of the following four possible values rather than the usual two possibilities – only true or only false:

- If there is leaving flow ($\phi^+(\alpha)$) for α and no entering flow ($\phi^-(\alpha)$) for α, then α is assigned the value *true*.
- If there is no leaving flow ($\phi^+(\alpha)$) for α and there is entering flow ($\phi^-(\alpha)$) for α, then α is assigned the value *false*.
- If there is leaving flow ($\phi^+(\alpha)$) for α and at the same time there is entering flow ($\phi^-(\alpha)$) for α, then α is given the value *contradictory*.

  Note that, alternatives in most real decision making problems share this contradictory characteristic. That is, most alternatives in each outranking decision problem are endowed with both leaving and entering flows.
- If there is neither leaving flow ($\phi^+(\alpha)$) nor entering flow ($\phi^-(\alpha)$) for α, then α is allotted the value *unknown*.

4 The truth value set L is the Łukasiewicz structure on the real unit interval [0, 1] which is an injective MV-algebra. Thus, in paraconsistent Pavelka style fuzzy logic, the two evidences: a, b for α are in L and in the parlance of the PROMETHEE technique, we say $\langle \phi^+(\alpha), \phi^-(\alpha) \rangle$ are in L. In other words, the evidence couple $\langle \phi^+(\alpha), \phi^-(\alpha) \rangle$ for the alternative α is such $\langle \phi^+(\alpha), \phi^-(\alpha) \rangle \in L \times L$.

5 The set of injective MV-algebra valued evidence couples $\langle \phi^+(\alpha), \phi^-(\alpha) \rangle$ generates a corresponding set $\mathcal{M}$ of evidence matrices which is also an injective MV-algebra. The set $\mathcal{M}$ is expressed as

$$\mathcal{M} = \left\{ \begin{bmatrix} \left(\phi^+(\alpha)\right)^* \wedge \phi^-(\alpha) & \phi^+(\alpha) \odot \phi^-(\alpha) \\ \left(\phi^+(\alpha)\right)^* \odot \left(\phi^-(\alpha)\right)^* & \phi^+(\alpha) \wedge \left(\phi^-(\alpha)\right)^* \end{bmatrix} \mid \left\langle \phi^+(\alpha), \phi^-(\alpha) \right\rangle \in L \times L \right\}.$$

6 Since the truth value set L is the Łukasiewicz structure on the real unit interval [0, 1], for all $\phi^+(\alpha)$, $\phi^-(\alpha) \in L$, we have $\phi^+(\alpha) \odot \phi^-(\alpha) = \max[0, \phi^+(\alpha) + \phi^-(\alpha) - 1]$, $\phi^+(\alpha) \wedge \phi^-(\alpha) = \min[\phi^+(\alpha), \phi^-(\alpha)]$ and $(\phi^+(\alpha))^* = 1 - \phi^+(\alpha)$.

7 Eventually, we establish two rankings for the finite set of alternatives: one of the two rankings is based on the falsehood value $F(\alpha) = (\phi^+(\alpha))^* \wedge \phi^-(\alpha)$ of the evidence matrix, while the other ranking is according to the truth value $T(\alpha) = \phi^+(\alpha) \wedge (\phi^-(\alpha))^*$ of the same evidence matrix.

Thus unlike the traditional PROMETHEE approach, re-expressing the positive and the negative outranking flows in the way of the paraconsistent Pavelka logic enables decision-makers to have a deeper understanding of the pair of values $\langle \phi^+(\alpha), \phi^-(\alpha) \rangle$ in a manner that brings about flexibility in the ranking process. Flexibility emanates from the fact that this approach offers users two ranking procedures and based on the attitude of users towards uncertainty they opt for one of the two ways or even both. This, therefore, empowers decision-makers to make well-informed decisions that lead to maximal gains or minimal losses. In other words, this novel approach helps decision-makers in a much better way to figure out the course of action that befits their goal and perception of the decision problem. The rest of the paper is organised as follows: Section 2 briefly describes PROMETHEE I and II, Section 3 describes the algorithm of the improved technique, in Section 4, we apply the novel technique to a real life decision problem, in Section 5, we discuss the findings, and in Section 6, we conclude.

2 PROMETHEE I and II

2.1 The PROMETHEE methodology

2.1.1 Decision model construction

Suppose we have a finite set of alternative courses of action represented by $A = \{A_i\}$; a finite set of criteria denoted by C so that $C = \{c_j\}$ and a set of weights denoted by W (i.e., $W = \{w_j\}$) where $i = 1, \cdots, n, j = 1, \cdots, m$ and w_j, as the weight of the criterion c_j. What is the optimal alternative course of action?

The genesis of the PROMETHEE methodology is the evaluation table. The table is composed of information obtained from the three sets defined above (A, C, W) as well as the performance score of every decision alternative on each criterion c_j.

2.1.2 The weight

The weight signifies the importance, significance or value of a given criterion to the decision maker. Although PROMETHEE has not provided hard and fast rules or guidelines for determining the weights of criteria, it is believed that the decision-maker will be able to assign the right weight to the various criteria. As a way of assisting in this process, various weight calculation methods have been proposed in Macharis et al. (2004).

2.1.3 The preference function

The preference degrees are the basis of the PROMETHEE method. A preference degree takes a value within the real unit interval [0, 1] and indicates the extend to which one alternative course of action is preferred to another. Hence, when the preference degree is 0 means there is no preference between a pair of alternatives. And when the preference degree is 1 implies a total preference of one alternative over another. Intermediate values between 0 and 1 represent partial preference degrees. The preference degree on a criterion for any pair of alternatives is calculated through a prescribed preference function that translates the difference of the evaluations of the pair on the criterion into a real unit value lying in the interval [0, 1]. The most important consideration in the process of determining the preference degree on a criterion is the decision maker's perception or how he or she regards the differences between the evaluations of pairs of alternatives on the criterion. Thus, the preference function is what the decision maker or the decision expert who is assisting the decision-maker uses to calculate the preference degrees on a criterion. The deviation for various criteria c_j is calculated via

$$d_j(A_k, A_l) = \begin{cases} c_j(A_k) - c_j(A_l) \\ -[c_j(A_k) - c_j(A_l)] \end{cases}$$

where the deviation $c_j(A_k) - c_j(A_l)$ is for a maximisation criterion c_j, and $-[c_j(A_k) - c_j(A_l)]$ is for a minimisation criterion c_j.

The value of the preference degree denoted here by $P_j(A_k, A_l)$ for the criterion c_j is deduced through the equation $P_j(A_k, A_l) = F_j[d_j(A_k, A_l)]$, for $0 \leq P_j(A_k, A_l) \leq 1$ and F_j as a non-decreasing preference function.

To help decision-makers in their choice of preference functions, six basic types of generalised preference functions have been provided as pointed out in Brans and Vincke (1985), Brans et al. (1986) and Figueira et al. (2005). These are type 1: usual criterion, type 2: quasi-criterion, type 3: criterion with linear preference, type 4: level criterion, type 5: criterion with linear preference and indifference area, and type 6: Gaussian criterion. Each of these preference functions has 0 or 1 or 2 parameters. These parameters are denoted by q, p and s. The q and p are the threshold of indifference and the threshold of strict preference, respectively. The parameter s however is the point of inflexion and the only parameter for the Gaussian preference function.

Therefore, these three items of information: the evaluation table, weight, and the preference function together dictate the preference structure of the decision-maker. In fact, in the next subsection, a step by step details of this method are provided.

A step by step guide on the implementation of the PROMETHEE method is captured in the following algorithm.

2.1.4 The algorithm of the PROMETHEE I and II

The algorithm for ranking of alternatives is as follows:

1. Complete the evaluation table by providing the list of decision alternatives, the list of criteria (attributes), weights of the available criteria and the performance scores of each decision alternative over all the criteria.
2. Compute the deviations of all pairs of alternatives under each attribute c_j through the equations:
   a. $d_j(A_k, A_l) = c_j(A_k) - c_j(A_l)$ for each maximising criterion c_j,
   b. $d_j(A_k, A_l) = -[c_j(A_k) - c_j(A_l)]$ for each minimising criterion c_j.
3. Choose a preference function F_j from the six recommended prefence functions in Section 2.1.3 for each criterion c_j and determine the preference degree via $P_j(A_k, A_l) = F_j[d_j(A_k, A_l)]$ for each $A_k, A_l \in A$.
4. Compute the aggregate preference index of the decision alternative A_k over A_l for all attributes c_j via the equation:

$$\pi(A_k, A_l) = \sum_{j=1}^{m} w_j P_j(A_k, A_l).$$

5 Compute the leaving and the entering flows of the option A_k over every other option A_l via the equations

$$\phi^+(A_k) = \frac{1}{n-1} \sum_{A_l \in A} \pi(A_k, A_l),\ k = 1, \cdots, n$$

$$\phi^-(A_k) = \frac{1}{n-1} \sum_{A_l \in A} \pi(A_k, A_l),\ k = 1, \cdots, n,$$

respectively.

6 PROMETHEE I: Perform partial ranking according to the following three rules:

a $A_k P A_l$ if one of the following holds:
- $\phi^+(A_k) > \phi^+(A_l)$ and $\phi^-(A_k) < \phi^-(A_l)$
- $\phi^+(A_k) > \phi^+(A_l)$ and $\phi^-(A_k) = \phi^-(A_l)$
- $\phi^+(A_k) = \phi^+(A_l)$ and $\phi^-(A_k) < \phi^-(A_l)$.

b $A_k I A_l$ if $\phi^+(A_k) = \phi^+(A_l)$ and $\phi^-(A_k) = \phi^-(A_l)$.

c $A_k R A_l$ if one of the following two conditions holds:
- $\phi^+(A_k) > \phi^+(A_l)$ and $\phi^-(A_k) > \phi^-(A_l)$
- $\phi^+(A_l) > \phi^+(A_k)$ and $\phi^-(A_l) > \phi^-(A_k)$

where P, I, R represent preference, indifference and incomparability, correspondingly. Furthermore, in case there is one or more pairs of alternatives that are not comparable, it is proposed that one proceeds to 7.

7 Determine the net flow for each option using the equation:

$$\phi(A_i) = \phi^+(A_i) - \phi^-(A_i) \text{ for every } A_i \text{ in } A.$$

8 PROMETHEE II: Conduct a complete ranking according to the following two rules:

a $A_k P A_l$ iff $\phi(A_k) > \phi(A_l)$

b $A_k I A_l$ iff $\phi(A_k) = \phi(A_l)$ for every A_k, A_l in A.

3 The algorithm of the improved technique

In reference to the decision model constructed in Section 2.1.1, the algorithm of incorporating the essential features of paraconsistent Pavelka style fuzzy logic into the PROMETHEE method is as follows:

1 Complete the evaluation table by providing the list of decision alternatives, the list of criteria (attributes) and their respective weights as well as the performance scores of each decision alternative over all the attributes.

2 Compute the deviations of all pairs of alternatives under each attribute c_j through the equations:

   a $d_j(A_k, A_l) = c_j(A_k) - c_j(A_l)$ for each maximising criterion c_j

   b $d_j(A_k, A_l) = -[c_j(A_k) - c_j(A_l)]$ for each minimising criterion c_j.

3 Choose a preference function F_j from the six recommended prefence functions in Section 2.1.3 for each criterion c_j and determine the preference degree via $P_j(A_k, A_l) = F_j[d_j(A_k, A_l)]$ for each $A_k, A_l \in A$.

4 Compute the aggregate preference index of the decision alternative A_k over A_l for all attributes c_j via the equation:

$$\pi(A_k, A_l) = \sum_{j=1}^{m} w_j P_j(A_k, A_l).$$

5 Compute the leaving and the entering flows of the option A_k over every other option A_l via the equations

$$\phi^{+}(A_k) = \frac{1}{n-1} \sum_{A_l \in A} \pi(A_k, A_l), k = 1, \cdots, n$$

$$\phi^{-}(A_k) = \frac{1}{n-1} \sum_{A_l \in A} \pi(A_k, A_l), k = 1, \cdots, n,$$

respectively.

Recall that in paraconsistent logic, the pair $\langle \phi^{+}(A_k), \phi^{-}(A_k) \rangle$ is referred to as the evidence couple.

6 Any couple $\langle \phi^{+}(A_k), \phi^{-}(A_k) \rangle$ induces a corresponding evidence matrix we may denote by

$$M_k = \begin{bmatrix} F(A_k) & C(A_k) \\ U(A_k) & T(A_k) \end{bmatrix}$$

where A_k is a specific alternative course of action.

7 The set of evidence matrices for all options constitute a truth-value set, and so any two options are comparable by means of their appropriate evidence matrices. For instance, assume M_1, M_2 are the matrices for options A_1, A_2, respectively. Then, we say that option A_2 is preferable to option A_1 if $M_1 \leq M_2$. Moreover, $M_1 \leq M_2$ provided $M_1 \Rightarrow M_2 = M_1^{\perp} \oplus M_2 = 1 = T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

For more information about the binary operation $\Rightarrow$, see Inusah and Turunen (2021). In this truth value set, the matrix for F which is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the bottom element, and the matrix T is the top element. However, there may be some incomparable options and so their respective matrices too in the truth set will not be comparable. Thus, the set of evidence matrices is not in a linear order. In Definitions 3.1 and 3.2, we show how this incomparability challenge is resolved.

In reality, it is reasonable to compare a pair of alternative courses of action say A_l, A_k through their corresponding evidence matrices M, N. Suppose the evidence couples (a, b), (x, y) induce the evidence matrices M, N correspondingly. We say that $M \leq N$ provided $a \leq x$ and $y \leq b$. This further implies $x^* \leq a^*$, $b^* \leq y^*$. In fact, this kind of relationship between M, N shows that there is more proof (evidence) in support of A_k than the evidence in support of A_l, and there is more evidence against A_l than there is against A_k. Furthermore, from the inequalities $a \leq x$, $y \leq b$, $x^* \leq a^*$, and $b^* \leq y^*$, it is clear that $T(M) = a \wedge b^* \leq x \wedge y^* = T(N)$ and $F(N) = x^* \wedge y \leq a^* \wedge b = F(M)$, where $T(M)$ and $F(M)$ are the abbreviations of $T(A_l)$ and $F(A_l)$, respectively. Also, $T(N)$ and $F(N)$ are correspondingly the abbreviations of $T(A_k)$ and $F(A_k)$. However, the converse does not hold in general. That is, from $T(M) \leq T(N)$, $F(N) \leq F(M)$, we cannot conclude that $M \leq N$. For example, consider $\langle a, b\rangle = \langle 0.1, 0.8\rangle$, $\langle x, y\rangle = \langle 0.2, 0.9\rangle$. Indeed, these are incomparable evidence couples. But, they produce the following comparable matrices M, N

$$M = \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad N = \begin{bmatrix} 0.8 & 0.1 \\ 0 & 0.1 \end{bmatrix}.$$

If M, N are incomparable, then the problem gets compounded and to overcome the complexity we propose two definitions and their respective theorems as follows: as previously mentioned, we slightly modified Definition 4.1 in Inusah and Turunen (2021) for those parts where the matrices M and N are not comparable; then, the theorems that follow from them are very similar to Theorem 4.1 in Inusah and Turunen (2021).

Definition 3.1: Let two options A_l and A_k be correspondingly assigned the evidence couples $\langle a, b\rangle$ and $\langle x, y\rangle$ which generate the corresponding evidence matrices M and N. We say that A_k is preferable to A_l written as $A_l \preceq_F A_k$ whenever:

1 $M \leq N$

2 $M \nleq\ngeq N$ and $T(M) = a \wedge b^* < x \wedge y^* = T(N)$

3 $M \nleq\ngeq N, T(M) = a \wedge b^* = x \wedge y^* = T(N)$ and $F(N) < F(M)$.

In particular, if $M = N$, then A_k and A_l are equally preferable and it is represented by $A_k \equiv_T A_l$. If $M \nleq\ngeq N, T(M) = T(N)$ and $F(M) = F(N)$, then A_k and A_l are weakly equally preferable and it is written as $A_k \sim_T A_l$.

Explanation: Let two options A_l and A_k be correspondingly assigned the evidence couples $\langle a, b\rangle$ and $\langle x, y\rangle$ which generate the corresponding evidence matrices M and N; all possible evidence couples and for that matter evidence matrices for the two alternatives A_l, A_k can be put into two categories:

a Comparable evidence couples (or matrices): Whenever the evidence couples for the alternatives A_l, A_k are comparable, then their evidence matrices M, N are also comparable. In deed, M, N are comparable if either $M \leq N$ or $N \leq M$. Moreover, $M \leq N$ if and only if

$$M^{\perp} \oplus N = 1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly, $N \leq M$ if and only if

$$N^{\perp} \oplus M = 1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For example, suppose alternatives A_l, A_k are represented by the evidence matrices M, N respectively which in turn are generated by the evidence couples $\langle 0.2, 0.5\rangle$, $\langle 0.4, 0.2\rangle$, respectively. Which of these two alternatives dominates the other? From the evidence couples, it is clear that the two alternatives are comparable and in fact, alternative A_k is preferable alternative A_l. Now, we show the preferability of A_k to A_l using their respective evidence matrices M, N. Given the evidence couple for M to be $\langle 0.2; 0.5\rangle$, the evidence couple for the matrix $M^{\perp}$ is $\langle 0.2^*, 0.5^*\rangle = \langle 0.8, 0.5\rangle$. Hence, the evidence couple for the matrix $M^{\perp} \oplus N$ is $\langle 0.8 \oplus 0.4, 0.5 \odot 0.2\rangle = \langle 1, 0\rangle$. The corresponding evidence matrix is

$$M^{\perp} \oplus N = \begin{bmatrix} 1^* \wedge 0 & 1 \odot 0 \\ 1^* \odot 0^* & 1 \wedge 0^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, $M \leq N$. This means A_k preferable to A_l. Conversely, the matrix $N^{\perp}$ is obtained through the evidence couple $\langle 0.4, 0.2\rangle$ by $\langle 0.4^*, 0.2^*\rangle = \langle 0.6, 0.8\rangle$. So, the evidence couple for the matrix $N^{\perp} \oplus M$ is $\langle 0.6 \oplus 0.2, 0.8 \odot 0.5\rangle = \langle 0.8, 0.3\rangle$. The corresponding evidence matrix is

$$N^{\perp} \oplus M = \begin{bmatrix} 0.8^* \wedge 0.3 & 0.8 \odot 0.3 \\ 0.8^* \odot 0.3^* & 0.8 \wedge 0.3^* \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.7 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, $N \not\leq M$ meaning A_l is not preferable to A_k. Hence, A_k is preferable to A_l. Furthermore, if A_l and A_k have the same evidence couple as $\langle 0.8, 0.3\rangle$, then their evidence matrices will be the same as

$$M = N = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.7 \end{bmatrix}.$$

In this case, we conclude that the alternatives A_l and A_k are equally preferable.

b Incomparable evidence couples (or matrices): When the evidence couples associated with alternatives A_l and A_k are incomparable, their evidence matrices too are incomparable. That is, for every matrices M, N; $M \not\leq N$ and $N \not\leq M$. In other words, the matrices

$$M^{\perp} \oplus N \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$N^{\perp} \oplus M \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, under incomparable situations, we can establish the preferred alternative between any two alternatives A_l and A_k either by means of the truth values (i.e., the fourth component of every matrix) or by means of the falsehood values (i.e., the first component of every matrix).

So, via the truth values, we identify the dominant alternative as follows:

1 If the evidence matrices M, N for the alternatives A_l, A_k respectively are incomparable (i.e., $M \not\lesseqgtr N$) and $T(M) < T(N)$, then A_k is preferable to A_l. For example, assume the evidence couples for A_l, A_k are $\langle 0.2, 0.1\rangle$ and $\langle 1, 0.2\rangle$, respectively. Obviously, from the evidence couples, alternatives A_l, A_k are incomparable. Now, if M, N are the corresponding evidence matrices, then

$$M = \begin{bmatrix} 0.1 & 0 \\ 0.7 & 0.2 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.8 \end{bmatrix}.$$

Thus, $M \not\lesseqgtr N$, but $T(M) = 0.2 < 0.8 = T(N)$. Hence, A_k is preferable to A_l (i.e., $A_l \preceq_T A_k$).

2 If the evidence matrices M, N for A_l, A_k respectively are incomparable (i.e., $M \not\lesseqgtr N$) but $T(M) = T(N)$ and $F(N) < F(M)$, then A_k is preferable to A_l. For example, if the evidence couples for A_l, A_k are $\langle 0.4, 0.7\rangle$ and $\langle 0.3, 0.2\rangle$ respectively, then from the two evidence couples, it is clear that A_l, A_k are incomparable. The matrices are,

$$M = \begin{bmatrix} 0.6 & 0.1 \\ 0 & 0.3 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0.2 & 0 \\ 0.5 & 0.3 \end{bmatrix}.$$

Hence, $T(M) = T(N) = 0.3$ and $F(N) = 0.2 < 0.6 = F(M)$. Therefore, $A_l \preceq_T A_k$.

3 If the alternatives A_l, A_k are incomparable (i.e., $M \not\lesseqgtr N$) but $T(M) = T(N)$ and $F(M) = F(N)$, then we conclude that A_l, A_k are weakly equally preferable. For example, let us assume that the alternatives A_l, A_k are assigned the evidence couples $\langle 0.3, 0.6\rangle$ and $\langle 0.4, 0.7\rangle$, respectively. Then from the two evidence couples, it is observed that A_l, A_k are incomparable. The corresponding evidence matrices are

$$M = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 0.3 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0.6 & 0.1 \\ 0 & 0.3 \end{bmatrix}.$$

So, from M, N, we have $T(M) = T(N) = 0.3$ and $F(M) = F(N) = 0.6$. Hence, the alternatives A_l, A_k are weakly equally preferable.

Therefore, items a and b cover all the possible evidence matrices that any given two alternatives, say A_l, A_k can generate from their evidence couples.

Theorem 1: The relation $\equiv_T$ is an equivalence relation with respect to the set of options denoted by A. However, whereas the relation $\sim_T$ is not an equivalence relation, that of $\preceq_T$ is a quasi-order relation on A.

The proof of Theorem 1 is found in Appendix A.

Definition 3.2: Let two options A_l and A_k be correspondingly assigned the evidence couples $\langle a, b\rangle$ and $\langle x, y\rangle$ which generate the corresponding evidence matrices M and N. We say that A_k is preferable to A_l written as $A_l \preceq_F A_k$ whenever:

1 $M \leq N$

2 $M \not\leq\not\geq N$ and $F(N) = x^* \wedge y < a^* \wedge b = F(M)$

3 $M \not\leq\not\geq N, F(N) = x^* \wedge y = a^* \wedge b = F(M)$ and $T(M) < T(N)$.

In particular, if $M = N$, then A_k and A_l are equally preferable and it is represented by $A_k \equiv_F A_l$. If $M \not\leq\not\geq N, F(M) = F(N)$ and $T(M) = T(N)$ then, A_k and A_l are weakly equally preferable and it is written as $A_k \sim_F A_l$.

The explanation of Definition 3.2 is similar to that of Definition 3.1. In fact, the two definitions are more less dual.

However, it is important to point out that Definitions 3.1 and 3.2 present different relations. For instance, assume the evidence matrices M, N are induced by the couples $\langle 0.1, 0.5\rangle$, $\langle 0.4, 0.7\rangle$, respectively. Then, M, N are not comparable and $T(M) = 0.1 \wedge 0.5 = 0.1 < 0.3 = 0.4 \wedge 0.3 = T(N)$ whereas, $F(N) = 0.6 \wedge 0.7 = 0.6 \not< 0.5 = 0.9 \wedge 0.5 = F(M)$.

Theorem 2: The relation $\equiv_F$ is an equivalence relation with respect to the options in A. However, whereas the relation $\sim_F$ is not an equivalence relation, the relation $\preceq_F$ is a quasi-order relation on A.

The proof of Theorem 2 is in Appendix B.

It is also observed that almost all the incomparable matrices M, N we encounter in practical applications have the order $T(M) < T(N)$ and $F(M) < F(N)$. But in theory, this is not always the case. However, we have the following proposition.

Proposition 3: If $M \not\leq\not\geq N$ and $T(M) \leq T(N)$, then $F(M) < F(N)$ or $F(N) < F(M)$ or $F(M) = F(N)$.

Proof: The proof of Proposition 3 is trivial. □

As an example to corroborate this proposition, consider that matrices M, N are generated respectively by the evidence couples:

1 $\langle 0.3, 0.1\rangle, \langle 0.6, 0.2\rangle$
2 $\langle 0.2, 0.1\rangle, \langle 1, 0.2\rangle$
3 $\langle 0, 0\rangle, \langle 1, 0.1\rangle$.

Remark 1: In practice, most incomparable matrices M, N with $T(M) < T(N)$ also have $F(M) < F(N)$. Thus, between two evidence matrices M, N; if N is observed to have higher truth and falsehood values than M then they are incomparable.

8 Rank the alternatives based on Definition 3.1 or rank them based on Definition 3.2 or both.

4 Ranking five mobile phone networks in the national capital, Accra – Ghana

We acquired the data about the quality of service to customers by five mobile phone networks operating in the Greater Accra region of Ghana from the National Communication Authority (NCA).

- Alternatives: In order not to reveal the true identities of the five mobile networks, we denote them by A_1, A_2, A_3, A_4 and A_5.
- Criteria: The criteria and weights of criteria as determined by the authority are call setup time (denoted by C_1), call completion rate (C_2), call congestion rate (C_3) and call drop rate (C_4). The criteria have the same weight of 0.25.
- Evaluation table: The table showing the service quality of each of the five mobile operators is Table 1.
- Preference function: The choice of a preference function is determined largely by the features of every application. The Gaussian criterion has been chosen for this data. The Gaussian criterion or the Gausssian preference function is a continuous and increasing preference function for all deviations on a criterion. Unlike other preference functions, the increase in this case follows an exponential function. This function has only one parameter s which is the point of inflection. In terms of the threshold values q, p, s is an intermediate value between them.

  1 Ranking by the usual PROMETHEE method.

  Based on the PROMETHEE procedure, we generated the following pairs of leaving and entering flows $\langle \phi^+(A_j), \phi^-(A_j)\rangle$ for each alternative:

  $$A_1 = \langle 0.13, 0.21\rangle,\ A_2 = \langle 0.43, 0.0\rangle,\ A_3 = \langle 0.15, 0.62\rangle,\ A_4 = \langle 0.19, 0.12\rangle,$$
  $$A_5 = \langle 0.19, 0.15\rangle.$$

  From the pairs, it is clear that A_1 and A_3 are incomparable, i.e., $A_1 R A_3$. Therefore, we apply PROMETHEE II, and the net flows are as follows:

  $$\phi(A_1) = -0.08,\ \phi(A_2) = 0.43,\ \phi(A_3) = -0.47,\ \phi(A_4) = 0.07 \text{ and } \phi(A_5) = 0.04.$$

  The corresponding complete ranking is:

$A_2 \quad P \quad A_4 \quad P \quad A_5 \quad P \quad A_1 \quad P \quad A_3.$

Hence, A_2 has emerged the optimal alternative.

2 Ranking by the proposed integrated approach.

The evidence couples $A_1 = \langle 0.13, 0.21\rangle$, $A_2 = \langle 0.43, 0.0\rangle$, $A_3 = \langle 0.15, 0.62\rangle$, $A_4 = \langle 0.19, 0.12\rangle$, $A_5 = \langle 0.19, 0.15\rangle$ generated the following evidence matrices:

$$A_1 = \begin{bmatrix} 0.21 & 0 \\ 0.66 & 0.13 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0.57 & 0.43 \end{bmatrix}, A_3 = \begin{bmatrix} 0.62 & 0 \\ 0.23 & 0.15 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.12 & 0 \\ 0.69 & 0.19 \end{bmatrix}, A_5 = \begin{bmatrix} 0.15 & 0 \\ 0.66 & 0.19 \end{bmatrix}.$$

Now, we rank the set of alternatives A using Definition 3.1 and Theorem 1. By Definition 3.1, if there are incomparable evidence matrices like A_1 and A_3, then the one with the higher truth value is preferable. Studying the evidence matrices, it is clear that there are neither equally preferable options ($A_k \equiv_T A_l$) nor weakly equally preferable ones ($A_k \sim_T A_l$). So, the third relation, A_k is preferable to A_l, denoted by $A_l \preceq_T A_k$ has been analysed and the findings have revealed the following complete ranking of the set of alternatives, A:

$A_1 \preceq_T A_3 \preceq_T A_5 \preceq_T A_4 \preceq_T A_2.$

Similarly, if the decision maker prefers a lower falsehood value, then we go by Definition 3.2 and Theorem 2 and the resulting complete ranking of the finite set A is as follows:

$A_3 \preceq_F A_1 \preceq_F A_5 \preceq_F A_4 \preceq_F A_2.$

Table 1 Evaluation table of service quality

Criteria	*Type of criteria*	*Alternatives A_1, A_2, A_3, A_4, A_5*
C_1	Min.	15.12, 12.09, 11.67, 13.86, 15.28
C_2	Max.	80, 96, 41, 81, 88
C_3	Min.	17, 3, 27, 12, 10
C_4	Min.	3, 1, 32, 8, 2

5 Discussion

From the analysis of the case study, it is clear that the ranking obtained using Definition 3.1 is slightly different from the one obtained by Definition 3.2, and the difference is about the worst option between A_1 and A_3. Whereas Definition 3.1 has settled on A_1 as the worst one, Definition 3.2 says otherwise. Incidentally, both Definition 3.2 and the normal PROMETHEE method have delivered the same ranking. Nonetheless, ranking by each of the three ways, has confirmed A_2 as the optimal alternative.

More importantly, by our novel approach, a set of alternatives can be ranked in three distinct ways: either by means of the truth values of the various alternatives as found in their corresponding evidence matrices or by their falsehood values or both.

Using the truth value procedure, the option or options with the highest truth value is adjudged the optimal alternative. However, by falsehood values, the alternative or alternatives with the least falsehood value are regarded the most efficient options. This implies by applying either of these two procedures (truth values or falsehood values) first the other one can be used to confirm the ranking emanating from the first procedure, and this thus makes our new technique superior to the usual PROMETHEE method. Moreover, where there are incomparable alternatives, it is often the case in real life applications that the alternative with the higher truth value is as well the alternative with the higher falsehood value (see Remark 1) and in such a circumstance, risk-averse decision-makers will opt for ranking based on falsehood values so that the choice with the least falsehood value becomes the best one.

Risk-neutral and risk-seeking users, on the other hand, will prefer ranking by truth values and settle on the alternative with the greatest truth value. Hence, decision-makers who adopt this modified technique are able to make better informed choices and take responsibility for the outcome of their decisions as this approach (via the truth and falsehood values) gives decision-makers a clue as to the degree of uncertainty or risk associated with each procedure they choose. Further, if the values of the unknown, and contradiction in the 2-by-2 matrix are deemed extremely high, particularly the value of the contradiction then it is a tacit advice or indication that further and more efficient information seeking and screening tools, where possible, need to be employed for more factual information. This way, values of these two components decrease as they get metamorphosed (converted) into the falsehood and truth values for more accurate ranking. The same cannot be said about the usual PROMETHEE method, which after delivering a complete ranking for a set of alternatives with incomparable options nothing is known or can be said about them again – every item of information gets lost.

It is also essential to add that when all the alternatives in a set are comparable, both the traditional PROMETHEE and the proposed method generate the same ranking. In fact, ordering by truth values and by falsehood values yield the same ranking as the ordinary PROMETHEE technique. Therefore, the proposed method herein is at least as good as the standardised PROMETHEE I and II and can also at least serve as a close substitute for it.

6 Conclusions

An enhanced method of complete ranking under the PROMETHEE methodology has been introduced. The idea is that every option A_l has a collection of proof (evidence) in support of A_l, and a collection of evidence that is not in support of A_l. So, the collection of evidence in support of A_l and the one that is not in support of A_l are respectively the pros and the cons of A_l. By means of the aggregate preference indices, these evidences are combined to form an evidence couple $\langle \phi^+(A_l), \phi^-(A_l)\rangle$ for the option A_l. In the parlance of the PROMETHEE technique, the components of the evidence couple are correspondingly called the positive and the negative outranking flows. This evidence couple $\langle \phi^+(A_l), \phi^-(A_l)\rangle$ generates an evidence matrix M for A_l. Naturally, option A_k with the evidence couple $\langle \phi^+(A_k), \phi^-(A_k)\rangle$ and a corresponding evidence matrix N is better than A_l if there is more evidence in support of A_k than there is in support of A_l and there is less evidence not in support A_k than there is not in support of A_l. This is what we have when the relationship between M, N is such that $M \leq N$. This implies A_k weights better than A_l on both the scales of the pros and the cons. However, if there is more evidence in support of A_k than there is in support of A_l and at the same time there is more evidence not in support of A_k than there is not in support A_l or the other way round, then the two matrices M, N are not comparable. When we are faced with such a scenario, we can obtain the preferred option in one of two-ways: either we compare the truth values of M, N and choose the matrix with the higher truth value or we compare the falsehood values of M, N and opt for the matrix with the lower falsehood value. Both ways generate a complete order for the available options. We can also implement both ways to verify or confirm what the possible best option or best options are. Moreover, using both ways becomes more imperative when the decision maker is interested in identifying the best few alternatives in endeavours such as searching for the best energy mix for electricity generation in an economy. Furthermore, this novel technique is more informative in the sense that it reveals to us as decision-makers the size of each of the four possible states (values), namely *falsehood*, *contradictory*, *unknown* and *truth* in any given evidence couple $\langle \phi^+(A_l), \phi^-(A_l) \rangle$ that users can lean on to make more reasonable and more productive decisions.

As a way of showing the efficiency of this novel technique, a real case study on five telecommunication networks in Accra, Ghana has been studied and the findings compared with that of the traditional PROMETHEE I and II.

Acknowledgements

This research is partially supported by the European Cooperation in Science and Technology (COST) Action DigForASP – CA17124.

References

Belnap, N.D. (1977) ‘A useful four-valued logic’, in Dunn, J.M. and Epstein, G. (Eds.): *Modern Uses of Multiple-valued Logic*.

Belohlavek, R. (2015) ‘Pavelka-style fuzzy logic in retrospect and prospect’, *Fuzzy Sets and Systems*, Vol. 281, pp.61–72.

Brans, J.P. and Vincke, P. (1985) ‘A preference ranking organisation method: (the PROMETHEE method for multiple criteria decision-making)’, *Management Science*, Vol. 31, No. 6, pp.647–656.

Brans, J.P., Vincke, P. and Mareschal, B. (1986) ‘How to select and how to rank projects: the PROMETHEE method’, *European Journal of Operational Research*, Vol. 24, pp.228–238.

De Keyser, W. and Peeters, P. (1996) ‘A note on the use of PROMETHEE multicriteria methods’, *European Journal of Operational Research*, Vol. 89, No. 3, pp.457–461.

Figueira, J., Greco, S. and Ehrgott, M. (2005) *Multiple Criteria Decision Analysis State of the Art Surveys*, Chapter 5, pp.163–189, Springer Science + Business Media, Inc., Boston.

Gabbay, D.M. and Woods, J. (2007) *Handbook of the History of Logic. The Many Valued and Nonmonotonic Turn in Logic*, pp.129–200, Elsevier, North-Holland.

Inusah, A. and Turunen, E. (2021) ‘A paraconsistent many-valued similarity method for multi-attribute decision making'', *Fuzzy Sets and Systems*, Vol. 409, pp.128–152.

Macharis, C., Springael, J., De Brucker, K. and Verbeke, A. (2004) 'PROMETHEE and AHP: the design of operational synergies in multicriteria analysis. Strengthening PROMETHEE with ideas of AHP', *European Journal of Operational Research*, Vol. 153, No. 2, pp.307–317.

Perny, P. and Tsoukias, A. (1998) 'On the continuous extensions of a four valued logic for preference modelling', in *Proceedings of the IPMU 1998 Conference*, Paris, pp.302–309.

Turunen, E., Öztürk, M. and Tsoukiás, A. (2010) 'Paraconsistent semantics for Pavelka style fuzzy sentential logic', *Fuzzy Sets and Systems*, Vol. 161, No. 4, pp.1926–1940.

Appendix A

The proof of Theorem 1

Proof: It is obvious that the relation $\equiv_T$ is an equivalence relation. For the reason that $A_l \sim_T A_k$, $A_k \sim_T A_l$ do not mean that $A_l \sim_T A_l$, we conclude that $\sim_T$ is not symmetric, and it is not reflexive either. Thus, $\sim_T$ is not an equivalence relation. The relation $\preceq_T$, on the other hand, defines a quasi-order relation since in the first place it is obviously reflexive for all $A_i \in A$. Secondly, we prove that it is transitive by supposing that $A_l \preceq_T A_k$ and $A_k \preceq_T A_z$, where A_z is denoted by the matrix P which is in turn generated by the couple $\langle p, q\rangle$. Then, to prove the fact that $A_l \preceq_T A_z$ for all A_l, A_k, $A_z \in A$, we have a number of cases and sub-cases to look at. These cases and sub-cases are as follows:

Case 1 The case $M \leq N$, $N \leq P$ trivially implies $M \leq P$, therefore $A_l \preceq_T A_z$.

Case 2a If $M \not\leq\not\geq N$, $N \not\leq\not\geq P$, and $T(M) = a \wedge b^* < x \wedge y^* = T(N)$, and $T(N) = x \wedge y^* < p \wedge q^* = T(P)$, then $T(M) = a \wedge b^* < p \wedge q^* = T(P)$. Now, the assumption $P \leq M$ is equivalent to $p \leq a$, $b \leq q$, where the second (in-)equality implies $q^* \leq b^*$, which in turn implies $T(P) = p \wedge q^* \leq a \wedge b^* = T(M)$, a contradiction. Therefore, either $M \leq P$ or $M \not\leq\not\geq P$. In both cases, $A_l \preceq_T A_z$.

Case 2b Let $M \not\leq\not\geq N$ and $T(M) < T(N)$, $N \leq P$. There are two sub-cases:

Sub-case I (i) $a < x$, (ii) $b < y$, (iii) $y^* < b^*$, (iv) $a \wedge b^* < x \wedge y^*$, (v) $x \leq p$, (vi) $q \leq y$, (vii) $y^* \leq q^*$.

By (i) and (v), $a < p$. If $q \leq b$, then $M \leq P$, hence $A_l \preceq_T A_z$. If $b < q$, then $M \not\leq\not\geq P$. By (v) and (vii), we observe $x \wedge y^* \leq p \wedge q^*$. Recalling (iv), we have $T(M) < T(P)$. Therefore, $A_l \preceq_T A_z$.

Sub-case II (i) $x < a$, (ii) $y < b$, (iii) $b^* < y^*$, (iv) $a \wedge b^* < x \wedge y^*$, (v) $x \leq p$, (vi) $q \leq y$, (vii) $y^* \leq q^*$.

By (vi) and (ii), $q < b$. If $a \leq p$, then $M \leq P$, consequently $A_l \preceq_T A_z$. If $p < a$, then $M \not\leq\not\geq P$. By (v) and (vii), we observe $x \wedge y^* \leq p \wedge q^*$. Recalling (iv), we have $T(M) < T(P)$. Therefore, $A_l \preceq_T A_z$.

Case 2c Let $M \le N$, $N \not\lessgtr P$, $T(N) < T(P)$. Again, there are two sub-cases:

Sub-case I (i) $a \le x$, (ii) $y \le b$, (iii) $b^* < y^*$, (iv) $p < x$, (v) $q < y$, (vi) $y^* < q^*$, (vii) $x \wedge y^* < p \wedge q^*$.

By (v) and (ii), $q < b$. If $a \le p$, then $M \le P$, hence $A_l \preceq_T A_z$. If $p < a$, then $M \not\lessgtr P$ and we reason as follows. By (i) and (iii), $a \wedge b^* \le x \wedge y^*$, which by (vii) implies $T(M) = a \wedge b^* < p \wedge q^* = T(P)$. Therefore, $A_l \preceq_T A_z$.

Sub-case II (i) $a \le x$, (ii) $y \le b$, (iii) $b^* \le y^*$, (iv) $x < p$, (v) $y \le q$, (vi) $q^* < y^*$, (vii) $x \wedge y^* < p \wedge q^*$.

By (i) and (iv), $a < p$. If $q \le b$, then $M \le P$, therefore $A_l \preceq_T A_z$. If $b < q$, we have $M \not\lessgtr P$ and we observe that the assumption $y^* = x \wedge y^*$ leads to $y^* = x \wedge y^* < p \wedge q^* \le q^*$, which contradicts (vi). Therefore, $x = x \wedge y^*$, and the following holds by (i) and (vii), $T(M) = a \wedge b^* \le x = x \wedge y^* < p \wedge q^* = T(P)$. Therefore, $A_l \preceq_T A_z$.

Case 3a Let $M \le N$, $N \not\lessgtr P$, $T(N) = x \wedge y^* = p \wedge q^* = T(P)$, $F(P) = p^* \wedge q < x^* \wedge y = F(N)$. This combination has two sub-cases:

Sub-case I (i) $a \le x$, (ii) $x^* \le a^*$, (iii) $y \le b$, (iv) $b^* \le y^*$, (v) $x < p$, (vi) $y < q$, (vii) $q^* < y^*$, (viii) $x \wedge y^* = p \wedge q^*$, (ix) $p^* \wedge q < x^* \wedge y$.

By (i) and (v), $a < p$. If $q \le b$, then $M \le P$ and therefore $A_l \preceq_T A_z$. If $b < q$, then $M \not\lessgtr P$. Now, through (i) and (iv), $a \wedge b^* \le x \wedge y^*$. And by applying (viii), $a \wedge b^* \le p \wedge q^*$. Thus, if $a \wedge b^* < p \wedge q^*$ [i.e., $T(M) < T(P)$], then $A_l \preceq_T A_z$. If $a \wedge b^* = p \wedge q^*$, that is $T(M) = T(P)$, we invoke (ii) and (iii) getting $x^* \wedge y \le a^* \wedge b$. And by recalling (ix), we have $F(P) = p^* \wedge q < a^* \wedge b = F(M)$. Therefore, $A_l \preceq_T A_z$.

Sub-case II (i) $a \le x$, (ii) $x^* \le a^*$, (iii) $y \le b$, (iv) $b^* \le y^*$, (v) $p < x$, (vi) $q < y$, (vii) $y^* < q^*$, (viii) $x \wedge y^* = p \wedge q^*$, (ix) $p^* \wedge q < x^* \wedge y$.

From (iii) and (vi), $q < b$. If $a \le p$, then $M \le P$ and $A_l \preceq_T A_z$. However, if $p < a$, then $M \not\lessgtr P$. Thus, we reason that by applying (i) and (iv), we get $a \wedge b^* \le x \wedge y^*$. By (viii), $a \wedge b^* = p \wedge q^*$, hence $A_l \preceq_T A_z$. However, if $a \wedge b^* = x \wedge y^*$ holds implies by (viii), $T(M) = a \wedge b^* = p \wedge q^* = T(P)$. Moreover, via (ii) and (iii), $x^* \wedge y \le a^* \wedge b$, and invoking (ix), $F(P) = p^* \wedge q < a^* \wedge b = F(M)$. Hence, $A_l \preceq_T A_z$.

Case 3b If $M \not\lessgtr N$, $T(M) = a \wedge b^* = x \wedge y^* = T(N)$, $F(N) = x^* \wedge y < a^* \wedge b = F(M)$ and $N \le P$. Here too, there are two sub-cases:

Sub-case I (i) $a < x$, (ii) $b < y$, (iii) $y^* < b^*$, (iv) $a \wedge b^* = x \wedge y^*$, (v) $x^* \wedge y < a^* \wedge b$, (vi) $x \le p$, (vii) $p^* \le x^*$, (viii) $q \le y$, (ix) $y^* \le q^*$.

By (i) and (vi), $a < p$, and if $q \leq b$ then $M \leq P$, hence $A_l \preceq_T A_z$. If $b < q$ then $M \not\leq\not\geq P$ and we reason that by applying (vi) and (ix), we obtain $x \wedge y^* \leq p \wedge q^*$. And by (iv), $a \wedge b^* \leq p \wedge q^*$. Now, if $a \wedge b^* < p \wedge q^*$, that is, $T(M) < T(P)$, then $A_l \preceq_T A_z$. But, if $a \wedge b^* = p \wedge q^*$ which means $T(M) = T(P)$, then, we take it that by applying (vii) and (viii), we have $p^* \wedge q < x^* \wedge y$, and by invoking (v), we obtain $p^* \wedge q \leq a^* \wedge b$, that is $F(P) < F(M)$, thus, $A_l \preceq_T A_z$.

Sub case II (i) $x < a$, (ii) $y < b$, (iii) $b^* < y^*$, (iv) $a \wedge b^* = x \wedge y^*$, (v) $x^* \wedge y < a^* \wedge b$, (vi) $x \leq p$, (vii) $p^* \leq x^*$, (viii) $q \leq y$, (ix) $y^* \leq q^*$.

Considering (ii) and (viii), we have $q < b$. So, if it is the case that $a \leq p$ then $M \leq P$ and so $A_l \preceq_T A_z$. But, in case $p < a$, $M \not\leq\not\geq P$. Thus, by (vi) and (ix), we have $x \wedge y^* \leq p \wedge q^*$. And by (iv), $a \wedge b^* \leq p \wedge q^*$. If $a \wedge b^* < p \wedge q^*$ [i.e., $T(M) < T(P)$], then $A_l \preceq_T A_z$. On the other hand, if $a \wedge b^* = p \wedge q^*$ [i.e., $T(M) = T(P)$], then by (vii) and (viii), $p^* \wedge q \leq x^* \wedge y$. And by (v), $p^* \wedge q < a^* \wedge b$. That is, $F(P) < F(M)$ and therefore $A_l \preceq_T A_z$.

Case 3c If $M \not\leq\not\geq N, T(M) = a \wedge b^* = x \wedge y^* = T(N), F(N) = x^* \wedge y < a^* \wedge b = F(M)$, and $N \not\leq\not\geq P, T(N) = x \wedge y^* = p \wedge q^* = T(P)$ and $F(P) = p^* \wedge q < x^* \wedge y = F(N)$. Now, since $T(M) = T(N)$ and $T(N) = T(P)$, $T(M) = T(P)$. Also, $F(P) < F(N)$ and $F(N) < F(M)$, thus, $F(P) < F(M)$. Let us assume $P \leq M$. Then, $T(P) \leq T(M) = T(N)$, a contradiction. So, $P \not\leq M$. Thus, either $M \leq P$ or $M \not\leq\not\geq P$. In both scenarios, $A_l \preceq_T A_z$.

Case 4 $M \not\leq\not\geq N, T(M) = a \wedge b^* < x \wedge y^* = T(N); N \not\leq\not\geq P, T(N) = x \wedge y^* = p \wedge q^* = T(P)$, and $F(P) = p^* \wedge q < x^* \wedge y = F(N)$.

Since $T(M) < T(N)$ and $T(N) = T(P)$, $T(M) < T(P)$. Let us suppose that $P \leq M$; this means $T(P) \leq T(M) < T(N)$, which is a contradiction to the fact that $T(N) = T(P)$. Thus, $P \not\leq M$; so either $M \leq P$ or $M \not\leq\not\geq P$. In both cases, $A_l \preceq_T A_z$.

Case 5 $M \not\leq\not\geq N, T(M) = a \wedge b^* = x \wedge y^* = T(N)$ and $F(N) = x^* \wedge y < a^* \wedge b = F(M); N \not\leq\not\geq P, T(N) = x \wedge y^* < p \wedge q^* = T(P)$.

Given that $T(M) = T(N)$ and $T(N) < T(P)$, we conclude that $T(M) < T(P)$. Assume $P \leq M$. Then, $T(P) \leq T(M) = T(N)$. Thus, $T(P) \leq T(N)$ contradicts the fact that $T(N) < T(P)$; hence, $P \not\leq M$. Therefore, either $M < P$ or $M \not\leq\not\geq P$. Again in both cases, $A_l \preceq_T A_z$.

Therefore, by 1–5, the relation $\preceq_T$ is transitive, and the proof is complete. □

Appendix B

The proof of Theorem 2

Proof: In fact, the proof of Theorem 2 is much similar to that of Theorem 1. For instance, the proof of transitivity of the relation $\preceq_F$ for the case $M \leq N$, $N \lneqq\!\!\!\!\!\!\geqq P$, and $F(P) = p^* \wedge q < x^* \wedge y = F(N)$ is composed of two sub-cases as follows:

Sub-case I (i) $a \leq x$, (ii) $x^* \leq a^*$, (iii) $y \leq b$, (iv) $x < p$, (v) $p^* < x^*$, (vi) $y < q$, (vii) $p^* \wedge q < x^* \wedge y$.

By (i) and (iv), $a < p$. If $q \leq b$ then $M \leq P$, hence, $A_l \preceq_F A_z$. If $b < q$, then $M \lneqq\!\!\!\!\!\!\geqq P$. Therefore, we reason that by (ii) and (iii,) $x^* \wedge y \leq a^* \wedge b$. And by (vii), we obtain $F(P) = p^* \wedge q < a^* \wedge b = F(M)$. Thus, $A_l \preceq_F A_z$.

Sub-case II (i) $a \leq x$, (ii) $x^* \leq a^*$, (iii) $y \leq b$, (iv) $p < x$, (v) $x^* < p^*$, (vi) $q < y$, (vii) $p^* \wedge q < x^* \wedge y$.

From (iii) and (vi), $q < b$. If $a \leq p$, then $M \leq P$ and $A_l \preceq_F A_z$. But, if $p < a$, we have $M \lneqq\!\!\!\!\!\!\geqq P$. By applying (ii) and (iii), $x^* \wedge y \leq a^* \wedge b$ is obtained. And by (vii), we get $p^* \wedge q < a^* \wedge b$. Therefore, $A_l \preceq_F A_z$.

Another example: The proof of transitivity of the case $M \lneqq\!\!\!\!\!\!\geqq N$, $F(M) = a^* \wedge b = x^* \wedge y = F(N)$, $T(M) = a \wedge b^* < x \wedge y^* = T(N)$ and $N \leq P$ on $\preceq_F$. There are two sub-cases in this proof too:

Sub-case I (i) $a < x$, (ii) $b < y$, (iii) $y^* < b^*$, (iv) $a^* \wedge b = x^* \wedge y$, (v) $x \leq p$, (vi) $p^* \leq x^*$, (vii) $q \leq y$, (viii) $y^* \leq q^*$, (ix) $a \wedge b^* < x \wedge y^*$.

By (i) and (v), $a < p$. If $q \leq b$, then $M \leq P$ and so $A_l \preceq_F A_z$. However, if $b < q$ implies $M \lneqq\!\!\!\!\!\!\geqq P$. So, by (vi) and (vii), $p^* \wedge q \leq x^* \wedge y$. And by (iv), $p^* \wedge q \leq a^* \wedge b$ is attained. Therefore, if $p^* \wedge q < a^* \wedge b$ then, $A_l \preceq_F A_z$. And if $p^* \wedge q = a^* \wedge b$, then we look at it that by (v) and (viii); $x \wedge y^* \leq p \wedge q^*$. By (ix), we have $a \wedge b^* < p \wedge q^*$. Thus, $A_l \preceq_F A_z$.

Sub-case II (i) $x < a$, (ii) $y < b$, (iii) $b^* < y^*$, (iv) $a^* \wedge b = x^* \wedge y$, (v) $x \leq p$, (vi) $p^* \leq x^*$, (vii) $q \leq y$, (viii) $y^* \leq q^*$, (ix) $a \wedge b^* < x \wedge y^*$.

From (ii) and (vii), $q < b$. If $a \leq p$, we have $M \leq P$, and therefore $A_l \preceq_F A_z$. If $p < a$, then $M \lneqq\!\!\!\!\!\!\geqq P$. Thus, $p^* \wedge q \leq x^* \wedge y$ by applying (vi) and (vii). By (iv), $p^* \wedge q \leq a^* \wedge b$. In case $p^* \wedge q < a^* \wedge b$, that is $F(P) < F(M)$ we conclude that $A_l \preceq_F A_z$. If however $F(P) = p^* \wedge q = a^* \wedge b = F(M)$ then by (v) and (viii), $x \wedge y^* \leq p \wedge q^*$. And by (ix), $T(M) = a \wedge b^* < p \wedge q^* = T(P)$. Thus, $A_l \preceq_F A_z$.

Hence, the relation $\preceq_F$ is transitive, and the proof is complete. □