Mathematical model of terrorism: case study of Boko Haram

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Abstract: A deterministic mathematical model that incorporates hostages taken by terrorists, arrested terrorists and the impact of negotiation to release arrested terrorists in exchange for hostages have been formulated in this paper to investigate the dynamics of terrorism in a region. Terrorist free equilibrium point of the model exists, and using the Jacobian matrix and Lyapunov function approach, it is locally and globally (in a special case) asymptotically stable when the reproduction number is less than unity. Negotiation to swap captives is shown to be beneficial provided the rate of such negotiations lies within a restricted range. Real-life data on Boko Haram terrorism have been used to test the model. Increasing counter-terrorism rate to 11 times and reducing the recruitment rate into Boko Haram by 4.2 times the baseline values, together with ensuring that hostages are not taken, will lead to the control of Boko Haram terrorism.

Keywords: terrorism; Boko-Haram; counter-terrorism; mathematical modelling; intervention.

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1 Introduction

The existence of the human race is faced by many threats that are emanating from different sources. Typical of these threats is the phenomenon of terrorism whose definition is quite elusive, see for instance Bruce (2013) and Bjørgo (2005). According to the report in Bjørgo (2005) there are more than 200 definitions of terrorism and the most acceptable definition is that *terrorism is a set of methods or strategies of combat rather than an identifiable ideology or movement, and that terrorism involves premeditated use of violence against (at least primarily) non-combatants to achieve a psychological effect of fear on others than the immediate targets*. Some of the characteristic features of terrorism are; detrimental to the economy, ripping peace and security, rendering many homeless, destruction of lives and properties, physical and psychological trauma through hostage-taking, sectarian animosity, and so on. Nations around the world are witnessing a heavy regime of terrorist activities leading to one degree of destruction to the other. Terrorism has deverstating effects on economy around the world which are evident in the rise in inflations and exchange rates (Jonathan and Makram, 2020). It is worthy of note that terrorists are taking advantages of scientific revolution to carry out their activities; one of the effects of revolution to terrorist propagation is in the use of non-traditional weapons such as chemical, biological, radiological and nuclear weapons (CBRN), see Binder and Ackerman (2019) and Tyler and Thomas (2020). The report of the Global Terrorism Index (GTI) published in 2019 reveals that in 2018, out of the 163 countries analysed, Iraq, Afghanistan, Nigeria, Syria, and Pakistan remain the top five countries most affected by terrorism, see GTI (2019). In February 2021, GTI published its report for the year 2020 which outlined that seven out of the ten nations that withnessed a major widespread in terrorism despite its fall in most regions in the year 2020 were in Sub-Saharan Africa (GTI, 2020). The report attributes the primary driver of terrorism to conflict; as countries which already have history of conflicts recorded 96 percent of the total death due to terrorism in 2019. Different countries have different policies to counter this act and these policies have been appraised or criticised. Typical policies for countering terrorism include violent and non-violent strategies. The non-violent strategies may include; negotiations with terrorists, government campaigns asking the susceptible group to adopt non-susceptible status, prevention of radicalisation, stopping radicalisation as it is occurring, and de-radicalisation programs. The violent means may include the military and the law enforcement approaches, see for instance Santoprete (2019), Clutterbuck (2015), Selim (2016), O'Halloran (2017) and Udwadia et al. (2006). Report in Arin et al. (2019) noted that countries such as the USA, Israel and Colombia have strict non-negotiation policies with terrorist and hostage takers because such negotiations may encourage terrorists to pursue their goals using violence. Despite this official policy, the report reveals that both the USA and Israel have paid ransom after the kidnapping of three US citizens during the Reagan administration in 1987 and the case of negotiation when Israel's children were kidnapped in Maalot. This might mean that a certain degree of negotiation with terrorists is still an option.

Many groups are designated as terrorists due to their activities. Some examples reported in GTI (2015) are: Alqeda, ISIL, and Al-Shabaab. As reported by Salaam (2012), Jama'atu Ahlus-Sunnah Lidda'Awati Wal Jihad popularly known as Boko Haram, a group originally formed in the year 2002 in Maiduguri, Borno State Nigeria has since emerged as one of the violent challengers against the nation. According to

Salaam, the group did not name itself Boko Haram, rather, the name stems from the public view of the sect's basic preaching and beliefs. Between December 2003 and July 2009, members of the Boko Haram sect have conducted several attacks on police, civilians, and the Nigerian military, see Kulungu (2019), Counter Extremism Project (2020) and Salaam (2012). To this end, it is not surprising that on 4th June 2013 the then President of Nigeria Goodluck Ebele Jonathan officially declared Boko Haram a terrorist organisation, as reported in Premium Times Newspaper (2019). Due to its ruthlessness, GTI (2015) crowned Boko Haram as the most deadly terrorist group in the world. Parts of the efforts by the Nigerian government to contain the deadly group includes counterterrorism, negotiations and reintegration effort as the long-term strategy to deal with the Boko Haram, see GTI (2018) and This Day Newspaper (2020). Boko Haram as reported in GTI (2018), specialises in an uncommon terrorist tactics which involve mass hostage-takings and extensive use of children and women as suicide bombers. Some of the kidnapped individuals are either forced to fight or provide support to the group's activities, see Maz et al. (2020) and Hinshaw and Parkinson (2016). Accordint to Salaam (2012), Hinshaw and Parkinson (2016) and Zenn (2014), the recruitment strategies of the group are both through voluntary or coercion. Those who join voluntarily may be motivated by ideological and religious reasons, see Pate (2014) and Kulungu (2019). According to Maz et al. (2020), monetary incentives have also been used to attract recruits into the sect.

Mathematical modelling is a versatile tool in studying many real-life phenomena. For example, Freeman et al. (1996) developed a model that shows how crime becomes concentrated in an area. Short et al. (2008) presented a statistical model of criminal behaviour in which the parameter values that lead to the evolution of stable hotspots of terrorism were determined. Other models of crime can be found in McMillon et al. (2014) and Pate (2014). A dynamical model of terrorism was first studied in the seminal work of Castillo-Chavez and Song (2003). Udwadia et al. (2006) developed a dynamical model of terrorism in Gaza Strip/West Bank. Ediel in Pinker (2009) developed a terror threat information model that captures the uncertainty in timing and location of terror attacks to create a mathematical framework for analysing counter-terrorism decision making. Gutfraind (2010) modelled terrorist organisations using dynamical systems where the terrorist group was divided into; the leaders and the foot soldiers only. Butler (2011), modelled the dynamics of recruitment of Hezbollah terrorist in Lebanon, and based on the model suggests that indirectly targeting the recruitment system is more effective in countering the menace of Hezbollah. McCluskey and Santoprete (2018) modelled the spread of extremist ideology through the radicalisation process, where they divided the total population into susceptible, extremists, and recruiters. Santoprete (2019) built on the model introduced in McCluskey and Santoprete (2018) by adding a treatment compartment to describe de-radicalisation and used basic reproduction number to evaluate counter-terrorism strategies. Santoprete and Xu (2018) included vaccinated class in their model, and used it to analyse two counter violent extremism strategies; prevention program and de-radicalisation program. Okoye et al. (2020) developed a dynamical model of terrorism where the authors used reproduction number similar to those in epidemiology, to determine whether terrorists can be eradicated in a given area or not. Other models developed to study terrorism includes Cummings and Weerasinghe (2017), Cherif et al. (2009), Gambo (2020), Udoh and Oladejo (2019) and Zahedzadeh (2017).

It is important to mention that the aforementioned studies and the references contained in them have contributed immensely towards understanding the dynamics of terrorism and some mechanisms to curtail its menace. However, none of the aforementioned studies consider the following:

1 the dynamics of abducted victims of terrorists

2 global sensitivity analysis to find the most important parameters in the dynamics of their models

3 terrorist induced death rates in the model.

The prevalence in terrorism with its associated effects on human lives and properties has shown that there is an urgent need for a renewed commitment to finding better strategies to control or quell this ever-growing threat which can be described as a global epidemic. The contribution in this paper is to present a deterministic mathematical model for studying the dynamics of terrorism with particular emphasis on Boko Haram using an autonomous system of differential equations. The new insight that can be obtained can help policymakers in designing effective counter-terrorism measures.

The current study extends the works of Santoprete (2019), Gambo (2020) and Okoye et al. (2020) by

1 incorporating abducted (hostaged) populations and captured (arrested terrorist) in the model

2 conducting global sensitivity analysis to find the most important parameters in the dynamics of the model

3 incorporating parameters to model the impacts of negotiations, arrest and de-radicalisation of captured terrorists.

2 Model formulation

Human population in this work is classified into susceptible (S), non-susceptible (N), abducted (A), and the terrorists (T). The terrorist population is further classified into: leaders (L), foot soldiers (F), and the captured (arrested foot soldiers) individuals (C). Susceptible in this context are individuals who are more vulnerable to be influenced by the sect's ideology. Non-susceptible are individuals who cannot be easily influenced by the terrorists. Foot soldiers refer to the rank and file terrorists whose responsibilities include receiving and carrying out instructions from the leaders, and their appearance is notable in the battlefield. Leaders are those with the authority to give instructions and to take major decisions. The captured terrorist are those terrorists arrested by the security agencies while the abducted population refers to those susceptible and non-susceptible whom the terrorists seize and take hostage. The state variables of the model are presented on Table 1.

2.1 Model assumptions

To formulate the model, the following assumptions are made:

1 Leaders can only emerge through promotion from the foot soldiers population.

2 Considering the long-term imprisonment of convicted terrorists, arrested leaders (who would be convicted eventually) are assumed to remain arrested until death, hence they are considered removed since they play no further role in the dynamics of terrorism.

3 The human population in the region is reasonably mixed so that every individual has equal chance of being victimised.

4 Recruitment into the terrorists population is from the susceptible or the abducted populations only.

5 Arrested or captured foot soldiers can be released through negotiations or be de-radicalised to assume non-susceptible or susceptible states.

6 Public enlightenment through government's campaign (vaccination) can convince some susceptible to adopt non-susceptible status.

Table 1 The six state variables of the model

Variable	*Description*
S	Susceptible humans
N	Non-susceptible humans
A	Abducted humans
F	Foot soldiers
L	Leaders
C	Captured terrorists

2.2 Dynamics of the non-terrorist population

2.2.1 Susceptible (S)

The population of the susceptible individuals is increasing by recruitment through birth or immigration at a rate ψ, a fraction of individuals α_3 that are released from abduction through negotiation at a rate γ, a fraction of individuals $(1-\alpha)$, released from detention through de-radicalisation at a rate ω. The susceptible population decreases due to direct recruitment to the foot soldiers at a rate η, the abduction of a fraction α_1 by foot soldiers at a rate g, and by government's campaign that convinces the susceptible to adopt non-susceptible status at a rate α_2. Susceptible population can die naturally at a rate μ_3 or through terrorist-induced death at a rate μ_t.

The change in the susceptible population is therefore,

$$\frac{dS}{dt} = \psi - (\mu_3 + \mu_t)\, S - g\alpha_1 F + \gamma\alpha_3 A - (\eta + \alpha_2)SF + \omega C(1-\alpha). \qquad (1)$$

2.2.2 Non-susceptible (N)

The population of the non-susceptible individuals is increasing by recruitment through birth or immigration at the rate Λ, a fraction of individuals $(1-\alpha_3)$ that are released from abduction through negotiation at a rate γ, a fraction of individuals α, released from detention through de-radicalisation at a rate ω, government's campaign that convinces the susceptible to adopt non-susceptible status at a rate α_2. The population in this class is decreasing by the fraction $(1-\alpha_1)$ of the individuals abducted by the terrorists at a rate g, the natural death rate μ_3 and the terrorist induced death rate μ_t.

The change in the non-susceptible population is therefore,

$$\frac{dN}{dt} = \Lambda - (\mu_3 + \mu_t) N + \gamma(1-\alpha_3)A + \alpha_2 SF + \alpha\omega C - g(1-\alpha_1)F. \tag{2}$$

2.2.3 *Abducted (A)*

The population of the abducted individuals is increasing through abduction from the susceptible and non-susceptible populations at a rate g. This population is decreasing by the recruitment into the foot soldiers at a rate δ, release of the abducted through negotiations at a rate γ, the natural death rate μ_3 and the terrorist induced death rate μ_t. The change in the abducted population is therefore,

$$\frac{dA}{dt} = gF - (\mu_t + \mu_3 + \delta + \gamma)A. \tag{3}$$

2.3 *Dynamics of the terrorist population*

2.3.1 *Foot soldiers (F)*

The population of the foot soldiers is increasing by recruitment from the susceptible and abducted classes at the rates η and δ respectively, and the release of captured foot soldiers at the rate γ. The population is decreasing by the arrests and promotion of the foot soldiers at the rates β and ξ respectively and death due to natural causes and counterterrorism at the rates μ_3 and μ_c respectively.

The change in the population of the foot soldiers is therefore,

$$\frac{dF}{dt} = \eta SF + \delta A + (\gamma - \mu_c - \mu_3 - \xi - \beta)F. \tag{4}$$

2.3.2 *Leaders (L)*

The population of the leaders is increasing by the promotion of the foot soldiers at the rate ξ and decreasing by the removal of the leaders through death due to natural causes at a rate μ_3, through counterterrorism at the rate $\sigma_L\mu_c$, where $0 < \sigma_L < 1$ is a modification parameter to account for the reduced death rate of the leaders in the face of counterterrorism. The change in the population of the leaders is therefore,

$$\frac{dL}{dt} = \xi F - (\mu_3 + \sigma_L\mu_c) L. \tag{5}$$

2.3.3 Captured terrorists (C)

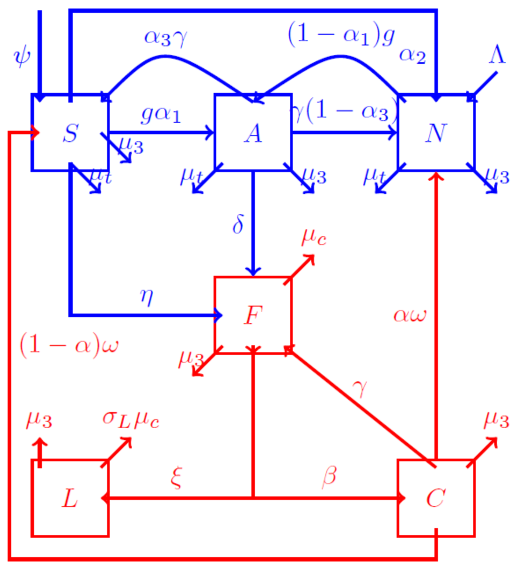
The population of the captured terrorists is increasing by the arrest of foot soldiers at a rate β and decreasing by the release of captured terrorists through negotiations and de-radicalisation at the rates γ and ω respectively and natural death at a rate μ_3.

The change in the population of the captured terrorists is therefore,

$$\frac{dC}{dt} = -(\omega + \mu_3)C - (\gamma - \beta)F. \tag{6}$$

The flow diagram of the model is given in Figure 1 and the parameter description is given on Table 2.

Figure 1 Flow diagram of the model (see online version for colours)



We further simplify the model by introducing $\mu = \mu_t + \mu_3$, $\mu_1 = \mu_c + \mu_3$, and $\mu_2 = \sigma_L \mu_c + \mu_3$ it then follows that the model equations are given by

$$\begin{aligned}
\frac{dN}{dt} &= \gamma\left(1-\alpha_3\right)A - N\mu + \Lambda + FS\alpha_2 + \omega\,\alpha\,C - g\left(1-\alpha_1\right)F \\
\frac{dS}{dt} &= -FS\left(\eta+\alpha_2\right) + C\omega\,\left(1-\alpha\right) - g\alpha_1 F + \gamma\,\alpha_3 A - S\mu + \psi, \\
\frac{dF}{dt} &= FS\eta + A\delta + F\left(\gamma - \beta - \mu_1 - \xi\right), \\
\frac{dA}{dt} &= -A\left(\gamma+\delta+\mu\right) + Fg, \\
\frac{dC}{dt} &= -C\left(\omega+\mu_3\right) - F\left(\gamma-\beta\right), \\
\frac{dL}{dt} &= F\xi - L\mu_2.
\end{aligned} \tag{7}$$

Table 2 Model parameters and their descriptions

Parameter	*Description and dimension*
ψ	Recruitment rate into the susceptible population through birth or immigration, time^{-1}
Λ	Recruitment rate into the non-susceptible population through birth or immigration, time^{-1}
μ_t	Terrorist induced death rate of non-terrorist population, time^{-1}
μ_c	Death rate of the foot soldiers due to counterterrorism, time^{-1}
μ_3	Human natural death rate, time^{-1}
η	Rate of recruitment of foot soldiers from the susceptible humans, time^{-1}
δ	Rate of recruitment of foot soldiers from the abducted humans, time^{-1}
ξ	Rate of promotion of foot soldiers, time^{-1}
β	Rate of arrests of foot soldiers, time^{-1}
γ	Rate of negotiation between the terrorists and the government, time^{-1}
ω	Rate of de-radicalisation of captured foot soldier, time^{-1}
g	Rate of abduction of individuals from the non-terrorist populations, time^{-1}
α_2	Rate at which susceptible adopts non-susceptible status (vaccination), time^{-1}
σ_L	Modification parameter to account for the reduce death/removal rate of the terrorist leaders, dimensionless
α	Fraction of individuals that are released to the non-susceptible class after de-radicalisation, dimensionless
α_1	Fraction of individuals abducted by foot soldiers from the susceptible class, dimensionless
α_3	Fraction of individuals that are released from abduction to the susceptible class, dimensionless

3 Terrorist free equilibrium point

Theorem 1: Model (7) has a unique terrorist free equilibrium point

$$E_{\text{free}} = (N^*, S^*, F^*, A^*, C^*, L^*) = \left(\frac{\Lambda}{\mu}, \frac{\psi}{\mu}, 0, 0, 0, 0\right)$$

which is locally asymptotically stable if $R_0 < 1$, where $R_0 = \frac{\eta\psi(\delta+\mu+\gamma)+\delta\mu g}{\mu(\delta+\mu+\gamma)(\beta-\gamma+\mu_1+\xi)}$.

Proof: In proving Theorem 1, we set $A = C = F = L = 0$ in model (7) and solving the remaining equations we obtain the terrorist free equilibrium point as

$$E_{\text{free}} = (N^*, S^*, F^*, A^*, C^*, L^*) = \left(\frac{\Lambda}{\mu}, \frac{\psi}{\mu}, 0, 0, 0, 0\right) \tag{8}$$

which establishes the first part of the proof. To prove the second part, we consider the Jacobian of model (7) evaluated at E_{free} which is given by

$$J^* = \begin{bmatrix} -\mu & 0 & \frac{\psi\,\alpha_2}{\mu} - g\,(1-\alpha_1) & \gamma\,(1-\alpha_3) & \omega\,\alpha & 0 \\ 0 & -\mu & -\frac{\psi\,(\eta+\alpha_2)}{\mu} - g\alpha_1 & \gamma\,\alpha_3 & \omega\,(1-\alpha) & 0 \\ 0 & 0 & \frac{\psi\,\eta}{\mu} - \beta + \gamma - \xi - \mu_1 & \delta & 0 & 0 \\ 0 & 0 & g & -\delta - \mu - \gamma & 0 & 0 \\ 0 & 0 & -\gamma + \beta & 0 & -\omega - \mu_3 & 0 \\ 0 & 0 & \xi & 0 & 0 & -\mu_2 \end{bmatrix}. \tag{9}$$

Four eigenvalues of J^* are $-\mu, -\mu, -\omega - \mu_3, -\mu_2$. The other two eigenvalues can be found from

$$J_1^* = \begin{bmatrix} \frac{\psi\,\eta}{\mu} - (\beta - \gamma + \mu_1 + \xi) & \delta \\ g & -(\delta + \mu + \gamma) \end{bmatrix}. \tag{10}$$

The conditions for the eigenvalues of equation (10) to be negative is that trace of $J_1^* < 0$ and determinant of $J_1^* > 0$. That is

$$\frac{\psi\,\eta - (\beta - \gamma + \mu_1 + \xi)\,\mu - (\delta + \mu + \gamma)\,\mu}{\mu} < 0,$$
$$\frac{\mu\,((\beta - \gamma + \mu_1 + \xi)\,(\delta + \mu + \gamma) - g\delta) - \eta\,\psi\,(\delta + \mu + \gamma)}{\mu} > 0. \tag{11}$$

For the inequalities (11) to be satisfied we must have

$$\frac{\eta\psi}{\mu} < \min\Big\{(\beta - \gamma + \mu_1 + \xi) - \frac{\delta g}{(\delta + \mu + \gamma)},$$
$$(\beta - \gamma + \mu_1 + \xi) + (\delta + \mu + \gamma)\}$$
$$= (\beta - \gamma + \mu_1 + \xi) - \frac{\delta g}{(\delta + \mu + \gamma)}. \tag{12}$$

Inequality (12) implies that $\frac{\eta\psi(\delta+\mu+\gamma)+\delta\mu g}{\mu(\delta+\mu+\gamma)(\beta-\gamma+\mu_1+\xi)} < 1$. Thus, E_{free} is locally asymptotically stable if $R_0 < 1$.

The threshold quantity R_0 is similar to the basic reproduction number in epidemiological models and might be obtained using the next-generation approach. The quantity is expected to provide a condition for the eradication of the terrorist. The conclusion in many studies is that if $R_0 < 1$, terrorist population can be eradicated and that terrorist population will persist if $R_0 > 1$, see Okoye et al. (2020) and Santoprete (2019). In Subsection 6.4 we conducted a global sensitivity analysis to determine the important parameters governing R_0 and hence the dynamics of terrorism.

One of the parameters governing R_0 in this study is the rate of negotiations. There is a strong argument that negotiating with terrorists will encourage more terrorist attacks. Here we want to find out whether negotiation rate is of any benefit in mitigating the impact of terrorism by studying R_0. To do that we consider the difference between reproductions with and without negotiation rate. Thus we state the following:

Theorem 2: Assuming that $R_5 > 1$ and $\gamma < (\mu + \delta)(R_5 - 1)$ then rate of negotiation can decrease the reproduction number, where $R_5 = \frac{\delta\, g\mu\,(\beta+\mu_1+\xi)}{(\delta+\mu)(\eta\,\psi\,\delta+\delta\, g\mu+\eta\,\psi\,\mu)}$.

Proof: To prove Theorem 2, we let the reproduction number without negotiation to be defined after setting $\gamma = 0$ by $R_0^0 = \frac{\delta\, g\mu+\eta\,\psi\,(\delta+\mu)}{\mu\,(\beta+\mu_1+\xi)(\delta+\mu)}$. For negotiation to have positive impact on the reproduction number, we must have $R_0 - R_0^0 < 0$. After simplification, this reduces to $(\eta\,\psi\,\delta + \delta\, g\mu + \eta\,\psi\,\mu)\gamma + (\delta + \mu)(\eta\,\psi\,\delta + \delta\, g\mu + \eta\,\psi\,\mu) - \delta\, g\mu(\beta + \mu_1 + \xi < 0$. Further simplifications yields $\gamma < (\mu + \delta)(R_5 - 1)$.

4 Persistence equilibrium point

We now investigate the conditions under which the equilibrium point of model (7) has all non-zero state variables. We refer to this as terrorist persistence equilibrium point denoted by $E_p = (N^{**}, S^{**}, F^{**}, C^{**}, L^{**})$. To find E_p we first obtain the equilibrium solutions of A^{**}, C^{**}, L^{**} from the forth, fifth and sixth equations of model (7) as

$$A^{**} = \frac{F^{**}g}{\delta + \mu + \gamma}, \tag{13}$$

$$C^{**} = \frac{F^{**}\,(-\gamma + \beta)}{\omega + \mu_3}, \tag{14}$$

$$L^{**} = \frac{F^{**}\xi}{\mu_2}. \tag{15}$$

Using equations (13) and (14) and the second and third equations in model (7) we obtain

$$S^{**} = \frac{(\beta - \gamma + \mu_1 + \xi)(\delta + \mu + \gamma) - \delta\, g}{\eta\,(\delta + \mu + \gamma)}, \tag{16}$$

$$F^{**} = \frac{(1 - R_0)\,\mu\,(\beta - \gamma + \mu_1 + \xi)}{(R_1 - 1)(\eta\, g\alpha_1 + (\beta - \gamma + \mu_1 + \xi)(\eta + \alpha_2))}, \tag{17}$$

where

$$R_1 = \frac{g\,(\omega + \mu_3)(\eta\,\gamma\,\alpha_3 + \delta\,\eta + \delta\,\alpha_2) + \eta\,\omega\,(1 - \alpha)(\delta + \mu + \gamma)(-\gamma + \beta)}{(\delta + \mu + \gamma)(\omega + \mu_3)(\eta\, g\alpha_1 + (\beta - \gamma + \mu_1 + \xi)(\eta + \alpha_2))}. \tag{18}$$

From the first equation of model (7) we solved for N^{**} and also using the fourth and fifth equations in model (7) we obtain

$$N^{**} = \frac{F^{**}Q + (\omega + \mu_3)(\Lambda\,\eta\,(\delta + \mu + \gamma) - F^{**}(\delta\, g\alpha_2 + \eta\, g\,(1 - \alpha_1)(\delta + \mu + \gamma))}{(\omega + \mu_3)\,\eta\,(\delta + \mu + \gamma)\,\mu}, \tag{19}$$

where

$$Q = \alpha\,\eta\,\omega\,(\delta + \mu + \gamma)(\beta - \gamma) + (\omega + \mu_3)(\eta\, g\gamma\,(1 - \alpha_3) + (\beta - \gamma + \mu_1 + \xi)(\delta + \mu + \gamma)\,\alpha_2).$$

Note that $N^{**} > 0$ provided

$$R_2 = \frac{\Lambda \eta\, (\delta + \mu + \gamma)\, (\omega + \mu_3) + F^{**} Q}{F^{**}\, (\omega + \mu_3)\, (\delta\, g \alpha_2 + \eta\, g\, (1 - \alpha_1)\, (\delta + \mu + \gamma))} > 1.$$

We need the following lemma:

Lemma 3: If $R_0 < 1$ then $(\delta + \mu + \gamma)\,(\beta - \gamma + \mu_1 + \xi) - \delta g > 0$.

Proof: To prove Lemma 3 we note that $R_0 < 1$ implies that $\psi\eta < \mu(\delta + \mu + \gamma)(\beta - \gamma + \mu_1 + \xi) - \delta g$. Since $\psi\eta > 0$, it must mean that $(\delta + \mu + \gamma)\,(\beta - \gamma + \mu_1 + \xi) > \delta g$.

Thus we state the following results:

Theorem 4: Suppose $R_2 > 1$:

1 If $R_1 > 1$ and $R_0 < 1$, then model (7) has a unique terrorist persistent equilibrium point given by E_p.

2 If $R_0 > 1$, and $R_1 < 1, R_A > 1$ then model (7) has a unique terrorist persistent equilibrium point, where $R_A = \frac{(\beta-\gamma+\mu_1+\xi)(\delta+\mu+\gamma)}{\delta g}$.

3 No terrorist persistent equilibrium point otherwise.

The implication of the first part of Theorem 4 is that the requirement for terrorism to be controlled when $R_0 < 1$ no longer holds. This is because E_{free} and E_p may co-exist. In the next section, we investigate the condition under which bifurcation may occur.

5 Backward bifurcation

The phenomenon of backward bifurcation, a situation where a stable terrorist free equilibrium point co-exists with a stable persistence equilibrium point when the reproduction is less than unity has been observed in the dynamical model of terrorism reported in Castillo-Chavez and Song (2003). We now investigate the condition under which bifurcation will occur in model (7). We state the following:

Theorem 5: Model (7) will undergo the phenomenon of backward bifurcation at $R_0 = 1$ if $R_4 > 1$, where

$$R_4 = \frac{\mu\, (g\gamma\, (\omega + \mu_3)\, \alpha_3 + \omega\, (1 - \alpha)\, (\delta + \mu + \gamma)\, (\beta - \gamma))}{(\delta + \mu + \gamma)\, (\omega + \mu_3)\, (g\mu\, \alpha_1 + \psi\, (\eta^* + \alpha_2))}. \tag{20}$$

Proof: To prove Theorem 5, we choose η as the bifurcation parameter and the critical value of this parameter which makes $R_0 = 1$ is

$$\eta^* = \frac{((\beta - \gamma + \mu_1 + \xi)\, (\delta + \mu + \gamma) - \delta\, g)\, \mu}{(\delta + \mu + \gamma)\, \psi}. \tag{21}$$

We let $x_1 = N,\ x_2 = S,\ x_3 = F,\ x_4 = A,\ x_5 = C$ and $x_6 = L$. Further more, let $\hat{f} = [f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6]$ be the left hand sides of model (7). The eigenvalues of the Jacobian of the model evaluated at E_{free} and at $\eta = \eta^*$

are $\left(0, -\mu_2, -(\omega+\mu_3), -\frac{\delta\, g+(\delta+\mu+\gamma)^2}{\delta+\mu+\gamma}, -\mu, -\mu\right)$. Clearly, one eigenvalue is zero and the remaining have negative real parts. The components of the right (w_i), and the left eigenvectors (v_i), $i = 1, ..., 6$ are $w_1 = \frac{\mu_2(R_3-1)g(1-\alpha_1)}{\mu\,\xi}$, $w_2 = \frac{\mu_2(R_4-1)(g\mu\,\alpha_1+\psi\,(\eta+\alpha_2))}{\xi\,\mu^2}$, $w_3 = \frac{\mu_2}{\xi}$, $w_4 = \frac{\mu_2 g}{\xi\,(\delta+\mu+\gamma)}$, $w_5 = \frac{\mu_2(-\gamma+\beta)}{\xi\,(\omega+\mu_3)}$, $w_6 = 1$ and $v_1 = v_2 = v_5 = v_6 = 0$, $v_3 = \delta+\gamma+\mu$ and $v_4 = \delta$, where

$$R_3 = \frac{\alpha\,\mu\,\omega\,(\delta+\mu+\gamma)(-\gamma+\beta)+(\omega+\mu_3)(g\mu\,\gamma\,(1-\alpha_3)+\psi\,(\delta+\mu+\gamma)\,\alpha_2)}{\mu\,g\,(1-\alpha_1)(\delta+\mu+\gamma)(\omega+\mu_3)}$$

and

$$R_4 = \frac{\mu\,(g\gamma\,(\omega+\mu_3)\,\alpha_3+\omega\,(1-\alpha)(\delta+\mu+\gamma)(-\gamma+\beta))}{(\delta+\mu+\gamma)(\omega+\mu_3)(g\mu\,\alpha_1+\psi\,(\eta^*+\alpha_2))}.$$

The bifurcation coefficients, a and b, are given, respectively, by

$$\begin{aligned} a &= \sum_{k,i,j=1}^{6} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i x_j} \\ &= \frac{2{\mu_2}^2\,(R_4-1)(\delta+\mu+\gamma)(g\mu\,\alpha_1+\psi\,(\eta^*+\alpha_2))\,\eta^*}{\xi^2\mu^2} \end{aligned}$$

and

$$b = \sum_{k,i=1}^{6} v_k w_i \frac{\partial^2 f_k}{\partial \eta x_i} = \frac{2\,(\delta+\mu+\gamma)\,\mu_2\psi}{\xi\,\mu}.$$

It is clear that b is always positive, the type of bifurcation that occurs at $R_0 = 1$ depends only on the sign of a. Hence, the result follows from Theorem 8 of Castillo-Chavez and Song (2003).

The appearance of a backward bifurcation implies that the elimination of the terrorist population is difficult. It is possible for a small number of terrorists to successfully penetrate the general population, see Castillo-Chavez and Song (2003). One of the immediate consequence of the results presented in Sections 4 and 5 is summarised in Corollary 6.

Corollary 6: Assuming that there is no abduction from the non-terrorist population $(g = 0)$ and no de-radicalisation $(\omega = 0)$ then model (7) will not undergo backward bifurcation.

Proof: Corollary 6 can be easily proved by setting $g = \omega = 0$ in equations (18) or (20).

5.1 Non-existence of backward bifurcation

In this subsection, we investigate the existence and stability of positive equilibria in which the abducted class is absent $(A = 0)$. Note that the terrorist free equilibrium point is same as stated under Theorem 1. We state the following result:

Theorem 7: Assuming that terrorists do not take hostages and there is no de-radicalisation, then the terrorist free equilibrium point is globally asymptotically stable.

Proof: The proof is based on the following Lyapunov function.

$$V = \left(N - \frac{\Lambda}{\mu}\left(1 + \ln\left(\frac{N\mu}{\Lambda}\right)\right) + C + L\right) k_1 + \left(S - \frac{\psi}{\mu}\left(1 + \ln\left(\frac{S\mu}{\psi}\right)\right)\right) k_2 + k_3 F, \tag{22}$$

where $k_1 = \mu\,(\beta + \xi + \mu_1)\,(1 - R_0)$, $k_2 = \frac{\mu\,(\beta\,\eta + \eta\,\xi + (\beta + \xi + \mu_1)\alpha_2)}{\eta\,(\eta + \alpha_2)}$ and $k_3 = \beta\,\mu + \mu\,\xi + \psi\,\alpha_2$. Differentiating equation (22) with respect to time and using equation (7) we obtain

$$\frac{dV}{dt} = \left(\left(\left(1 - \frac{\Lambda}{\mu\,N}\right) k_1\alpha_2 - k_2\eta\,Y_1 + k_3\eta\right) S + \frac{k_2\psi\,\eta\,Y_1}{\mu} - k_3 Y_2 + (\beta + \xi)\,k_1\right) F + \left(2\,\Lambda - \mu\,N - \frac{\Lambda^2}{\mu\,N}\right) k_1 + \left(2\,\psi - \mu\,S - \frac{\psi^2}{\mu\,S}\right) k_2 + (-C\mu_3 - L\mu_2)\,k_1. \tag{23}$$

Here $Y_1 = \eta + \alpha_2$, $Y_2 = \beta + \xi + \mu_1$. Upon further simplification we obtain

$$\frac{dV}{dt} = -\,(\beta + \xi + \mu_1)\,(1 - R_0)\left(\mu\,(C\mu_3 + L\mu_2) + \frac{(\Lambda - N\mu)^2}{N} + \frac{(\beta\,\eta + \eta\,\xi + (\beta + \xi + \mu_1)\,\alpha_2)\,(S\mu - \psi)^2}{\eta\,(\eta + \alpha_2)\,S\,(1 - R_0)\,(\beta + \xi + \mu_1)} + \frac{\Lambda\,F S\alpha_2}{N}\right). \tag{24}$$

Now, $\frac{dV}{dt} \le 0$ for $R_0 \le 1$ with equality holding only at E_{free}. The largest invariant set for which $\frac{dV}{dt} = 0$ consists of just the equilibrium E_{free}. The theorem then follows from LaSalle's Invariance Principle.

5.2 Persistence equilibrium point: special case

We are interested in finding the equilibrium point of model (7) in the special case when terrorists do not abduct people and there is no de-radicalisation. Under this condition we set $A = 0$, $\delta = 0$, $g = \omega = 0$ and $\gamma = 0$ in model (7) to obtain a reduced model. We set the left-hand side of the reduced model equal to zero and after some algebraic manipulations we obtained the boundary equilibrium point as $E_{pA} = (N_1^{**}, S_1^{**}, F_1^{**}, 0, C_1^{**}, L_1^{**})$ given by

$$E_{pA} = \left(\frac{\Lambda\,\eta^2\,(\eta + \alpha_2) + \alpha_2\,(R_0 - 1)\,\mu\,(\beta + \xi + \mu_1)}{\eta^2\,(\eta + \alpha_2)\,\mu}, \frac{\psi}{\mu\,R_0}, \frac{(R_0 - 1)\,\mu}{\eta\,(\eta + \alpha_2)}, 0, \frac{\beta\,F_1^{**}}{\mu_3}, \frac{\xi\,F_1^{**}}{\mu_2}\right). \tag{25}$$

The Jacobian matrix of the reduced model evaluated at E_{pA} is given by

$$J_1^{**} = \begin{bmatrix} -\mu & \frac{\mu\,(R_0-1)\alpha_2}{\eta\,(\eta+\alpha_2)} & \frac{(\beta+\xi+\mu_1)\alpha_2}{\eta} & 0 & 0 & 0 \\ 0 & -\mu\,(R_0-1)-\mu & -(\beta+\xi+\mu_1)\,(\eta+\alpha_2) & 0 & 0 & 0 \\ 0 & \frac{\mu\,(R_0-1)}{\eta+\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & \beta & 0 & -\mu_3 & 0 \\ 0 & 0 & \xi & 0 & 0 & -\mu_2 \end{bmatrix}. \tag{26}$$

Four eigenvalues of J_1^{**} are $-\mu, -\mu, -\mu_2, -\mu_3$ the other eigenvalues can be obtained from

$$J_{11}^{**} = \begin{bmatrix} -\mu\,(R_0-1)-\mu & -(\beta+\xi+\mu_1)\,(\eta+\alpha_2) \\ \frac{\mu\,(R_0-1)}{\eta+\alpha_2} & 0 \end{bmatrix}. \tag{27}$$

The trace of J_{11}^{**} is $-\mu\,(R_0-1)-\mu$ and the determinant is $\mu\,(R_0-1)\,(\beta+\xi+\mu_1)$. Thus, we state the following:

Theorem 8: The persistence equilibrium point of the reduced model exist if $R_0 > 1$ and is locally asymptotically stable.

6 Global sensitivity analysis and numerical intervention strategies

In this section, we use real data and information on Boko Haram to simulate and to perform global uncertainty and sensitivity analysis and to consider some numerical intervention strategies on model (7). These investigations can be used to provide a better understanding of the theoretical results obtained from the model analysis. We believe that the new insight that might be obtained can assist policymakers especially in Nigeria and neighbouring countries to design efficient strategy in curbing the threat pose by Boko Haram.

Finding suitable data for model simulation and sensitivity analysis is a big challenge because in many situations such information is not available or is covered with uncertainty. For these reasons, it is important to conduct sampling and sensitivity analysis to determine parameters that have a substantial influence on model output. To perform our investigations, we need to assign baseline parameter values, ranges, and distribution types for sensitivity analysis.

6.1 Baseline parameter values

We used information from the literature to estimate parameter values for simulation and sensitivity analysis of our model where such information is found. For example, the areas most ravaged by Boko Haram conflicts in Nigeria are Adamawa, Borno, and Yobe states. We refer to these states as the affected region. We obtained the population of the affected region in 2013 from the report in Demographic Statistics Bulletin (2017). Also from the findings of GTI reported in GTI (2015), 1,595 and 6,644 civilians were killed by Boko Haram fighters in 2013 and 2014 respectively. Also, from Figure 5 of the findings of the Nigerian security tracker reported in Campbell and Harwood (2018), about 584 and 6,308 members of Boko Haram were reportedly killed in 2013 and 2014 respectively. From the report in Hinshaw and Parkinson (2016), 10,000 boys were abducted and forced to become Boko Haram fighters between 2011 and 2018. With the abduction of the schoolgirls and other unreported cases, we assumed that 12,000 people were recruited from the abducted population to become Boko Haram fighters in seven years. Also from the report in Zenn (2014), Boko Haram may have up to 50,000 members. To this end, we assumed that the number of Boko Haram fighters is 45,000 in 2013. Based on the report in Zenn (2014), there were 1,758 Boko Haram incidents between 2011 and 2018. From the report in The Guardian Newspaper (2017), there was a negotiation between the Nigerian government and Boko Haram that led to the release of 82 kidnapped schoolgirls in exchange for five Boko Haram members. Since there are two other instances where kidnapped schoolgirls were released by Boko Haram, we estimate that there were three successful negotiations between Boko Haram and the Nigerian Government. We estimated the negotiation rate γ using the definition given in Arin et al. (2019). The rest of the important information and assumptions required for estimation of the model parameters and their sources are given in Table 3 and the parameter values are given on Table 4.

Table 3 Relevant information required for parameter estimation

Item	*Quantity*	*Period*	*Source*
Population of the affected area	11,645,046	2013	Demographic Statistics Bulletin (2017)
Non-terrorist killed	1,595	2013	GTI (2015)
Non-terrorist abducted	20,000	2011 to 2018	Assumed
Population of foot soldiers	45,000	2013	Estimated from Zenn (2014)
Foot soldiers killed	584	2013	Campbell (2020)
Foot soldiers recruited from abducted	12,000	2011 to 2018	Estimated
Foot soldiers recruited from non-terrorist population	51	Per month	Assumed
Captured foot soldiers	2,000	2011 to 2018	Assumed
De-radicalised foot Soldiers	900	2011 to 2018	This Day Newspaper (2020)
Total number of Boko Haram incidents	1,758	2011 to 2018	Campbell and Harwood (2018)
Successful negotiation between Boko Haram and government	3	2011 to 2018	Estimated

From the report in Demographic Statistics Bulletin (2017), it was estimated that the average recruitment by birth into the affected region is about 41,935 children per month. However, the impact of emigration was not considered in that estimate. To cushion the effect of emigration, we used assumed recruitment rates into the nonsusceptible and susceptible populations as Λ = 10,000 per month ψ = 25,000 per month. The reason for $\psi > \Lambda$ is to accommodate the suggestion made in Udwadia et al. (2006) that terrorist from other places move to the affected area and settle first as susceptible before joining the terrorist organisation. We also assumed that the susceptible and non-susceptible populations loses equal number of abducted individuals and received equal number of individuals released from abduction and de-radicalisation. Assume further that the rate of promotion of the foot soldiers to be proportional to the death rate of the leaders. So that $\xi = k\,(\mu_3 + \sigma_L \mu_c)$. By choosing k to have the same magnitude as μ_3 we have $\xi = 1.0851 \times 10^{-6}$ and we arbitrarily assigned the value of α_2. Information on Table 3 have been used as a basis for estimating the baseline values for the rest of the parameters presented on Table 4 as

$$\text{Parameter value} = \frac{\text{Number of affected individuals in a compartment}}{N_c \times \text{number of months}},$$

where N_c = total number of individuals in the given compartment. For example, the death rate of the foot soldiers $\mu_c = \frac{584}{45{,}000 \times 12} = 0.001081481\ \text{month}^{-1}$. Human natural death have been estimated in this paper based on 80 years life expectancy, see Santoprete (2019) and the references therein.

Table 4 Baseline parameter values, range for sensitivity analysis and distribution types

Parameter	*Baseline value month*$^{-1}$	*Low*	*High*	*Distribution*
α	0.5	0	1	Uniform
α_1	0.5	0	1	Uniform
α_2	1.2932×10^{-6}	8.4058×10^{-7}	1.6812×10^{-6}	Triangular
α_3	0.5	0	1	Uniform
β	0.0005	0.000325	0.00065	Triangular
δ	0.007143	0.004643	0.009286	Uniform
η	3.2329×10^{-10}	2.1×10^{-10}	4.2×10^{-10}	Uniform
γ	2.03153×10^{-5}	1.32×10^{-5}	2.64×10^{-5}	Triangular
g	2.0446×10^{-5}	1.33×10^{-5}	2.66×10^{-5}	Triangular
Λ	10,000	6,500	13,000	Triangular
μ_t	1.1414×10^{-5}	7.419×10^{-6}	4.7939×10^{-5}	Triangular
μ_c	0.00108	0.00070296	0.011896	Triangular
μ_3	0.001041667	0.0010417	0.0013889	Uniform
ω	0.005357	0.003482	0.006964	Uniform
ψ	25,000	16,250	32,500	Triangular
σ_L	1×10^{-3}	0.00065	0.0013	Uniform
ξ	1.0851×10^{-6}			

6.2 Model simulation

In this section, the baseline parameter values have been used to simulate our model and the results are presented in Figures 2 and 3.

Figure 2 presents the simulation results of the model (7) using the initial condition indicated. The initial number of Leaders have been chosen in line with the report of Counter Extremism Project (2020) which indicates that the leadership structure of Boko Haram is made up of 31 members. Since the group is divided into two factions, as report in Onapajo and Ozden (2020), the number of leaders is assumed to be 62. The result correspond to $R_0 = 2.9216$. This might explain why Boko Haram terrorists remain active despite the various counter-terrorism measures already applied. We observe a rise in the population of the non-susceptible group while there is a drastic decline in the population of the susceptible. The depletion of the population of the susceptible is primarily due to the combination of a high rate of vaccination and recruitment into the Boko Haram population. The number of foot soldiers reaches its peak around 126 months but remain relatively high even after more than 200 months, while the arrested or captured population follows an oscillatory pattern. Decline is observed in the populations of the leaders and the abducted. The decline in the abducted population indicates that some of the abducted are forced or convince to join the foot soldiers which helps in sustaining the terrorist group.

Figure 2 Numerical simulation of model using the initial condition (N, S, F, A, C, L) = (7,808,734, 2,205,826, 20,000, 25,000, 2,000, 62) using the baseline values on Table 4 (see online version for colours)

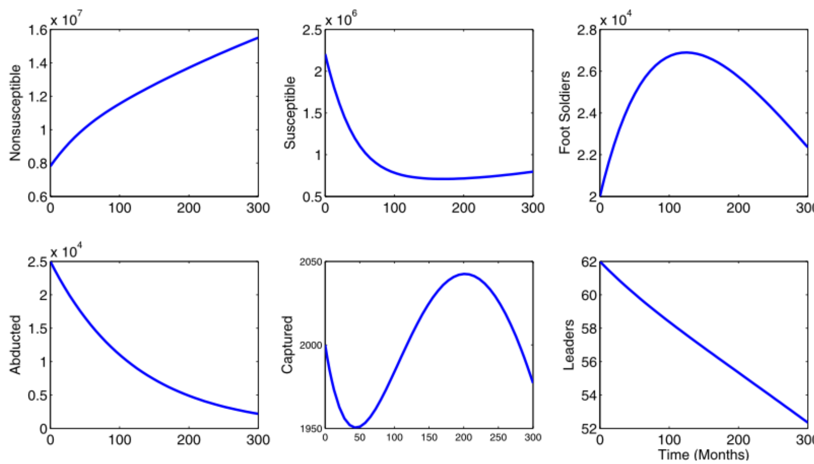
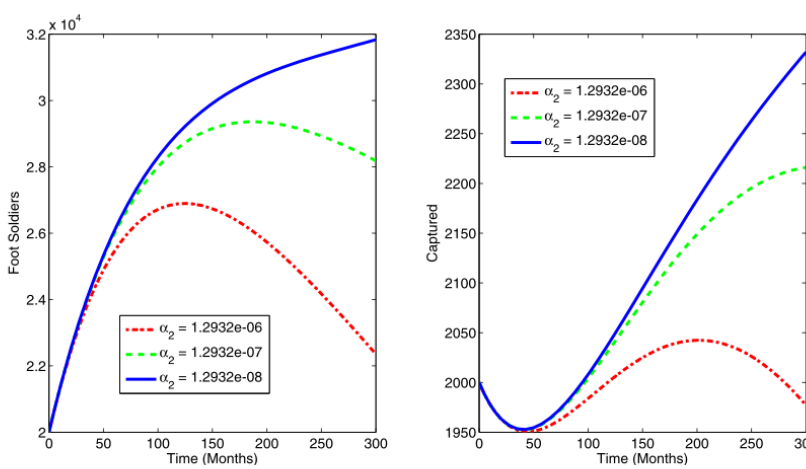


Figure 3 Effects of vaccination on the foot soldiers and the arrested terrorist (see online version for colours)



In Figure 3 we present the impact of vaccination as a non-violent strategy to contain Boko Haram. Shown on the figure are the foot soldiers and the arrested Boko Haram members by simulating model (7) using the baseline parameter values except for α_2 which is varied as indicated on the figure. The figure clearly indicates that in the first 35 months, the value of α_2 does not seems to have any significant impacts on the dynamics of the foot soldiers and arrested Boko Haram members. After this period, the smaller the value of α_2, the higher the number of foot soldiers and the arrested Boko Haram members.

6.3 Global sensitivity analysis

We begin this subsection by assigning ranges and probability distribution functions to the model parameters.

The report in Campbell and Harwood (2018) indicates that the number of civilian and Boko Haram deaths are 4.2 and 11 times higher in 2014 than in 2013 respectively. These fatality values correspond to the peak death rates due to Boko Haram activities in over ten years of its activities as reported in GTI (2018). Thus, we set the low and high values of μ_t, ψ, λ, g, δ, η and γ to 1/4.2 and 4.2 times the corresponding baseline values respectively. Similarly, the low and high values of μ_c and β are set to 1/11 and 11 times the corresponding baseline values respectively. We assumed that α, α_1 and α_3 are uniformly distributed in [0, 1]. We also assumed that the human natural death rate is uniformly distributed between the ages of 60 and 80 years. For the rest of the parameters, the low and the high values are set to 65% (except for α_2, with low value 0.001 of the baseline value) and 135% of the baseline values respectively in line with the work reported in Bala and Gimba (2019).

The shape of the fatalities profile depicted on Graph 5 of Campbell and Harwood (2018) is approximately triangular. Thus we assume triangular distributions for both μ_t and μ_c. As suggested by the report in Hoare et al. (2008), we assumed the probability distributions of the parameters as either uniform or triangular and we set the peak at the average values of high and low, see Table 4.

We used Latin hypercube sampling (LHS) to draw 1,000 samples for each of the model parameters (except ξ which is a dependent parameter) resulting in 1,000 by 16 matrix, where each row defines a unique parameter set. The rows were used to calculate the response functions, R_0 and R_4. Partial rank correlation coefficient (PRCC) was used to characterise the statistical contribution of the relevant parameter to the response functions as was done in some epidemiological models, see for instance Agusto et al. (2013) and Gimba and Bala (2017). The output of these calculations are depicted as tornado plots and presented in Figures 4 and 5.

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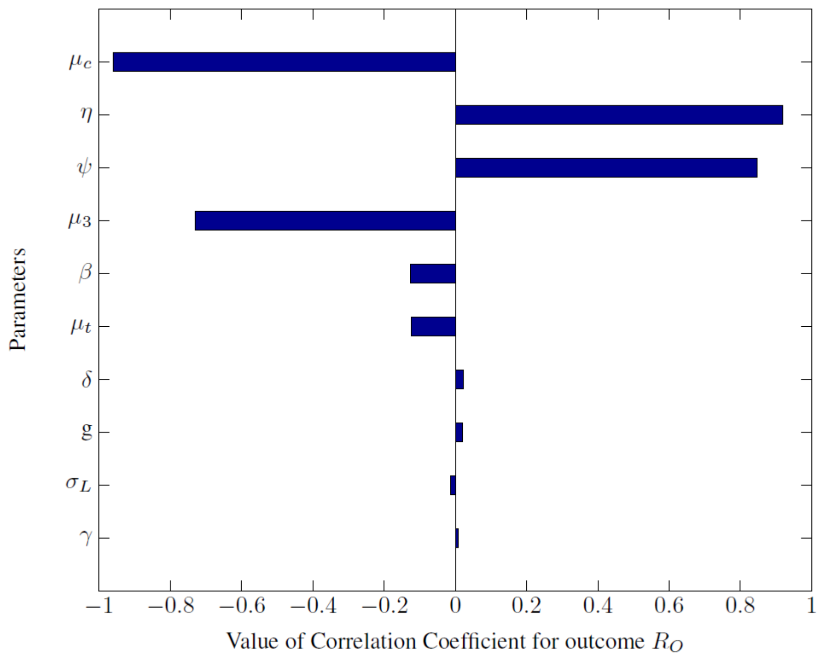
Figure 4 Tornado plot showing the sensitivities of the model parameters affecting the reproduction number (see online version for colours)



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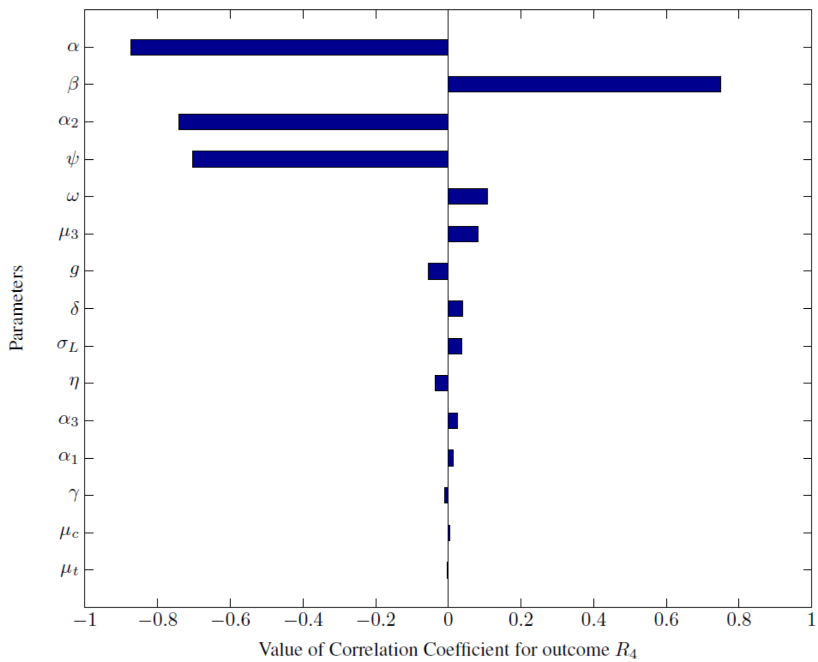
The top five most sensitive parameters affecting R_0 are μ_c, η, ψ, μ_3, and β in that order as shown in Figure 4. To reduce the value of R_0, we need to reduce η, ψ or increase the values of μ_3, μ_c and β. The implication of this is that targeting the recruitment into the Boko Haram and susceptible populations will make the most significant impact in reducing the menace of Boko Haram. In addition to these, counter terrorism measures resulting in eliminating the Boko Haram members should be enhanced.

From Figure 5, the top five most sensitive parameters affecting R_4 are α, β, α_2, ψ and ω. The implication of this is that backward bifurcation can be significantly reduced by ensuring that fraction of de-radicalised individuals joining the non-susceptible group is large. From our model analysis conducted so far, we find that effective strategy for controlling the menace of Boko Haram will require finding a strategy such that $R_0 < 1$, $R_1 < 1$, $R_4 < 1$. This will mean that persistence equilibrium point will not co-exists with the terrorist free equilibrium point and there is no backward bifurcation. We investigate this in the next subsection.

Figure 5 Tornado plot showing the sensitivities of the model parameters affecting R_4 (see online version for colours)



6.4.1 Numerical intervention strategies

From the sensitivity analysis results of R_0 and R_4, we choose recruitment into the Boko Haram population η, counterterrorism parameters μ_c and β as the intervention parameters or decision variables to formulate an optimisation strategy for finding optimal values of the decision variables that will minimised the multi-objective problem:

$$\text{Minimise} \quad \left.\begin{array}{l} F_i(\mathbf{X}), i = 1, ..., 3, \\ X_i^{(l)} \leq X_i \leq X_i^{(u)} \end{array}\right\}. \tag{28}$$

where $F_1 = R_0$, $F_2 = R_1$, $F_3 = R_4$, $X_1 = \mu_c$, $X_2 = \beta$, $X_3 = \eta$, $\mathbf{X} = (X_1, X_2, X_3)^T$. The constraints represents the variable bounds, $X_i^{(l)}$ and $X_i^{(u)}$ restricting each decision variable X_i. These bounds constitute a decision space. Here, we set $X_1^{(l)}$ = 9.8316 $\times$ 10^{-5}, $X_2^{(l)}$ = 4.81 $\times$ 10^{-5}, $X_3^{(l)}$ = 7.6975 $\times$ 10^{-11}, and $X_1^{(u)}$ = 0.01190, $X_2^{(u)}$ = 0.0058, $X_3^{(u)}$ = 1.3578 $\times$ 10^{-9}, these values corresponds to the low and high of the corresponding parameter value given on Table 4. There are many algorithms for solving multi-objective optimisation problems see for instance Deb (2001). In this paper, we used MATLAB *fminimax* optimisation function which is a multi-objective solver that uses a goal attainment method to ensure distribution of numerical estimation of gradients of objective functions and minimises the largest value of a set of multivariable functions when initial estimate is assigned, see MATLAB Optimization Toolbox R2011b (2011). Goal attainment method proposed in Gembicki and Haimes (1975) minimises a set of objectives $F(x) = \{F_1(x), F_2(x), ..., F_k(x)\}$ that are associated with a set of desired goals $\mathbf{X} = \{X_1, X_2, ..., X_k\}^T$. Using the *fminimax* function to solve equation (28), the optimised values of the decision variables are: X_1^* = 0.01190, X_2^* = 0.0058, X_3^* = 7.6975 $\times$ 10^{-11}, and the optimised values of the objective functions are F_1 = 0.098466405170748, F_2 = 0.000079032806562, and F_3 = 0.000956271165662. The result is quite expected in the case of R_0 because the values X_1, X_2 corresponds to the high values of μ_c and β respectively, while the value of X_3 corresponds to the low value of η. Significantly, the optimised decision variables reduces the values of all objective functions to below unity.

7 Discussion

Terrorist organisations around the world share certain characteristics in their patterns of violence and mode of operations, however, Boko Haram additionally adopts uncommon tactics such as massive hostage-taking and recruitment from the hostage. In this report, we modelled the dynamics of terrorism and we used real-life data to study Boko Haram terrorist organisation. We did not consider desertion and prison terms for captured foot soldiers. Also, the leaders do not seem to play a significant role in the dynamics of the model as was done in Gutfraind (2010) through the inclusion of terrorist strength. The reason for this omission is to reduce the nonlinearity of the model equations. According to the report in Onapajo and Ozden (2020), de-radicalisation can be successful or unsuccessful. In our model, we consider only the former. For this reason, de-radicalisation rate is not appearing in the reproduction number.

The inclusion of the abducted compartment and negotiation rates makes our model different from the ones in the literature. From the sensitivity analysis conducted, we find that the most effective approach for eradication of terrorism comes from sustained effort to decrease the population of the foot soldiers follow by limiting the recruitment into the Boko Haram population, see Figure 4. This finding does not fully support the work reported in Butler (2011) and Castillo-Chavez and Song (2003). This is because eliminating the terrorist population through counterterrorism was not considered in those reports. The implication of these findings is that the combination of violent and non-violent strategies will produce desirable effects on the reduction of the menace of Boko Haram. To this end, an effective strategy to control Boko Haram will require:

1 increasing the baseline values of the death and arrest rates of the foot soldiers to 11 times the baseline value

2 reducing the recruitment rate into the foot soldiers from the susceptible population by 4.2 times the baseline value

3 stop hostages from being taken.

Items 1 to 2 above will guarantee that reproduction numbers calculated within the parameters range on Table 4 will fall below unity. Item 3 will guarantee that there is no backward bifurcation. We conclude that if items 1 to 3 are followed, Boko Haram will be controlled. This finding is significant and has not been reported in previous studies to the best of our knowledge. Two other parameters that are key in mitigating the impacts of Boko Haram are recruitment into the susceptible population and natural death. The policy implication of this is that governments around the Boko Haram prone areas should tighten security around the border areas.

From the results on Figure 3 we recommend that the government should not relent in its vaccination campaign as a non-violent strategy even if it is not yielding the desired results over a short time period. There is a long time benefit of the campaign if it is sustained. From our results in Section 3, we find that negotiation to release the abducted individuals in exchange for captured terrorists is meaningful, provided the negotiation rate is within a restricted range. This finding does not fully support the report in the regression model reported in Arin et al. (2019) for sample terrorist negotiation running from 1978 to 2005 and comprising 1,435 events in 125 countries. In that report, it was found that moderate rates of negotiation increase the number of future terror events, while higher negotiation rates tend to decrease the number of future terror events, this is because the parameter for negotiation in our model requires the government to free arrested terrorists in exchange of those abducted. An increase in the death rate due to counterterrorism will decrease the reproduction number in line with work (Okoye et al., 2020; Santoprete, 2019; Santoprete and Xu, 2018). Unlike the models in McCluskey and Santoprete (2018), Okoye et al. (2020), Santoprete (2019) and Santoprete and Xu (2018), results in this paper shows that $R_0 > 1$ does not guarantee the existence of terrorist persistent equilibrium point and $R_0 < 1$ is not sufficient to defeat terrorist organisation, because of the appearance of backward bifurcation.

8 Conclusions

In this paper, a dynamical model of terrorism have been formulated to incorporate the hostage and arrested terrorist populations as compartments in the model and included parameters to model the impacts of negotiations and de-radicalisation into the model. Data specific to Boko Haram have been used to test the model. Global sensitivity analysis have been conducted to find the most important parameters in the dynamics of terrorism. We derived a condition of backward bifurcation to occur and also a threshold condition to be satisfied for negotiation to be useful in curbing the menace of terrorism. The results shows that the terrorist free equilibrium point is globally asymptotically stable under a special condition.

References

Agusto, F.B., Del Valle, S.Y., Blayneh, K.W., Ngonghala, C.N., Goncalves, M.J., Li, N., Zhao, R. and Gong, H. (2013) 'The impact of bed-net use on malaria prevalence', *Journal of Theoretical Biology*, Vol. 3, No. 20, pp.58–65.

Arin, K.P., Feess, E., Kuhlenkasper, T. and Reich, O.F. (2019) 'Negotiating with terrorists: the costs of compliance', *Southern Economic Journal*, Vol. 86, No. 1, pp.305–317.

Bala, S. and Gimba, B. (2019) 'Global sensitivity analysis to study the impacts of bed-nets, drug treatment, and their efficacies on a two-strain malaria model', *Mathematical and Computational Applications*, Vol. 24, No. 1, p.32.

Binder, M.K. and Ackerman, G.A. (2019) 'Pick your POICN: introducing the profiles of incidents involving CBRN and non-state actors (POICN) database', *Studies in Conflict and Terrorism*, pp.1–24 [online] https://doi.org/10.1080/1057610X.2019.1577541.

Bjørgo, T. (2005) *Root Causes of Terrorism: Myths, Reality and Ways Forward*, 2 Park Square, Milton Park, Abingdon, Oxon OX14 4RN.

Bruce, G. (2013) *Definition of Terrorism: Social and Political Effects*, JMVH [online] https://jmvh.org/article/definitionof-terrorism-social-andpolitical-effects/ (accessed 24 July 2020).

Butler, L.B. (2011) *Hezbollah: The Dynamics of Recruitment*, A Monograph Submitted to the Schools of Advanced Military Studies, United States Army Command and General Staff College, Fort Leavenworth, Kansa.

Campbell, J. and Harwood, A. (2018) *Boko Haram's Deadly Impact* [online] https://www.cfr.org/article/boko-harams-deadly-impact (accessed 3 July 2020).

Campbell, J. (2020) *Nigeria Security Tracker* [online] https://www.cfr.org/nigeria/nigeria-security-tracker/p29483 (accessed 3 July 2020).

Castillo-Chavez, C. and Song, B. (2003) 'Models for the transmission dynamics of fanatic behaviors', in *Bioterrorism: Mathematical Modeling Applications in Homeland Security*, pp.155–172, SIAM.

Cherif, A., Yoshioka, H., Ni, W. and Bose, P. (2009) *Terrorism: Mechanisms of Radicalization Processes, Control of Contagion and Counter-Terrorist Measures*, arXiv preprint arXiv: 0910.5272.

Clutterbuck, L. (2015) *Deradicalization Programs and Counterterrorism: A Perspective on the Challenges and Benefits*, Middle East Institute, No. 10.

Counter Extremism Project (2020) *Boko Haram* [online] https://www.counterextremism.com/threat/boko-haram (accessed 3 July 2020).

Cummings, P. and Weerasinghe, C. (2017) 'Modeling the characteristics of radical ideological growth using an agent based model methodology', *MODSIM World*.

Deb, K. (2001) *Multi-Objective Optimization using Evolutionary Algorithms*, Wiley, Chicester, UK.

Demographic Statistics Bulletin (2017) *National Bureau of Statistics* [online] https://nigerianstat.gov.ng/download/775 (accessed 3 July 2020).

Freeman, S., Grogger, S. and Sonstelie, J. (1996) 'The spatial concentration of crime', *Journal of Urban Economics*, Vol. 40, No. 2, pp.216–231.

Gambo, A. (2020) 'Mathematical modeling of dynamics behavior of terrorism and control', *Caspian Journal of Mathematical Sciences*, Vol. 9, No. 1, pp.68–85.

Gembicki, F. and Haimes, Y. (1975) 'Approach to performance and sensitivity multiobjective optimization: the goal attainment method', *IEEE Trans Autom Control.*, 22 August, Vol. 20, No. 6, pp.769–777.

Gimba, B. and Bala, S.I. (2017) 'Modeling the impact of bed-net use and treatment on malaria transmission dynamics', *International Scholarly Research Notices*.

Global Terrorism Index (GTI) (2015) *Measuring and Understanding the Impact of Terrorism* [online] https://reliefweb.int/report/world/global-terrorism-index-2015-measuring-and-understanding-impact-terrorism (accessed 3 July 2020).

Global Terrorism Index (GTI) (2018) *Measuring the Impact of Terrorism* [online] https://visionofhumanity.org/reports (accessed 9 February 2019).

Global Terrorism Index (GTI) (2019) *Measuring the Impact of Terrorism* [online] https://reliefweb.int/report/world/global-terrorism-index-2019 (accessed 3 July 2020).

Global Terrorism Index (GTI) (2020) *Measuring the Impact of Terrorism* [online] https://reliefweb.int/report/world/global-terrorism-index-2020-measuring-impact-terrorism (accessed 24 August 2021).

Gutfraind, A. (2010) *Mathematical Terrorism*, A PhD dissertation, Faculty of Graduate School of Cornell University.

Hinshaw, D. and Parkinson, J. (2016) *The 10,000 Kidnapped Boys of Boko Haram*, The Wall Street [online] https://www.wsj.com/articles/the-kidnapped-boys-of-boko-haram-1471013062 (accessed 3 July 2020).

Hoare, A., Regan, D.G. and Wilson, D.P. (2008) 'Sampling and sensitivity analyses tools (SASAT) for computational modelling', *Theoretical Biology and Medical Modelling*, Vol. 5, No. 1, p.1.

Jonathan, B. and Markram, E. (2020) *Forecast Performance in Times of Terrorism*, Federal Reserve Bank of Dallas, 22 August.

Kulungu, M. (2019) 'Does Boko Haram pose a threat to the US?', *Counter Terrorist Trends and Analysis*, International Centre for Political Violence and Terrorism Research, Vol. 11, No. 2, pp.5–13 [online] https://www.jstor.org/stable/26627975 (accessed 6 August 2020).

MATLAB Optimization Toolbox R2011b (2011) *Optimization Toolbox User's Guide*, The MathWorks Inc., 3 Apple Hill DriveNatick, MA 01760-2098, 22 August.

Maz, K.D., Koldas, U. and Aksit, S. (2020) 'Challenges of countering terrorist recruitment in the Lake Chad region: the case of Boko Haram', *Religions*, Vol. 11, No. 2, p.96.

McCluskey, C.C. and Santoprete, M. (2018) 'A bare-bones mathematical model of radicalization', *Journal of Dynamics & Games*, Vol. 5, No. 3, p.243.

McMillon, D., Simon, C.P. and Morenoff, J. (2014) 'Modeling the underlying dynamics of the spread of crime', *PloS One*, Vol. 9, No. 4, p.e88923.

O'Halloran, P. (2017) 'The challenges of evaluating attitudinal change: a case study of the effectiveness of international countering violent extremism (CVE) programs', Paper presented at the *Canadian Political Association Conference*, Toronto, Ontario, 3 May.

Okoye, C., Collins, O. and Mbah, G. (2020) 'Mathematical approach to the analysis of terrorism dynamics', *Security Journal*, Vol. 33, No. 7, pp.1–12.

Onapajo, H. and Ozden, K. (2020) 'Non-military approach against terrorism in Nigeria: deradicalization strategies and challenges in countering Boko Haram', *Security Journal*, DOI: 10.1057/s41284-020-00238-2.

Pate, A. (2014) *Boko Haram: An Assessment of Strengths, Vulnerabilities, and Policy Options*, Report to the Strategic Multilayer Assessment Office, Department of Defense, and the Office of University Programs, Department of Homeland Security, College Park, MD, January.

Pinker, E.J. (2009) 'A mathematical analysis of short-term responses to threats of terrorism', in *Mathematical Methods in Counterterrorism*, pp.141–160, Springer-Verlag/Wien.

Premium Times Newspaper (2019) *Jonathan Officially Declares Boko Haram A Terrorist Organisation* [online] https://www.premiumtimesng.com/news/137586-jonathan-officially-declares-boko-haram-a-terrorist-organisation.html (accessed 15 April 2019).

Salaam, A.O. (2012) 'Boko Haram: beyond religious fanaticism', *Journal of Policing, Intelligence and Counter Terrorism*, Vol. 7, No. 2, pp.147–162.

Santoprete, M. (2019) 'Countering violent extremism: a mathematical model', *Applied Mathematics and Computation*, 1 October, Vol. 358, pp.314–329, DOI: 10.1016/j.amc.2019.04.054.

Santoprete, M. and Xu, F. (2018) 'Global stability in a mathematical model of de-radicalization', *Physica A: Statistical Mechanics and its Applications*, 1 November, Vol. 509, pp.151–161, DOI: 10.1016/j.physa.2018.06.027.

Selim, G. (2016) 'Approaches for countering violent extremism at home and abroad', *The ANNALS of the American Academy of Political and Social Science*, Vol. 668, No. 1, pp.94–101.

Short, M.B., D'Orsogna, M.R., Pasour, V.B., Tita, G.E., Brantingham, P.J., Bertozzi, A.L. and Chayes, L.B. (2008) 'A statistical model of criminal behavior', *Mathematical Models and Methods in Applied Sciences*, Vol. 18, No. Supp. 1, pp.1249–1267.

The Guardian Newspaper (2017) *Chibok Schoolgirls Families Await as 82 are Freed by Boko-Haram* [online] https://www.theguardian.com/world/2017/may/07/chibok-schoolgirls-familes-await-as-82-are-freed-by-boko-haram-exchange-prisons (accessed 9 February 2019).

This Day Newspaper (2020) 'NIMC Registers 900 Repentant Boko Haram Members [online] https: //www.thisdaylive.com/index.php/2020/06/16/nimc-registers-900-repentant-boko-haram-members/ (accessed 3 July 2020).

Tyler, A.C. and Thomas, R.G. (2020) 'Modeling terrorist attack cycles as a stochastic process: analyzing chemical, biological, radiological, and nuclear (CBRN) incidents', *Journal of Applied Security Research*, DOI: 10.1080/19361610.2020.1761743.

Udoh, I. and Oladejo, N. (2019) 'Optimal human resources allocation in counter-terrorism (CT) operation: a mathematical deterministic model', *International Journal of Advances in Scientific Research and Engineering*, Vol. 5, pp.96–115, DOI: 10.31695/IJASRE.2019.33008.

Udwadia, F., Leitmann, G. and Lambertini, L. (2006) 'A dynamical model of terrorism', *Discrete Dynamics in Nature and Society*, Vol. 2006, Article ID 85653, pp.1–32, Hindawi Publishing Corporation, DOI: 10.1155/DDNS/2006/85653.

Zahedzadeh, G. (2017) 'Containing terrorism: a dynamic model', *Journal of Strategic Security*, Vol. 10, No. 2, pp.48–59.

Zenn, J. (2014) *Boko Haram: Recruitment, Financing, and Arms Trafficking in the Lake Chad Region* [online] https://www.ctc.usma.edu/boko-haram-recruitment-financing-and-arms-trafficking-in-the-lake-chad-region/ (accessed 19 June 2020).