Eventual periodicity of solutions for some discrete max-type system of third order

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Abstract: This paper is concerned with the eventually periodicity of the following max-type difference equation system

$$x_{n+1} = \max\left\{\frac{A}{x_n y_{n-1}}, x_{n-2}\right\},$$
$$y_{n+1} = \max\left\{\frac{A}{y_n x_{n-1}}, y_{n-2}\right\},$$

where $n \in \mathbb{N}$, $A \in \mathbb{R}$, and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary non-zero numbers.

Keywords: periodic solutions; difference equations; max-type system.

Reference to this paper should be made as follows: Ma, H. and Wang, H. (2022) 'Eventual periodicity of solutions for some discrete max-type system of third order', *Int. J. Dynamical Systems and Differential Equations*, Vol. 12, No. 1, pp.1–35.

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1 Introduction

Difference equations are pervasive in mathematics and understanding the behaviour of such equations gives insight to many interesting problems, see Din et al. (2012), Elsayed et al. (2013), Elsayed and Eleissawy(2012) and Ibrahim andTouafek (2014). Max-type difference equations, which appeared for the first time in control theory, have attracted extensively attention recently (Qin et al., 2012; Xiao and Shi, 2013; Touafek and Haddad, 2015; Yazlik et al., 2015; Ibrahim and Touafek, 2014; Ibrahim, 2016). Ibrahim (2016) examined the periodicity and formularisation of the solutions for a system of semi-max-type difference equations of second order in the form

$$x_{n+1} = \max\left\{\frac{A_n}{y_{n-1}}, x_{n-1}\right\},$$
$$y_{n+1} = \min\left\{\frac{B_n}{x_{n-1}}, y_{n-1}\right\}, \tag{1}$$

where $n \in \mathbb{N}_0,\ \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(A_n)_{n\in\mathbb{N}_0},\ (B_n)_{n\in\mathbb{N}_0}$ are two-periodic positive sequences, and initial values $x_0,\ x_{-1},\ y_0,\ y_{-1} \in (0,\ +\infty)$. Williams (2016) has investigated the general solutions and periodic solutions of the following max-type difference equation system

$$x_{n+1} = \max\left\{y_{n-1}^2, \frac{A}{y_{n-1}}\right\},$$
$$y_{n+1} = \max\left\{x_{n-1}^2, \frac{A}{x_{n-1}}\right\}, \tag{2}$$

where $n \in \mathbb{N}_0, x_{-1} = \alpha, y_{-1} = \beta, x_0 = \lambda$ and $y_0 = \mu$ are constants and $A > 0$.

In this paper, we study the eventually periodicity of the following max-type difference equation system

$$x_{n+1} = \max\left\{\frac{A}{x_n y_{n-1}}, x_{n-2}\right\},$$
$$y_{n+1} = \max\left\{\frac{A}{y_n x_{n-1}}, y_{n-2}\right\}, \tag{3}$$

where $n \in \mathbb{N}$, $A \in \mathbb{R} \setminus \{0\}$, and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary non-zero numbers.

2 Preliminaries

Firstly, we give two definitions.

Definition 1: The sequence $\{x_n, y_n\}_{n=-k}^{\infty}$ is eventually periodic with period p if there is an $n_0 \in \{-k, \ \cdots, \ -1, \ 0, \ 1, \ \cdots\}$ such that for all $n \geq n_0$,

$$x_{n+p} = x_n, \ \ y_{n+p} = y_n.$$

Definition 2: The sequence $\{x_n, y_n\}_{n=-k}^{\infty}$ is eventually positive (negative) if there is an $n_0 \in \{-k, \ \cdots, \ -1, \ 0, \ 1, \ \cdots\}$ such that for all $n \geq n_0$,

$$x_n > (<)0, \ \ y_n > (<)0.$$

In order to get the eventually periodic solutions of (3), the following lemma is needed.

Lemma 1: *Assume that* $\{x_n, y_n\}_{n=-2}^{\infty}$ *is a solution of (3) and there is* $k_0 \in \mathbb{N}_0 \cup \{-2, -1\}$ *such that*

$$x_{k_0} = x_{k_0+3}, \ \ x_{k_0+1} = x_{k_0+4}, \ \ x_{k_0+2} = x_{k_0+5}, \tag{4}$$

$$y_{k_0} = y_{k_0+3}, \ \ y_{k_0+1} = y_{k_0+4}, \ \ y_{k_0+2} = y_{k_0+5}, \tag{5}$$

then this solution is eventually periodic with period three.

Proof: To prove this lemma, we just need to prove that the following equations are true.

$$x_{k_0} = x_{k_0+3m}, \ \ x_{k_0+1} = x_{k_0+1+3m}, \ \ x_{k_0+2} = x_{k_0+2+3m}, \tag{6}$$

$$y_{k_0} = y_{k_0+3m}, \ \ y_{k_0+1} = y_{k_0+1+3m}, \ \ y_{k_0+2} = y_{k_0+2+3m}, \tag{7}$$

for every $m \in \mathbb{N}$.

We use the method of induction. For $m = 1$, (6) and (7) become (4) and (5), so the result holds. Assume that (4) and (5) hold for $1 \leq m \leq m_0$, by using (3)-(7), we have

$$\begin{aligned}
x_{k_0+3(m_0+1)} &= \max\left\{\frac{A}{x_{k_0+3m_0+2}y_{k_0+3m_0+1}}, x_{k_0+3m_0}\right\} \\
&= \max\left\{\frac{A}{x_{k_0+2}y_{k_0+1}}, x_{k_0}\right\} = x_{k_0+3} = x_{k_0}, \\
y_{k_0+3(m_0+1)} &= \max\left\{\frac{A}{y_{k_0+3m_0+2}x_{k_0+3m_0+1}}, y_{k_0+3m_0}\right\} \\
&= \max\left\{\frac{A}{y_{k_0+2}x_{k_0+1}}, y_{k_0}\right\} = y_{k_0+3} = y_{k_0}, \\
x_{k_0+1+3(m_0+1)} &= \max\left\{\frac{A}{x_{k_0+3m_0+3}y_{k_0+3m_0+2}}, x_{k_0+3m_0+1}\right\} \\
&= \max\left\{\frac{A}{x_{k_0+3}y_{k_0+2}}, x_{k_0+1}\right\} = x_{k_0+4} = x_{k_0+1}, \\
y_{k_0+1+3(m_0+1)} &= \max\left\{\frac{A}{y_{k_0+3m_0+3}x_{k_0+3m_0+2}}, y_{k_0+3m_0+1}\right\}
\end{aligned}$$

$$= \max\left\{\frac{A}{y_{k_0+3}x_{k_0+2}}, y_{k_0+1}\right\} = y_{k_0+4} = y_{k_0+1},$$

$$x_{k_0+2+3(m_0+1)} = \max\left\{\frac{A}{x_{k_0+3m_0+4}y_{k_0+3m_0+3}}, x_{k_0+3m_0+2}\right\}$$

$$= \max\left\{\frac{A}{x_{k_0+4}y_{k_0+3}}, x_{k_0+2}\right\} = x_{k_0+4} = x_{k_0+2},$$

$$y_{k_0+2+3(m_0+1)} = \max\left\{\frac{A}{y_{k_0+3m_0+4}x_{k_0+3m_0+3}}, y_{k_0+3m_0+2}\right\}$$

$$= \max\left\{\frac{A}{y_{k_0+4}x_{k_0+3}}, y_{k_0+2}\right\} = y_{k_0+4} = y_{k_0+2}.$$

For the sake of argument, we will give the initial values for three different situations as the following.

$(H1)$ All of the initials values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are negative;

$(H2)$ All of the initials values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are positive;

$(H3)$ At least one of the initials values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ is greater than zero and at least one of the initial values is less than zero.

3 Periodic solutions of (3) for the case $A > 0$

In this section, we will discuss the eventually periodic solutions of (3) for the case $A > 0$.

Theorem 1: *Suppose that $A > 0$ and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ satisfy (H1), then every solution of (3) is eventually periodic with period 3.*

Proof: Since $A > 0$ and $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 < 0$, then

$$x_1 = \max\left\{\frac{A}{x_0y_{-1}}, x_{-2}\right\} = \frac{A}{x_0y_{-1}}, \quad y_1 = \max\left\{\frac{A}{y_0x_{-1}}, y_{-2}\right\} = \frac{A}{y_0x_{-1}}.$$

(I) Suppose that $x_0y_{-1} \geq y_0x_{-1}$, and

(i) If $\frac{x_0}{y_0} \geq 1$, then

$$x_2 = \max\left\{\frac{x_0y_{-1}}{y_0}, x_{-1}\right\} = x_{-1}, \quad y_2 = \max\left\{\frac{y_0x_{-1}}{x_0}, y_{-1}\right\} = \frac{y_0x_{-1}}{x_0};$$

$$x_3 = \max\{y_0, x_0\} = y_0, \quad y_3 = \max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}}, y_0\right\} = y_0;$$

$$x_4 = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{x_0 A}{y_0^2 x_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\left\{\frac{y_0 x_{-1}}{x_0}, x_{-1}\right\} = \frac{y_0 x_{-1}}{x_0},\ \ y_5 = \max\left\{x_{-1}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_6 = \max\{x_0, y_0\} = y_0,\ \ y_6 = \max\{y_0, y_0\} = y_0;$$
$$x_7 = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{x_0 A}{y_0^2 x_{-1}}\right\} = \frac{x_0 A}{y_0^2 x_{-1}},\ \ y_7 = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{x_0 A}{y_0^2 x_{-1}};$$
$$x_8 = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0},\ \ y_8 = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_9 = \max\{y_0, y_0\} = y_0,\ \ y_9 = \max\{y_0, y_0\} = y_0;$$
$$x_{10} = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{x_0 A}{y_0^2 x_{-1}}\right\} = \frac{x_0 A}{y_0^2 x_{-1}},\ \ y_{10} = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{x_0 A}{y_0^2 x_{-1}}\right\} = \frac{x_0 A}{y_0^2 x_{-1}}.$$

Hence $x_5 = y_5 = x_8 = y_8,\ x_6 = y_6 = x_9 = y_9,\ x_7 = y_7 = x_{10} = y_{10}$, by Lemma 1 and induction method, the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = \frac{y_0 x_{-1}}{x_0};\ \ x_{3n} = y_{3n} = y_0;\ \ x_{3n+1} = y_{3n+1} = \frac{x_0 A}{y_0^2 x_{-1}},\ \ n = 2, 3, \ldots.$$

(ii) If $0 < \frac{x_0}{y_0} < 1$, and

(a) $\frac{x_0^2 y_{-1}}{y_0 x_{-1}} \le y_0$, then

$$x_3 = \max\{y_0, x_0\} = x_0,\ \ y_3 = \max\left\{\frac{x_0^2 y_{-1}}{y_0 x_{-1}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{y_0 x_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\{x_{-1}, x_{-1}\} = x_{-1},\ \ y_5 = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_6 = \max\{y_0, x_0\} = x_0,\ \ y_6 = \max\{x_0, y_0\} = x_0;$$
$$x_7 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}},\ \ y_7 = \max\left\{\frac{A}{x_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{x_0 x_{-1}};$$
$$x_8 = \max\left\{\frac{y_0 x_{-1}}{x_0}, x_{-1}\right\} = x_{-1},\ \ y_8 = \max\left\{x_{-1}, \frac{y_0 x_{-1}}{x_0}\right\} = x_{-1};$$
$$x_9 = \max\{x_0, x_0\} = x_0,\ \ y_9 = \max\{y_0, x_0\} = x_0;$$
$$x_{10} = \max\left\{\frac{A}{x_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{x_0 x_{-1}},\ \ y_{10} = \max\left\{\frac{A}{x_0 x_{-1}}, \frac{A}{x_0 x_{-1}}\right\} = \frac{A}{x_0 x_{-1}};$$

$$x_{11} = \max\{x_{-1}, x_{-1}\} = x_{-1}, \;\; y_{11} = \max\{x_{-1}, x_{-1}\} = x_{-1};$$
$$x_{12} = \max\{x_0, x_0\} = x_0, \;\; y_{12} = \max\{x_0, x_0\} = x_0;$$
$$x_{13} = \max\left\{\frac{A}{x_0x_{-1}}, \frac{A}{x_0x_{-1}}\right\} = \frac{A}{x_0x_{-1}}, \;\; y_{13} = \max\left\{\frac{A}{x_0x_{-1}}, \frac{A}{x_0x_{-1}}\right\} = \frac{A}{x_0x_{-1}}.$$

Hence $x_8 = y_8 = x_{11} = y_{11}$, $x_9 = y_9 = x_{12} = y_{12}$, $x_{10} = y_{10} = x_{13} = y_{13}$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = x_{-1}; \;\; x_{3n} = y_{3n} = x_0; \;\; x_{3n+1} = y_{3n+1} = \frac{A}{x_0x_{-1}}, \;\; n = 3, 4, \ldots.$$

(b) $\frac{x_0^2y_{-1}}{y_0x_{-1}} \geq y_0$, then

$$x_3 = \max\{y_0, x_0\} = x_0, \;\; y_3 = \max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}}, y_0\right\} = \frac{x_0^2y_{-1}}{y_0x_{-1}};$$
$$x_4 = \max\left\{\frac{A}{y_0x_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{y_0x_{-1}}, \;\; y_4 = \max\left\{\frac{y_0A}{x_0^2y_{-1}}, \frac{A}{y_0x_{-1}}\right\} = \frac{y_0A}{x_0^2y_{-1}};$$
$$x_5 = \max\left\{\frac{y_0^2x_{-1}^2}{x_0^2y_{-1}}, x_{-1}\right\} = x_{-1}, \;\; y_5 = \max\left\{\frac{x_0y_{-1}}{y_0}, \frac{y_0x_{-1}}{x_0}\right\} = \frac{x_0y_{-1}}{y_0};$$
$$x_6 = \max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}}, x_0\right\} = x_0, \;\; y_6 = \max\left\{\frac{y_0^2x_{-1}}{x_0y_{-1}}, \frac{x_0^2y_{-1}}{y_0x_{-1}}\right\} = \frac{y_0^2x_{-1}}{x_0y_{-1}};$$
$$x_7 = \max\left\{\frac{y_0A}{x_0^2y_{-1}}, \frac{A}{y_0x_{-1}}\right\} = \frac{y_0A}{x_0^2y_{-1}}, \;\; y_7 = \max\left\{\frac{x_0y_{-1}A}{y_0^2x_{-1}^2}, \frac{y_0A}{x_0^2y_{-1}}\right\} = \frac{x_0y_{-1}A}{y_0^2x_{-1}^2};$$
$$x_8 = \max\left\{\frac{x_0^3y_{-1}^2}{y_0^3x_{-1}}, x_{-1}\right\} = x_{-1}, \;\; y_8 = \max\left\{\frac{y_0^2x_{-1}^2}{x_0^2y_{-1}}, \frac{x_0y_{-1}}{y_0}\right\} = \frac{y_0^2x_{-1}^2}{x_0^2y_{-1}};$$
$$x_9 = \max\left\{\frac{y_0^2x_{-1}}{x_0y_{-1}}, x_0\right\} = x_0, \;\; y_9 = \max\left\{\frac{x_0^4y_{-1}^2}{y_0^3x_{-1}^2}, \frac{y_0^2x_{-1}}{x_0y_{-1}}\right\} = \frac{y_0^2x_{-1}}{x_0y_{-1}};$$
$$x_{10} = \max\left\{\frac{x_0y_{-1}A}{y_0^2x_{-1}^2}, \frac{y_0A}{x_0^2y_{-1}}\right\} = \frac{x_0y_{-1}A}{y_0^2x_{-1}^2}, \;\; y_{10} = \max\left\{\frac{x_0y_{-1}A}{y_0^2x_{-1}^2}, \frac{x_0y_{-1}A}{y_0^2x_{-1}^2}\right\} = \frac{x_0y_{-1}A}{y_0^2x_{-1}^2};$$
$$x_{11} = \max\{x_{-1}, x_{-1}\} = x_{-1}, \;\; y_{11} = \max\left\{\frac{y_0^2x_{-1}^2}{x_0^2y_{-1}}, \frac{y_0^2x_{-1}^2}{x_0^2y_{-1}}\right\} = \frac{y_0^2x_{-1}^2}{x_0^2y_{-1}};$$
$$x_{12} = \max\left\{\frac{y_0^2x_{-1}}{x_0y_{-1}}, x_0\right\} = x_0, \;\; y_{12} = \max\left\{x_0, \frac{y_0^2x_{-1}}{x_0y_{-1}}\right\} = x_0;$$
$$x_{13} = \max\left\{\frac{x_0y_{-1}A}{y_0^2x_{-1}^2}, \frac{x_0y_{-1}A}{y_0^2x_{-1}^2}\right\} = \frac{x_0y_{-1}A}{y_0^2x_{-1}^2}, \;\; y_{13} = \max\left\{\frac{A}{x_0x_{-1}}, \frac{x_0y_{-1}A}{y_0^2x_{-1}^2}\right\} = \frac{A}{x_0x_{-1}};$$
$$x_{14} = \max\left\{\frac{y_0^2x_{-1}^2}{x_0^2y_{-1}}, x_{-1}\right\} = x_{-1}, \;\; y_{14} = \max\left\{x_{-1}, \frac{y_0^2x_{-1}^2}{x_0^2y_{-1}}\right\} = x_{-1};$$
$$x_{15} = \max\{x_0, x_0\} = x_0, \;\; y_{15} = \max\left\{\frac{y_0^2x_{-1}}{x_0y_{-1}}, x_0\right\} = x_0;$$
$$x_{16} = \max\left\{\frac{A}{x_0x_{-1}}, \frac{x_0y_{-1}A}{y_0^2x_{-1}^2}\right\} = \frac{A}{x_0x_{-1}}, \;\; y_{16} = \max\left\{\frac{A}{x_0x_{-1}}, \frac{A}{x_0x_{-1}}\right\} = \frac{A}{x_0x_{-1}};$$
$$x_{17} = \max\{x_{-1}, x_{-1}\} = x_{-1}, \;\; y_{17} = \max\{x_{-1}, x_{-1}\} = x_{-1};$$
$$x_{18} = \max\{x_0, x_0\} = x_0, \;\; y_{18} = \max\{x_0, x_0\} = x_0;$$
$$x_{19} = \max\left\{\frac{A}{x_0x_{-1}}, \frac{A}{x_0x_{-1}}\right\} = \frac{A}{x_0x_{-1}}, \;\; y_{19} = \max\left\{\frac{A}{x_0x_{-1}}, \frac{A}{x_0x_{-1}}\right\} = \frac{A}{x_0x_{-1}}.$$

Hence $x_{14} = y_{14} = x_{17} = y_{17},\ \ x_{15} = y_{15} = x_{18} = y_{18},\ \ x_{16} = y_{16} = x_{19} = y_{19},$ by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = x_{-1};\ \ x_{3n} = y_{3n} = x_0;\ \ x_{3n+1} = y_{3n+1} = \frac{A}{x_0 x_{-1}},\ \ n = 5, 6, \ldots.$$

(II) Suppose that $x_0 y_{-1} \le y_0 x_{-1}$, the proof is similar to case 1, so we just give the result.

(i) If $\frac{x_0}{y_0} \ge 1$, and
(a) $x_0 \ge \frac{y_0^2 x_{-1}}{x_0 y_{-1}}$, then the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = y_{-1};\ \ x_{3n} = y_{3n} = y_0;\ \ x_{3n+1} = y_{3n+1} = \frac{A}{y_0 y_{-1}},\ \ n = 3, 4, \ldots.$$

(b) $x_0 \le \frac{y_0^2 x_{-1}}{x_0 y_{-1}}$, then the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = y_{-1};\ \ x_{3n} = y_{3n} = y_0;\ \ x_{3n+1} = y_{3n+1} = \frac{A}{y_0 y_{-1}},\ \ n = 5, 6, \ldots.$$

(ii) If $0 < \frac{x_0}{y_0} < 1$, then the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n} = y_{3n} = x_0;\ \ x_{3n+1} = y_{3n+1} = \frac{y_0 A}{x_0^2 y_{-1}},\ \ n = 2, 3, \ldots.$$

Remark 1: $A > 0$ and (H1) imply that every solution of (3) is eventually sign-changing.

Theorem 2: *Suppose that $A > 0$ and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ satisfy (H2), then every solution of (3) is eventually periodic with period three.*

Proof: As in the proof of Theorem 1, there are several cases which should be discussed because of the maximum property in system (3). While due to the similarity of the proof and the space limitations, here we just show the results of some cases.

(I) Assume that $y_{-2} \ge \frac{A}{y_0 x_{-1}} \ge \frac{A}{x_0 y_{-1}} \ge x_{-2}$, and

(i) If $\frac{x_0}{y_0} \ge 1$, then the solution is eventually periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n} = x_0;\ \ y_{3n-2} = y_{-2};\ \ y_{3n-1} = y_{-1};$$
$$y_{3n} = x_0,\ \ n = 1, 2, \ldots.$$

(ii) If $0 < \frac{x_0}{y_0} < 1$, and
(a) $x_0 \ge \frac{y_0 A}{x_0 y_{-1} y_{-2}}$, then the solution is eventually periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n} = x_0;$$

$$y_{3n-2} = y_{-2};\ \ y_{3n-1} = y_{-1};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(b) $x_0 \leq \frac{y_0 A}{x_0 y_{-1} y_{-2}}$, then the solution is eventually periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n} = \frac{y_0 A}{x_0 y_{-1} y_{-2}};$$
$$y_{3n-2} = y_{-2};\ \ y_{3n-1} = y_{-1};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(II) Assume that $\frac{A}{y_0 x_{-1}} \geq \frac{A}{x_0 y_{-1}} \geq x_{-2} \geq y_{-2}$, and

(i) If $\frac{x_0}{y_0} \geq 1$, then the solution is eventually periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n} = x_0;$$
$$y_{3n-2} = \frac{A}{y_0 x_{-1}};\ \ y_{3n-1} = y_{-1};\ \ y_{3n} = x_0,\ \ n = 1, 2, \ldots.$$

(ii) If $0 < \frac{x_0}{y_0} < 1$, and

(a) $x_0 \geq \frac{y_0^2 x_{-1} A}{x_0 y_{-1}}$, then the solution is eventually periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n} = x_0;$$
$$y_{3n-2} = \frac{A}{y_0 x_{-1}};\ \ y_{3n-1} = y_{-1};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(b) $x_0 \leq \frac{y_0^2 x_{-1} A}{x_0 y_{-1}}$, then the solution is eventually periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n} = \frac{y_0^2 x_{-1}}{x_0 y_{-1}};$$
$$y_{3n-2} = \frac{A}{y_0 x_{-1}};\ \ y_{3n-1} = y_{-1};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

Remark 2: $A > 0$ and (H2) imply that every solution of (3) is positive.

Theorem 3: *Suppose that $A > 0$ and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ satisfy (H3), then every solution of (3) is eventually periodic with period three.*

Proof: (I) suppose that $x_{-1}, y_{-2} > 0$, $x_{-2}, x_0, y_{-1}, y_0 < 0$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$

(i) if $\frac{A}{y_{-2} x_0} \geq y_{-1}$, then

$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1},\ \ y_2 = \max\left\{\frac{A}{y_{-2} x_0}, y_{-1}\right\} = \frac{A}{y_{-2} x_0};$$

(a) $\frac{x_0^2 y_{-1} y_{-2}}{A} \geq y_0$, we have

$$x_3 = \max\left\{\frac{A}{x_{-1}y_{-2}}, x_0\right\} = \frac{A}{x_{-1}y_{-2}},\ \ y_3 = \max\left\{\frac{x_0^2 y_{-1}y_{-2}}{A}, y_0\right\} = \frac{x_0^2 y_{-1}y_{-2}}{A};$$
$$x_4 = \max\left\{\frac{y_{-2}^2 x_{-1}x_0}{A}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A^2}{x_0^2 y_{-1}y_{-2}x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_5 = \max\left\{\frac{A}{x_0 y_{-2}}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{x_{-1}, \frac{A}{y_{-2}x_0}\right\} = x_{-1};$$
$$x_6 = \max\left\{\frac{A}{x_{-1}y_{-2}}, \frac{A}{x_{-1}y_{-2}}\right\} = \frac{A}{x_{-1}y_{-2}},\ \ y_6 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, \frac{x_0^2 y_{-1}y_{-2}}{A}\right\} = \frac{x_0 y_{-1}}{x_{-1}};$$
$$x_7 = \max\left\{y_{-2}, \frac{A}{x_0 y_{-1}}\right\} = y_{-2},\ \ y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_8 = \max\left\{\frac{x_{-1}A}{x_0 y_{-1}y_{-2}}, x_{-1}\right\} = x_{-1},\ \ y_8 = \max\{x_{-1}, x_{-1}\} = x_{-1};$$
$$x_9 = \max\left\{\frac{A}{x_{-1}y_{-2}}, \frac{A}{x_{-1}y_{-2}}\right\} = \frac{A}{x_{-1}y_{-2}},\ \ y_9 = \max\left\{\frac{A}{x_{-1}y_{-2}}, \frac{x_0 y_{-1}}{x_{-1}}\right\} = \frac{x_0 y_{-1}}{x_{-1}};$$
$$x_{10} = \max\{y_{-2}, y_{-2}\} = y_{-2},\ \ y_{10} = \max\left\{\frac{A}{x_0 y_{-1}}, y_{-2}\right\} = y_{-2};$$

Hence $x_5 = x_8,\ x_6 = x_9,\ x_7 = x_{10},\ y_5 = y_8,\ y_6 = y_9,\ y_7 = y_{10}$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n-1} = x_{-1};\ \ x_{3n} = \frac{A}{x_{-1}y_{-2}};\ \ x_{3n+1} = y_{-2};$$
$$y_{3n-1} = x_{-1};\ \ y_{3n} = \frac{x_0 y_{-1}}{x_{-1}};\ \ y_{3n+1} = y_{-2},\ \ n = 2, 3, \ldots.$$

(b) $\frac{x_0^2 y_{-1} y_{-2}}{A} \leq y_0$, the results are the same as (a).

(ii) if $\frac{A}{y_{-2}x_0} \leq y_{-1}$, then

$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1},\ \ y_2 = \max\left\{\frac{A}{y_{-2}x_0}, y_{-1}\right\} = y_{-1};$$

(a) $x_0 \geq y_0$, we have

$$x_3 = \max\left\{\frac{A}{x_{-1}y_{-2}}, x_0\right\} = \frac{A}{x_{-1}y_{-2}},\ \ y_3 = \max\{x_0, y_0\} = x_0;$$
$$x_4 = \max\left\{\frac{x_{-1}y_{-2}}{y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A}{x_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_5 = \max\{y_{-1}, x_{-1}\} = x_{-1},\ \ y_5 = \max\{x_{-1}, y_{-1}\} = x_{-1};$$
$$x_6 = \max\left\{\frac{A}{x_{-1}y_{-2}}, \frac{A}{x_{-1}y_{-2}}\right\} = \frac{A}{x_{-1}y_{-2}},\ \ y_6 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, x_0\right\} = \frac{x_0 y_{-1}}{x_{-1}};$$
$$x_7 = \max\left\{y_{-2}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, y_{-2}\right\} = \frac{A}{x_0 y_{-1}};$$
$$x_8 = \max\{x_{-1}, x_{-1}\} = x_{-1},\ \ y_8 = \max\left\{\frac{x_0 x_{-1}y_{-1}y_{-2}}{A}, x_{-1}\right\} = x_{-1};$$

$$x_9=\max\left\{\frac{x_0y_{-1}}{x_{-1}},\frac{A}{x_{-1}y_{-2}}\right\}=\frac{A}{x_{-1}y_{-2}},\quad y_9=\max\left\{\frac{x_0y_{-1}}{x_{-1}},\frac{x_0y_{-1}}{x_{-1}}\right\}=\frac{x_0y_{-1}}{x_{-1}};$$
$$x_{10}=\max\left\{y_{-2},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\quad y_{10}=\max\left\{\frac{A}{x_0y_{-1}},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}};$$

Hence $x_5=x_8,\ x_6=x_9,\ x_7=x_{10},\ y_5=y_8,\ y_6=y_9,\ y_7=y_{10},$ by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n-1}=x_{-1};\ x_{3n}=\frac{A}{x_{-1}y_{-2}};\ x_{3n+1}=\frac{A}{x_0y_{-1}};$$
$$y_{3n-1}=x_{-1};\ y_{3n}=\frac{x_0y_{-1}}{x_{-1}};\ y_{3n+1}=\frac{A}{x_0y_{-1}},\quad n=2,3,\ldots.$$

(b) $x_0\le y_0$, the results are the same as (a).

(II) Suppose that $y_0>0, x_{-2},x_{-1},x_0,y_{-2},y_{-1}<0$, and

(i) if $\frac{A}{y_0x_{-1}}\ge y_{-2}$, then

$$x_1=\max\left\{\frac{A}{x_0y_{-1}},x_{-2}\right\}=\frac{A}{x_0y_{-1}},\quad y_1=\max\left\{\frac{A}{y_0x_{-1}},y_{-2}\right\}=\frac{A}{y_0x_{-1}};$$
$$x_2=\max\left\{\frac{x_0y_{-1}}{y_0},x_{-1}\right\}=\frac{x_0y_{-1}}{y_0},\quad y_2=\max\left\{\frac{y_0x_{-1}}{x_0},y_{-1}\right\}=\frac{y_0x_{-1}}{x_0};$$

(a) $\frac{y_0^2x_{-1}}{x_0y_{-1}}\ge x_0$, we have

$$x_3=\max\left\{\frac{y_0^2x_{-1}}{x_0y_{-1}},x_0\right\}=\frac{y_0^2x_{-1}}{x_0y_{-1}},\quad y_3=\max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}},y_0\right\}=\frac{x_0^2y_{-1}}{y_0x_{-1}};$$
$$x_4=\max\left\{\frac{x_0^2y_{-1}A}{y_0^3x_{-1}^2},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\quad y_4=\max\left\{\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2},\frac{A}{y_0x_{-1}}\right\}=\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2};$$
$$x_5=\max\left\{\frac{y_0x_{-1}}{x_0},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\quad y_5=\max\left\{\frac{x_0^4y_{-1}^3}{y_0^4x_{-1}^2},\frac{y_0x_{-1}}{x_0}\right\}=\frac{y_0x_{-1}}{x_0};$$
$$x_6=\max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}},\frac{y_0^2x_{-1}}{x_0y_{-1}}\right\}=\frac{x_0^2y_{-1}}{y_0x_{-1}},\quad y_6=\max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}},\frac{x_0^2y_{-1}}{y_0x_{-1}}\right\}=\frac{x_0^2y_{-1}}{y_0x_{-1}};$$
$$x_7=\max\left\{\frac{A}{x_0y_{-1}},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\quad y_7=\max\left\{\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2},\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2}\right\}=\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2};$$
$$x_8=\max\left\{\frac{y_0x_{-1}}{x_0},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\quad y_8=\max\left\{\frac{x_0y_{-1}}{y_0},\frac{y_0x_{-1}}{x_0}\right\}=\frac{x_0y_{-1}}{y_0};$$
$$x_9=\max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}},\frac{x_0^2y_{-1}}{y_0x_{-1}}\right\}=\frac{x_0^2y_{-1}}{y_0x_{-1}},\quad y_9=\max\left\{y_0,\frac{x_0^2y_{-1}}{y_0x_{-1}}\right\}=\frac{x_0^2y_{-1}}{y_0x_{-1}};$$
$$x_{10}=\max\left\{\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\quad y_{10}=\max\left\{\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2},\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2}\right\}=\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2};$$
$$x_{11}=\max\left\{\frac{y_0x_{-1}}{x_0},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\quad y_{11}=\max\left\{\frac{x_0y_{-1}}{y_0},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0};$$

Hence $x_6 = x_9,\ x_7 = x_{10},\ x_8 = x_{11},\ y_6 = y_9,\ y_7 = y_{10},\ y_9 = y_{11}$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = \frac{x_0^2 y_{-1}}{y_0 x_{-1}};\ \ x_{3n+1} = \frac{A}{x_0 y_{-1}};\ \ x_{3n+2} = \frac{x_0 y_{-1}}{y_0};$$
$$y_{3n} = \frac{x_0^2 y_{-1}}{y_0 x_{-1}};\ \ y_{3n+1} = \frac{y_0^2 x_{-1} A}{x_0^3 y_{-1}^2};\ \ y_{3n+2} = \frac{x_0 y_{-1}}{y_0},\ \ n = 2, 3, \ldots.$$

(b) $\frac{y_0^2 x_{-1}}{x_0 y_{-1}} \leq x_0$, we have

$$x_3 = \max\left\{\frac{y_0^2 x_{-1}}{x_0 y_{-1}}, x_0\right\} = x_0,\ \ y_3 = \max\left\{\frac{x_0^2 y_{-1}}{y_0 x_{-1}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{x_0 y_{-1}};$$
$$x_5 = \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0},\ \ y_5 = \max\left\{y_{-1}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_6 = \max\{y_0, x_0\} = y_0,\ \ y_6 = \max\left\{\frac{x_0^2 y_{-1}}{y_0 x_{-1}}, y_0\right\} = y_0;$$
$$x_7 = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}};$$
$$x_8 = \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0},\ \ y_8 = \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_9 = \max\{y_0, y_0\} = y_0,\ \ y_9 = \max\left\{\frac{x_0^2 y_{-1}}{y_0 x_{-1}}, y_0\right\} = y_0;$$

Hence $x_4 = x_7,\ x_5 = x_8,\ x_6 = x_9,\ y_4 = y_7,\ y_5 = y_8,\ y_6 = y_9$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n+1} = \frac{A}{x_0 y_{-1}};\ \ x_{3n+2} = \frac{x_0 y_{-1}}{y_0};\ \ x_{3n+3} = y_0;$$
$$y_{3n+1} = \frac{A}{x_0 y_{-1}};\ \ y_{3n+2} = \frac{y_0 x_{-1}}{x_0};\ \ y_{3n+3} = y_0,\ \ n = 1, 2, \ldots.$$

(ii) if $\frac{A}{y_0 x_{-1}} \leq y_{-2}$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = \frac{x_0 y_{-1}}{y_0},\ \ y_2 = \max\left\{\frac{A}{y_{-2} x_0}, y_{-1}\right\} = \frac{A}{y_{-2} x_0};$$

(a) $\frac{y_0 A}{x_0 y_{-1} y_{-2}} \geq x_0$, we have

$$x_3 = \max\left\{\frac{y_0 A}{x_0 y_{-1} y_{-2}}, x_0\right\} = \frac{y_0 A}{x_0 y_{-1} y_{-2}},\ \ y_3 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = \frac{x_0^2 y_{-1} y_{-2}}{A};$$

$$x_4=\max\left\{\frac{x_0^2y_{-1}y_{-2}^2}{y_0A},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\ \ y_4=\max\left\{\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}},y_{-2}\right\}=\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}};$$
$$x_5=\max\left\{\frac{A}{x_0y_{-2}},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\ \ y_5=\max\left\{\frac{x_0^4y_{-1}^3y_{-2}^2}{y_0^2A^2},\frac{A}{y_{-2}x_0}\right\}=\frac{A}{y_{-2}x_0};$$
$$x_6=\max\left\{\frac{x_0^2y_{-1}y_{-2}}{A},\frac{y_0A}{x_0y_{-1}y_{-2}}\right\}=\frac{x_0^2y_{-1}y_{-2}}{A},$$
$$y_6=\max\left\{\frac{x_0^2y_{-1}y_{-2}}{A},\frac{x_0^2y_{-1}y_{-2}}{A}\right\}=\frac{x_0^2y_{-1}y_{-2}}{A};$$
$$x_7=\max\left\{\frac{A}{x_0y_{-1}},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\ \ y_7=\max\left\{\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}},\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}}\right\}=\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}};$$
$$x_8=\max\left\{\frac{A}{x_0y_{-2}},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\ \ y_8=\max\left\{\frac{x_0y_{-1}}{y_0},\frac{A}{x_0y_{-2}}\right\}=\frac{x_0y_{-1}}{y_0};$$
$$x_9=\max\left\{\frac{x_0^2y_{-1}y_{-2}}{A},\frac{x_0^2y_{-1}y_{-2}}{A}\right\}=\frac{x_0^2y_{-1}y_{-2}}{A},$$
$$y_9=\max\left\{y_0,\frac{x_0^2y_{-1}y_{-2}}{A}\right\}=\frac{x_0^2y_{-1}y_{-2}}{A};$$
$$x_{10}=\max\left\{\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},$$
$$y_{10}=\max\left\{\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}},\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}}\right\}=\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}};$$
$$x_{11}=\max\left\{\frac{A}{x_0y_{-2}},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\ \ y_{11}=\max\left\{\frac{x_0y_{-1}}{y_0},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0};$$

Hence $x_6=x_9,\ x_7=x_{10},\ x_8=x_{11},\ y_6=y_9,\ y_7=y_{10},\ y_8=y_{11},$ by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n}=\frac{x_0^2y_{-1}y_{-2}}{A};\ \ x_{3n+1}=\frac{A}{x_0y_{-1}};\ \ x_{3n+2}=\frac{x_0y_{-1}}{y_0};$$
$$y_{3n}=\frac{x_0^2y_{-1}y_{-2}}{A};\ \ y_{3n+1}=\frac{y_0A^2}{x_0^3y_{-1}^2y_{-2}};\ \ y_{3n+2}=\frac{x_0y_{-1}}{y_0},\ \ n=2,3,\ldots.$$

(b) $\frac{y_0A}{x_0y_{-1}y_{-2}}\leq x_0$, we have

$$x_3=\max\left\{\frac{y_0A}{x_0y_{-1}y_{-2}},x_0\right\}=x_0,\ \ y_3=\max\left\{\frac{x_0^2y_{-1}y_{-2}}{A},y_0\right\}=y_0;$$
$$x_4=\max\left\{y_{-2},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\ \ y_4=\max\left\{\frac{A}{x_0y_{-1}},y_{-2}\right\}=\frac{A}{x_0y_{-1}};$$
$$x_5=\max\left\{\frac{x_0y_{-1}}{y_0},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\ \ y_5=\max\left\{y_{-1},\frac{A}{y_{-2}x_0}\right\}=\frac{A}{y_{-2}x_0};$$
$$x_6=\max\{y_0,x_0\}=y_0,\ \ y_6=\max\left\{\frac{x_0^2y_{-1}y_{-2}}{A},y_0\right\}=y_0;$$
$$x_7=\max\left\{\frac{y_{-2}x_0}{y_0},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}},\ \ y_7=\max\left\{\frac{A}{x_0y_{-1}},\frac{A}{x_0y_{-1}}\right\}=\frac{A}{x_0y_{-1}};$$
$$x_8=\max\left\{\frac{x_0y_{-1}}{y_0},\frac{x_0y_{-1}}{y_0}\right\}=\frac{x_0y_{-1}}{y_0},\ \ y_8=\max\left\{\frac{x_0y_{-1}}{y_0},\frac{A}{x_0y_{-2}}\right\}=\frac{A}{x_0y_{-2}};$$

$$x_9 = \max\{y_0, y_0\} = y_0, \ \ y_9 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = y_0;$$

Hence $x_4 = x_7,\ x_5 = x_8,\ x_6 = x_9,\ y_4 = y_7,\ y_5 = y_8,\ y_6 = y_9$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n+1} = \frac{A}{x_0 y_{-1}}; \ \ x_{3n+2} = \frac{x_0 y_{-1}}{y_0}; \ \ x_{3n+3} = y_0;$$
$$y_{3n+1} = \frac{A}{x_0 y_{-1}}; \ \ y_{3n+2} = \frac{A}{x_0 y_{-2}}; \ \ y_{3n+3} = y_0, \ \ n = 1, 2, \ldots.$$

(III) Suppose that $x_{-1}, x_0, y_{-1} > 0$, $x_{-2}, y_{-2}, y_0 < 0$, and

(i) if $\frac{A}{y_0 x_{-1}} \geq y_{-2}$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1}, \ \ y_2 = \max\left\{\frac{y_0 x_{-1}}{x_0}, y_{-1}\right\} = y_{-1};$$
$$x_3 = \max\{y_0, x_0\} = x_0, \ \ y_3 = \max\{x_0, y_0\} = x_0;$$
$$x_4 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_4 = \max\left\{\frac{A}{x_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{x_0 x_{-1}};$$

(a) $x_{-1} \geq y_{-1}$, we have

$$x_5 = \max\{y_{-1}, x_{-1}\} = x_{-1}, \ \ y_5 = \max\{x_{-1}, y_{-1}\} = x_{-1};$$
$$x_6 = \max\{x_0, x_0\} = x_0, \ \ y_6 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, x_0\right\} = x_0;$$
$$x_7 = \max\left\{\frac{A}{x_0 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_7 = \max\left\{\frac{A}{x_0 x_{-1}}, \frac{A}{x_0 x_{-1}}\right\} = \frac{A}{x_0 x_{-1}};$$
$$x_8 = \max\{y_{-1}, x_{-1}\} = x_{-1}, \ \ y_8 = \max\{x_{-1}, x_{-1}\} = x_{-1};$$

Hence $x_3 = x_6,\ x_4 = x_7,\ x_5 = x_8,\ y_3 = y_6,\ y_4 = y_7,\ y_5 = y_8$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = x_0; \ \ x_{3n+1} = \frac{A}{x_0 y_{-1}}; \ \ x_{3n+2} = x_{-1};$$
$$y_{3n} = x_0; \ \ y_{3n+1} = \frac{A}{x_0 x_{-1}}; \ \ y_{3n+2} = x_{-1}, \ \ n = 1, 2, \ldots.$$

(b) $x_{-1} \leq y_{-1}$, we have

$$x_5 = \max\{y_{-1}, x_{-1}\} = y_{-1}, \ \ y_5 = \max\{x_{-1}, y_{-1}\} = y_{-1};$$
$$x_6 = \max\left\{\frac{x_0 x_{-1}}{y_{-1}}, x_0\right\} = x_0, \ \ y_6 = \max\{x_0, x_0\} = x_0;$$
$$x_7 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 x_{-1}}\right\} = \frac{A}{x_0 x_{-1}};$$
$$x_8 = \max\{y_{-1}, y_{-1}\} = y_{-1}, \ \ y_8 = \max\{x_{-1}, y_{-1}\} = y_{-1};$$

Hence $x_3 = x_6,\ x_4 = x_7,\ x_5 = x_8,\ y_3 = y_6,\ y_4 = y_7,\ y_5 = y_8$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = x_0;\ \ x_{3n+1} = \frac{A}{x_0 y_{-1}};\ \ x_{3n+2} = y_{-1};$$
$$y_{3n} = x_0;\ \ y_{3n+1} = \frac{A}{x_0 x_{-1}};\ \ y_{3n+2} = y_{-1},\ \ n = 1, 2, \ldots.$$

(ii) if $\frac{A}{y_0 x_{-1}} \le y_{-2}$, the results are the same as (i). We omit other cases since they are similar in proof of induction.

Remark 3: $A > 0$ and (H3) imply that (3) could have either eventually positive or eventually negative or eventually sign-changing solutions.

4 Periodic solutions of (3) for the case $A < 0$

In this section, we will discuss the eventually periodic solutions of (3) for the case $A < 0$.

Theorem 4: *Suppose that $A < 0$ and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ satisfy (H1), then every solution of (3) is eventually periodic with period three.*

Proof: Since $A < 0$ and $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 < 0$, by the induction and iterative method, we can obtain that $x_n, y_n < 0$ for every $n \in \mathbb{N}$. We let $B = -A,\ u_n = -x_n,\ v_n = -y_n$, then (3) becomes

$$u_{n+1} = \min\left\{\frac{B}{u_n v_{n-1}}, u_{n-2}\right\},$$
$$v_{n+1} = \min\left\{\frac{B}{v_n u_{n-1}}, v_{n-2}\right\}, \tag{8}$$

where $u_n, v_n > 0$ for $n = -2, -1, 0, \ldots$.

Now in order to prove the theorem, we only need to prove that the solutions of (8) are eventually periodic with period three. Similarly as in the proof of Theorem 1, considering the limit length of the paper, we shall go with the following one case, other cases can be treated similarly. Assume that $\frac{B}{u_0 v_{-1}} \ge v_{-2} \ge u_{-2} \ge \frac{B}{v_0 u_{-1}}$, then we have

$$u_1 = \min\left\{\frac{B}{u_0 v_{-1}}, u_{-2}\right\} = u_{-2},\ \ v_1 = \min\left\{\frac{B}{v_0 u_{-1}}, v_{-2}\right\} = \frac{B}{v_0 u_{-1}};$$
$$u_2 = \min\left\{\frac{B}{u_{-2} v_0}, u_{-1}\right\} = \frac{B}{u_{-2} v_0},\ \ v_2 = \min\left\{\frac{v_0 u_{-1}}{u_0}, v_{-1}\right\} = v_{-1};$$

(i) If $\frac{u_0}{v_0} \ge 1$, and

(a) $u_0 \ge \frac{v_0^2 u_{-1} u_{-2}}{B}$, then

$$u_3 = \min\left\{\frac{v_0^2 u_{-1} u_{-2}}{B}, u_0\right\} = \frac{v_0^2 u_{-1} u_{-2}}{B},\ \ v_3 = \min\left\{\frac{B}{v_{-1} u_{-2}}, v_0\right\} = v_0;$$

$$u_4=\min\left\{\frac{B^2}{v_0^2v_{-1}u_{-1}u_{-2}},u_{-2}\right\}=u_{-2},\ \ v_4=\min\left\{u_{-2},\frac{B}{v_0u_{-1}}\right\}=\frac{B}{v_0u_{-1}};$$
$$u_5=\min\left\{\frac{B}{u_{-2}v_0},\frac{B}{u_{-2}v_0}\right\}=\frac{B}{u_{-2}v_0},\ \ v_5=\min\left\{\frac{B}{u_{-2}v_0},v_{-1}\right\}=v_{-1};$$
$$u_6=\min\left\{\frac{v_0^2u_{-1}u_{-2}}{B},\frac{v_0^2u_{-1}u_{-2}}{B}\right\}=\frac{v_0^2u_{-1}u_{-2}}{B},\ \ v_6=\min\left\{\frac{B}{v_{-1}u_{-2}},v_0\right\}=v_0.$$

Hence $u_1=u_4,\ u_2=u_5,\ u_3=u_6, v_1=v_4,\ v_2=v_5,\ v_3=v_6$, by Lemma 1, the solution is eventually periodic with period three as the following

$$u_{3n-2}=u_{-2};\ \ u_{3n-1}=\frac{B}{u_{-2}v_0};\ \ u_{3n}=\frac{v_0^2u_{-1}u_{-2}}{B};\ \ \ v_{3n-2}=\frac{B}{v_0u_{-1}};$$
$$v_{3n-1}=v_{-1};\ \ v_{3n}=v_0,\ \ n=1,2,\ldots.$$

i.e.

$$x_{3n-2}=x_{-2};\ \ x_{3n-1}=\frac{B}{x_{-2}y_0};\ \ x_{3n}=\frac{y_0^2x_{-1}x_{-2}}{B};\ \ y_{3n-2}=\frac{B}{y_0x_{-1}};$$
$$y_{3n-1}=y_{-1};\ \ y_{3n}=y_0,\ \ n=1,2,\ldots.$$

(b) $u_0\le\frac{v_0^2u_{-1}u_{-2}}{B}$, then

$$u_3=\min\left\{\frac{v_0^2u_{-1}u_{-2}}{B},u_0\right\}=u_0,\ \ v_3=\min\left\{\frac{B}{v_{-1}u_{-2}},v_0\right\}=v_0;$$
$$u_4=\min\left\{\frac{B}{u_0v_{-1}},u_{-2}\right\}=u_{-2},\ \ v_4=\min\left\{u_{-2},\frac{B}{v_0u_{-1}}\right\}=\frac{B}{v_0u_{-1}};$$
$$u_5=\min\left\{\frac{B}{u_{-2}v_0},\frac{B}{u_{-2}v_0}\right\}=\frac{B}{u_{-2}v_0},\ \ v_5=\min\left\{\frac{v_0u_{-1}}{u_0},v_{-1}\right\}=v_{-1};$$
$$u_6=\min\left\{\frac{v_0^2u_{-1}u_{-2}}{B},u_0\right\}=u_0,\ \ v_6=\min\left\{\frac{B}{v_{-1}u_{-2}},v_0\right\}=v_0.$$

Hence $u_1=u_4,\ u_2=u_5,\ u_3=u_6, v_1=v_4,\ v_2=v_5,\ v_3=v_6$, by Lemma 1, the solution is eventually periodic with period three as the following

$$u_{3n-2}=u_{-2};\ \ u_{3n-1}=\frac{B}{u_{-2}v_0};\ \ u_{3n}=u_0;\ \ v_{3n-2}=\frac{B}{v_0u_{-1}};$$
$$v_{3n-1}=v_{-1};\ \ v_{3n}=v_0,\ \ n=1,2,\ldots.$$

i.e.

$$x_{3n-2}=x_{-2};\ \ x_{3n-1}=\frac{B}{x_{-2}y_0};\ \ x_{3n}=x_0;\ \ y_{3n-2}=\frac{B}{y_0x_{-1}};$$
$$y_{3n-1}=y_{-1};\ \ y_{3n}=y_0,\ \ n=1,2,\ldots.$$

(ii) If $0<\frac{u_0}{v_0}<1$, and

(a) $v_0 \geq \frac{B}{v_{-1}u_{-2}}$, then

$$u_3 = \min\left\{\frac{v_0^2 u_{-1}u_{-2}}{B}, u_0\right\} = u_0, \;\; v_3 = \min\left\{\frac{B}{v_{-1}u_{-2}}, v_0\right\} = \frac{B}{v_{-1}u_{-2}};$$
$$u_4 = \min\left\{\frac{B}{u_0 v_{-1}}, u_{-2}\right\} = u_{-2}, \;\; v_4 = \min\left\{\frac{v_0 v_{-1} u_{-2}^2}{B}, \frac{B}{v_0 u_{-1}}\right\} = \frac{B}{v_0 u_{-1}};$$
$$u_5 = \min\left\{v_{-1}, \frac{B}{u_{-2}v_0}\right\} = \frac{B}{u_{-2}v_0}, \;\; v_5 = \min\left\{\frac{v_0 u_{-1}}{u_0}, v_{-1}\right\} = v_{-1};$$
$$u_6 = \min\left\{\frac{v_0^2 u_{-1}u_{-2}}{B}, u_0\right\} = u_0, \;\; v_6 = \min\left\{\frac{B}{v_{-1}u_{-2}}, \frac{B}{v_{-1}u_{-2}}\right\} = \frac{B}{v_{-1}u_{-2}}.$$

Hence $u_1 = u_4,\; u_2 = u_5,\; u_3 = u_6, v_1 = v_4,\; v_2 = v_5,\; v_3 = v_6$, by Lemma 1, the solution is eventually periodic with period three as the following

$$u_{3n-2} = u_{-2}; \;\; u_{3n-1} = \frac{B}{u_{-2}v_0}; \;\; u_{3n} = u_0;$$
$$v_{3n-2} = \frac{B}{v_0 u_{-1}}; \;\; v_{3n-1} = v_{-1}; \;\; v_{3n} = \frac{B}{v_{-1}u_{-2}}, \;\; n = 1, 2, \cdots.$$

i.e.

$$x_{3n-2} = x_{-2}; \;\; x_{3n-1} = \frac{B}{x_{-2}y_0}; \;\; x_{3n} = x_0;$$
$$y_{3n-2} = \frac{B}{y_0 x_{-1}}; \;\; y_{3n-1} = y_{-1}; \;\; y_{3n} = \frac{B}{y_{-1}x_{-2}}, \;\; n = 1, 2, \cdots.$$

(b) $v_0 \leq \frac{B}{v_{-1}u_{-2}}$, then

$$u_3 = \min\left\{\frac{v_0^2 u_{-1}u_{-2}}{B}, u_0\right\} = u_0, \;\; v_3 = \min\left\{\frac{B}{v_{-1}u_{-2}}, v_0\right\} = v_0;$$
$$u_4 = \min\left\{\frac{B}{u_0 v_{-1}}, u_{-2}\right\} = u_{-2}, \;\; v_4 = \min\left\{u_{-2}, \frac{B}{v_0 u_{-1}}\right\} = \frac{B}{v_0 u_{-1}};$$
$$u_5 = \min\left\{\frac{B}{u_{-2}v_0}, \frac{B}{u_{-2}v_0}\right\} = \frac{B}{u_{-2}v_0}, \;\; v_5 = \min\left\{\frac{v_0 u_{-1}}{u_0}, v_{-1}\right\} = v_{-1};$$
$$u_6 = \min\left\{\frac{v_0^2 u_{-1}u_{-2}}{B}, u_0\right\} = u_0, \;\; v_6 = \min\left\{\frac{B}{v_{-1}u_{-2}}, v_0\right\} = v_0.$$

Hence $u_1 = u_4,\; u_2 = u_5,\; u_3 = u_6, v_1 = v_4,\; v_2 = v_5,\; v_3 = v_6$, by Lemma 1, the solution is periodic with period three as the following

$$u_{3n-2} = u_{-2}; \;\; u_{3n-1} = \frac{B}{u_{-2}v_0}; \;\; u_{3n} = u_0;$$
$$v_{3n-2} = \frac{B}{v_0 u_{-1}}; \;\; v_{3n-1} = v_{-1}; \;\; v_{3n} = v_0, \;\; n = 1, 2, \cdots.$$

i.e.

$$x_{3n-2} = x_{-2}; \;\; x_{3n-1} = \frac{B}{x_{-2}y_0}; \;\; x_{3n} = x_0;$$
$$y_{3n-2} = \frac{B}{y_0 x_{-1}}; \;\; y_{3n-1} = y_{-1}; \;\; y_{3n} = y_0, \;\; n = 1, 2, \ldots.$$

Remark 4: $A < 0$ and (H1) imply that every solution of (3) is negative.

Theorem 5: *Suppose that $A < 0$ and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ satisfy (H2), then every solution of (3) is periodic with period three.*

Proof: Since $A < 0$ and $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 > 0$, then we have

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = x_{-2}, \;\; y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_2 = \max\left\{\frac{A}{x_{-2} y_0}, x_{-1}\right\} = x_{-1}, \;\; y_2 = \max\left\{\frac{A}{y_{-2} x_0}, y_{-1}\right\} = y_{-1};$$
$$x_3 = \max\left\{\frac{A}{x_{-1} y_{-2}}, x_0\right\} = x_0, \;\; y_3 = \max\left\{\frac{A}{y_{-1} x_{-2}}, y_0\right\} = y_0,$$

from this and by induction we have $x_n, \; y_n > 0$ for $n \in \mathbb{N}$. Hence by Lemma 1,

$$x_{3n-2} = x_{-2}; \;\; x_{3n-1} = x_{-1}; \;\; x_{3n} = x_0; \;\; y_{3n-2} = y_{-2}; \;\; y_{3n-1} = y_{-1};$$
$$y_{3n} = y_0, \; n = 1, 2, \ldots.$$

Remark 5: $A < 0$ and (H2) imply that every solution of (3) is positive.

Theorem 6: *Suppose that $A < 0$ and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ satisfy (H3), then every solution of (3) is eventually periodic with period three.*

Proof: Since there are many categories, we will discuss only four situations, other cases can be treated similarly.
(I) Suppose that $x_0, \; y_0 > 0$, $x_{-2}, \; x_{-1}, \; y_{-2}, \; y_{-1} < 0$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = \frac{A}{x_0 y_{-1}}, \;\; y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = \frac{A}{y_0 x_{-1}};$$

(i) if $y_0 x_{-1} \geq x_0 y_{-1}$, then

$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1}; \;\; y_2 = \max\left\{\frac{y_0 x_{-1}}{x_0}, y_{-1}\right\} = \frac{y_0 x_{-1}}{x_0}.$$

(a) $\frac{x_0}{y_0} \geq 1$, and

(a1) $y_0 \geq \frac{x_0^2 y_{-1}}{y_0 x_{-1}}$, we have

$$x_3 = \max\{y_0, x_0\} = x_0,\ y_3 = \max\left\{\frac{x_0^2 y_{-1}}{y_0 x_{-1}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{y_0 x_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\{x_{-1}, x_{-1}\} = x_{-1},\ \ y_5 = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_6 = \max\{y_o, x_0\} = x_0,\ \ y_6 = \max\{x_0, y_0\} = x_0;$$
$$x_7 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}},\ \ y_7 = \max\left\{\frac{A}{x_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_8 = \max\left\{\frac{y_0 x_{-1}}{x_0}, x_{-1}\right\} = \frac{y_0 x_{-1}}{x_0},\ \ y_8 = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_9 = \max\{x_0, x_0\} = x_0,\ \ y_9 = \max\{x_0, x_0\} = x_0;$$
$$x_{10} = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}},\ \ y_{10} = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_{11} = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0},\ \ y_{11} = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0}.$$

Hence $x_6 = y_6 = x_9 = y_9,\ x_7 = y_7 = x_{10} = y_{10},\ x_8 = y_8 = x_{11} = y_{11}$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = y_{3n} = x_0;\ \ x_{3n+1} = y_{3n+1} = \frac{A}{y_0 x_{-1}};\ \ x_{3n+2} = y_{3n+2} = \frac{y_0 x_{-1}}{x_0},\ \ n = 2, 3, \ldots.$$

(a2) $y_0 \leq \frac{x_0^2 y_{-1}}{y_0 x_{-1}}$, the result is the same as (a1).

$$x_{3n} = y_{3n} = x_0;\ \ x_{3n+1} = y_{3n+1} = \frac{A}{y_0 x_{-1}};\ \ x_{3n+2} = y_{3n+2} = \frac{y_0 x_{-1}}{x_0},\ \ n = 2, 3, \ldots.$$

(b) $0 < \frac{x_0}{y_0} < 1$, we have

$$x_3 = \max\{y_0, x_0\} = y_0,\ \ y_3 = \max\left\{\frac{x_0^2 y_{-1}}{y_0 x_{-1}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{x_{-1}, \frac{y_0 x_{-1}}{x_0}\right\} = x_{-1};$$
$$x_6 = \max\{y_o, y_0\} = y_0,\ \ y_6 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, y_0\right\} = \frac{x_0 y_{-1}}{x_{-1}};$$
$$x_7 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{y_0 x_{-1}},\ \ y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_8 = \max\left\{\frac{y_0 x_{-1}^2}{x_0 y_{-1}}, x_{-1}\right\} = x_{-1},\ \ y_8 = \max\{x_{-1}, x_{-1}\} = x_{-1};$$

$$x_9 = \max\{y_o, y_0\} = y_0,\ \ y_9 = \max\left\{y_0, \frac{x_0y_{-1}}{x_{-1}}\right\} = \frac{x_0y_{-1}}{x_{-1}};$$
$$x_{10} = \max\left\{\frac{A}{y_0x_{-1}}, \frac{A}{y_0x_{-1}}\right\} = \frac{A}{y_0x_{-1}},\ \ y_{10} = \max\left\{\frac{A}{x_0y_{-1}}, \frac{A}{y_0x_{-1}}\right\} = \frac{A}{y_0x_{-1}}.$$

Hence $x_5 = x_8,\ x_6 = x_9,\ x_7 = x_{10},\ y_5 = y_8,\ y_6 = y_9,\ y_7 = y_{10},$ by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = x_{-1};\ \ x_{3n} = y_0;\ \ y_{3n} = \frac{x_0y_{-1}}{x_{-1}};$$
$$x_{3n+1} = y_{3n+1} = \frac{A}{y_0x_{-1}},\ \ n = 2, 3, \ldots.$$

(ii) if $y_0x_{-1} \le x_0y_{-1}$, then

$$x_2 = \max\left\{\frac{x_0y_{-1}}{y_0}, x_{-1}\right\} = \frac{x_0y_{-1}}{y_0};\ \ y_2 = \max\left\{\frac{y_0x_{-1}}{x_0}, y_{-1}\right\} = y_{-1}.$$

(a) $\frac{x_0}{y_0} \ge 1$, we have

$$x_3 = \max\left\{\frac{y_0^2x_{-1}}{x_0y_{-1}}, x_0\right\} = x_0,\ \ y_3 = \max\{x_0, y_0\} = x_0;$$
$$x_4 = \max\left\{\frac{A}{x_0y_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}},\ \ y_4 = \max\left\{\frac{y_0A}{x_0^2y_{-1}}, \frac{A}{y_0x_{-1}}\right\} = \frac{A}{y_0x_{-1}};$$
$$x_5 = \max\left\{y_{-1}, \frac{x_0y_{-1}}{y_0}\right\} = y_{-1},\ \ y_5 = \max\left\{\frac{y_0x_{-1}}{x_0}, y_{-1}\right\} = y_{-1};$$
$$x_6 = \max\left\{\frac{y_0x_{-1}}{y_{-1}}, x_0\right\} = \frac{y_0x_{-1}}{y_{-1}},\ \ y_6 = \max\{x_0, x_0\} = x_0;$$
$$x_7 = \max\left\{\frac{A}{y_0x_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}},\ \ y_7 = \max\left\{\frac{A}{x_0y_{-1}}, \frac{A}{y_0x_{-1}}\right\} = \frac{A}{x_0y_{-1}};$$
$$x_8 = \max\{y_{-1}, y_{-1}\} = y_{-1},\ \ y_8 = \max\left\{\frac{x_0y_{-1}^2}{y_0x_{-1}}, y_{-1}\right\} = y_{-1};$$
$$x_9 = \max\left\{x_0, \frac{y_0x_{-1}}{y_{-1}}\right\} = \frac{y_0x_{-1}}{y_{-1}},\ \ y_9 = \max\{x_0, x_0\} = x_0;$$
$$x_{10} = \max\left\{\frac{A}{y_0x_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}},\ \ y_{10} = \max\left\{\frac{A}{x_0y_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}}.$$

Hence $x_5 = x_8,\ x_6 = x_9,\ x_7 = x_{10},\ y_5 = y_8,\ y_6 = y_9,\ y_7 = y_{10},$ by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n-1} = y_{3n-1} = y_{-1};\ \ x_{3n} = \frac{y_0x_{-1}}{y_{-1}};\ \ y_{3n} = x_0;$$
$$x_{3n+1} = y_{3n+1} = \frac{A}{x_0y_{-1}},\ \ n = 2, 3, \ldots.$$

(b) $0 < \frac{x_0}{y_0} < 1$, and

(b1) $x_0 \le \frac{y_0^2 x_{-1}}{x_0 y_{-1}}$, we have

$$
\begin{aligned}
x_3 &= \max\left\{\frac{y_0^2 x_{-1}}{x_0 y_{-1}}, x_0\right\} = \frac{y_0^2 x_{-1}}{x_0 y_{-1}}, \quad y_3 = \max\{x_0, y_0\} = y_0;\\
x_4 &= \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \quad y_4 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{x_0 y_{-1}};\\
x_5 &= \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0}, \quad y_5 = \max\left\{\frac{x_0^2 y_{-1}^2}{y_0^2 x_{-1}}, y_{-1}\right\} = y_{-1};\\
x_6 &= \max\left\{y_0, \frac{y_0^2 x_{-1}}{x_0 y_{-1}}\right\} = y_0, \quad y_6 = \max\{x_0, y_0\} = y_0;\\
x_7 &= \max\left\{\frac{A}{y_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \quad y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}};\\
x_8 &= \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0}, \quad y_8 = \max\left\{\frac{x_0 y_{-1}}{y_0}, y_{-1}\right\} = \frac{x_0 y_{-1}}{y_0};\\
x_9 &= \max\{y_0, y_0\} = y_0, \quad y_9 = \max\{y_0, y_0\} = y_0;\\
x_{10} &= \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \quad y_{10} = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}};\\
x_{11} &= \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0}, \quad y_{11} = \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0}.
\end{aligned}
$$

Hence $x_6 = y_6 = x_9 = y_9,\ x_7 = y_7 = x_{10} = y_{10},\ x_8 = y_8 = x_{11} = y_{11}$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = y_{3n} = y_0; \quad x_{3n+1} = y_{3n+1} = \frac{A}{x_0 y_{-1}}; \quad x_{3n+2} = y_{3n+2} = \frac{x_0 y_{-1}}{y_0}, \quad n = 2, 3, \ldots.$$

(b2) $x_0 \ge \frac{y_0^2 x_{-1}}{x_0 y_{-1}}$, the result is the same as (b1).

$$x_{3n} = y_{3n} = y_0; \quad x_{3n+1} = y_{3n+1} = \frac{A}{x_0 y_{-1}}; \quad x_{3n+2} = y_{3n+2} = \frac{x_0 y_{-1}}{y_0}, \quad n = 2, 3, \ldots.$$

(II) Suppose that $x_0 > 0$, $x_{-2}, x_{-1}, y_{-2}, y_{-1}, y_0 < 0$, and
(i) if $\frac{A}{y_0 x_{-1}} \ge y_{-2}$, then

$$
\begin{aligned}
x_1 &= \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = \frac{A}{x_0 y_{-1}}, \quad y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = \frac{A}{y_0 x_{-1}};\\
x_2 &= \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = \frac{x_0 y_{-1}}{y_0}; \quad y_2 = \max\left\{\frac{y_0 x_{-1}}{x_0}, y_{-1}\right\} = \frac{y_0 x_{-1}}{x_0}.
\end{aligned}
$$

(a) $\frac{y_0^2 x_{-1}}{x_0 y_{-1}} \ge x_0$, we have

$$x_3 = \max\left\{\frac{y_0^2 x_{-1}}{x_0 y_{-1}}, x_0\right\} = \frac{y_0^2 x_{-1}}{x_0 y_{-1}}, \quad y_3 = \max\left\{\frac{x_0^2 y_{-1}}{y_0 x_{-1}}, y_0\right\} = \frac{x_0^2 y_{-1}}{y_0 x_{-1}};$$

$$x_4 = \max\left\{\frac{x_0^2y_{-1}A}{y_0^3x_{-1}^2}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}}, \quad y_4 = \max\left\{\frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2}, \frac{A}{y_0x_{-1}}\right\} = \frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2};$$
$$x_5 = \max\left\{\frac{y_0x_{-1}}{x_0}, \frac{x_0y_{-1}}{y_0}\right\} = \frac{y_0x_{-1}}{x_0}, \quad y_5 = \max\left\{\frac{x_0^4y_{-1}^3}{y_0^4x_{-1}^2}, \frac{y_0x_{-1}}{x_0}\right\} = \frac{y_0x_{-1}}{x_0};$$
$$x_6 = \max\left\{\frac{x_0^4y_{-1}^2}{y_0^3x_{-1}^2}, \frac{y_0^2x_{-1}}{x_0y_{-1}}\right\} = \frac{y_0^2x_{-1}}{x_0y_{-1}}, \quad y_6 = \max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}}, \frac{x_0^2y_{-1}}{y_0x_{-1}}\right\} = \frac{x_0^2y_{-1}}{y_0x_{-1}};$$
$$x_7 = \max\left\{\frac{x_0^2y_{-1}A}{y_0^3x_{-1}^2}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}}, \quad y_7 = \max\left\{\frac{A}{x_0y_{-1}}, \frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2}\right\} = \frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2};$$
$$x_8 = \max\left\{\frac{y_0x_{-1}}{x_0}, \frac{y_0x_{-1}}{x_0}\right\} = \frac{y_0x_{-1}}{x_0}, \quad y_8 = \max\left\{\frac{x_0^4y_{-1}^3}{y_0^4x_{-1}^2}, \frac{y_0x_{-1}}{x_0}\right\} = \frac{y_0x_{-1}}{x_0};$$

Hence $x_3 = x_6,\ x_4 = x_7,\ x_5 = x_8,\ y_3 = y_6,\ y_4 = y_7,\ y_5 = y_8$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = \frac{y_0^2x_{-1}}{x_0y_{-1}}; \quad x_{3n+1} = \frac{A}{x_0y_{-1}}; \quad x_{3n+2} = \frac{y_0x_{-1}}{x_0},$$
$$y_{3n} = \frac{x_0^2y_{-1}}{y_0x_{-1}}, \quad y_{3n+1} = \frac{y_0^2x_{-1}A}{x_0^3y_{-1}^2}; \quad y_{3n+2} = \frac{y_0x_{-1}}{x_0}, \quad n = 1, 2, \ldots.$$

(b) $\frac{y_0^2x_{-1}}{x_0y_{-1}} \le x_0$, we have

$$x_3 = \max\left\{\frac{y_0^2x_{-1}}{x_0y_{-1}}, x_0\right\} = x_0, \quad y_3 = \max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{A}{y_0x_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}}, \quad y_4 = \max\left\{\frac{A}{x_0y_{-1}}, \frac{A}{y_0x_{-1}}\right\} = \frac{A}{x_0y_{-1}};$$
$$x_5 = \max\left\{\frac{x_0y_{-1}}{y_0}, \frac{x_0y_{-1}}{y_0}\right\} = \frac{x_0y_{-1}}{y_0}, \quad y_5 = \max\left\{y_{-1}, \frac{y_0x_{-1}}{x_0}\right\} = \frac{y_0x_{-1}}{x_0};$$
$$x_6 = \max\{y_0, x_0\} = x_0, \quad y_6 = \max\left\{\frac{x_0^2y_{-1}}{y_0x_{-1}}, y_0\right\} = y_0;$$
$$x_7 = \max\left\{\frac{A}{y_0x_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}}, \quad y_7 = \max\left\{\frac{A}{x_0y_{-1}}, \frac{A}{x_0y_{-1}}\right\} = \frac{A}{x_0y_{-1}};$$
$$x_8 = \max\left\{\frac{x_0y_{-1}}{y_0}, \frac{x_0y_{-1}}{y_0}\right\} = \frac{x_0y_{-1}}{y_0}, \quad y_8 = \max\left\{y_{-1}, \frac{y_0x_{-1}}{x_0}\right\} = \frac{y_0x_{-1}}{x_0};$$

Hence $x_3 = x_6,\ x_4 = x_7,\ x_5 = x_8,\ y_3 = y_6,\ y_4 = y_7,\ y_5 = y_8$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = x_0; \quad x_{3n+1} = \frac{A}{x_0y_{-1}}; \quad x_{3n+2} = \frac{x_0y_{-1}}{y_0}, \quad y_{3n} = y_0, \quad y_{3n+1} = \frac{A}{x_0y_{-1}};$$
$$y_{3n+2} = \frac{y_0x_{-1}}{x_0}, \quad n = 1, 2, \ldots.$$

(ii) if $\frac{A}{y_0x_{-1}} \le y_{-2}$, then

$$x_1 = \max\left\{\frac{A}{x_0y_{-1}}, x_{-2}\right\} = \frac{A}{x_0y_{-1}}, \quad y_1 = \max\left\{\frac{A}{y_0x_{-1}}, y_{-2}\right\} = y_{-2};$$

$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = \frac{x_0 y_{-1}}{y_0}; \ \ y_2 = \max\left\{\frac{A}{y_{-2} x_0}, y_{-1}\right\} = \frac{A}{y_{-2} x_0}.$$

(a) $\frac{y_0 A}{x_0 y_{-1} y_{-2}} \geq x_0$, we have

$$\begin{aligned}
x_3 &= \max\left\{\frac{y_0 A}{x_0 y_{-1} y_{-2}}, x_0\right\} = \frac{y_0 A}{x_0 y_{-1} y_{-2}}, \ \ y_3 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = \frac{x_0^2 y_{-1} y_{-2}}{A};\\
x_4 &= \max\left\{\frac{x_0^2 y_{-1} y_{-2}^2}{y_0 A}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_4 = \max\left\{\frac{y_0 A^2}{x_0^3 y_{-1}^2 y_{-2}}, y_{-2}\right\} = \frac{y_0 A^2}{x_0^3 y_{-1}^2 y_{-2}};\\
x_5 &= \max\left\{\frac{A}{x_0 y_{-2}}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{A}{x_0 y_{-2}}, \ \ y_5 = \max\left\{\frac{x_0^4 y_{-1}^3 y_{-2}^2}{y_0^2 A^2}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};\\
x_6 &= \max\left\{\frac{x_0^4 y_{-1}^2 y_{-2}^2}{y_0 A^2}, \frac{y_0 A}{x_0 y_{-1} y_{-2}}\right\} = \frac{y_0 A}{x_0 y_{-1} y_{-2}},\\
&\qquad y_6 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, \frac{x_0^2 y_{-1} y_{-2}}{A}\right\} = \frac{x_0^2 y_{-1} y_{-2}}{A};\\
x_7 &= \max\left\{\frac{x_0^2 y_{-1} y_{-2}^2}{y_0 A}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{y_0 A^2}{x_0^3 y_{-1}^2 y_{-2}}\right\} = \frac{y_0 A^2}{x_0^3 y_{-1}^2 y_{-2}};\\
x_8 &= \max\left\{\frac{A}{x_0 y_{-2}}, \frac{A}{x_0 y_{-2}}\right\} = \frac{A}{x_0 y_{-2}}, \ \ y_8 = \max\left\{\frac{x_0^4 y_{-1}^3 y_{-2}^2}{y_0^2 A^2}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};
\end{aligned}$$

Hence $x_3 = x_6,\ x_4 = x_7,\ x_5 = x_8,\ y_3 = y_6,\ y_4 = y_7,\ y_5 = y_8$, by Lemma 1, the solution is eventually periodic with period three as the following

$$\begin{aligned}
x_{3n} &= \frac{y_0 A}{x_0 y_{-1} y_{-2}}; \ \ x_{3n+1} = \frac{A}{x_0 y_{-1}}; \ \ x_{3n+2} = \frac{A}{x_0 y_{-2}},\\
y_{3n} &= \frac{x_0^2 y_{-1} y_{-2}}{A}, \ \ y_{3n+1} = \frac{y_0 A^2}{x_0^3 y_{-1}^2 y_{-2}}; \ \ y_{3n+2} = \frac{A}{y_{-2} x_0}, \ \ n = 1, 2, \ldots.
\end{aligned}$$

(b) $\frac{y_0 A}{x_0 y_{-1} y_{-2}} \leq x_0$, we have

$$\begin{aligned}
x_3 &= \max\left\{\frac{y_0 A}{x_0 y_{-1} y_{-2}}, x_0\right\} = x_0, \ \ y_3 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = y_0;\\
x_4 &= \max\left\{y_{-2}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_4 = \max\left\{\frac{A}{x_0 y_{-1}}, y_{-2}\right\} = \frac{A}{x_0 y_{-1}};\\
x_5 &= \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0}, \ \ y_5 = \max\left\{y_{-1}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};\\
x_6 &= \max\{y_0, x_0\} = x_0, \ \ y_6 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = y_0;\\
x_7 &= \max\left\{y_{-2}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}}, \ \ y_7 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}};\\
x_8 &= \max\left\{\frac{x_0 y_{-1}}{y_0}, \frac{x_0 y_{-1}}{y_0}\right\} = \frac{x_0 y_{-1}}{y_0}, \ \ y_8 = \max\left\{y_{-1}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};
\end{aligned}$$

Hence $x_3 = x_6,\ x_4 = x_7,\ x_5 = x_8,\ y_3 = y_6,\ y_4 = y_7,\ y_5 = y_8$, by Lemma 1, the solution is eventually periodic with period three as the following

$$x_{3n} = x_0;\ \ x_{3n+1} = \frac{A}{x_0 y_{-1}};\ \ x_{3n+2} = \frac{x_0 y_{-1}}{y_0},\ \ y_{3n} = y_0,$$
$$y_{3n+1} = \frac{A}{x_0 y_{-1}};\ \ y_{3n+2} = \frac{A}{y_{-2} x_0},\ \ n = 1, 2, \ldots.$$

(III) Suppose that $x_{-2},\ x_{-1},\ x_0,\ y_0 > 0$, $y_{-2},\ y_{-1} < 0$, and

(i) if $\frac{A}{x_0 y_{-1}} \geq x_{-2}$, $\frac{A}{y_0 x_{-1}} \geq y_{-2}$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1};\ \ y_2 = \max\{x_{-1}, y_{-1}\} = x_{-1}.$$

(a) $x_0 \geq y_0$, we have

$$x_3 = \max\{y_0, x_0\} = x_0,\ y_3 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{A}{x_0 y_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{\frac{y_0 x_{-1}}{x_0}, x_{-1}\right\} = x_{-1};$$
$$x_6 = \max\{y_0, x_0\} = x_0,\ \ y_6 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = x_0,$$
$$y_{3n-2} = \frac{A}{y_0 x_{-1}},\ \ y_{3n-1} = x_{-1};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(b) $x_0 \leq y_0$, we have

$$x_3 = \max\{y_0, x_0\} = y_0,\ y_3 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\{x_{-1}, x_{-1}\} = x_{-1};$$
$$x_6 = \max\{y_0, y_0\} = y_0,\ \ y_6 = \max\left\{\frac{x_0 y_{-1}}{x_{-1}}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = y_0,\ \ y_{3n-2} = \frac{A}{y_0 x_{-1}},$$
$$y_{3n-1} = x_{-1};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(ii) if $\frac{A}{x_0 y_{-1}} \geq x_{-2}$, $\frac{A}{y_0 x_{-1}} \leq y_{-2}$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_2 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1};\ \ y_2 = \max\left\{\frac{A}{y_{-2} x_0}, y_{-1}\right\} = \frac{A}{y_{-2} x_0}.$$

(a) $\frac{A}{x_{-1} y_{-2}} \geq x_0$, we have

$$x_3 = \max\left\{\frac{A}{x_{-1} y_{-2}}, x_0\right\} = \frac{A}{x_{-1} y_{-2}},\ \ y_3 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{x_{-1} y_{-2}^2 x_0}{A}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_5 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{x_{-1}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};$$
$$x_6 = \max\left\{\frac{A}{x_{-1} y_{-2}}, \frac{A}{x_{-1} y_{-2}}\right\} = \frac{A}{x_{-1} y_{-2}},\ \ y_6 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = \frac{A}{x_{-1} y_{-2}},$$
$$y_{3n-2} = y_{-2},\ \ y_{3n-1} = \frac{A}{y_{-2} x_0};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(b) $\frac{A}{x_{-1} y_{-2}} \leq x_0$, we have

$$x_3 = \max\left\{\frac{A}{x_{-1} y_{-2}}, x_0\right\} = x_0,\ y_3 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{y_{-2}, \frac{A}{x_0 y_{-1}}\right\} = \frac{A}{x_0 y_{-1}},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_5 = \max\left\{\frac{x_0 y_{-1}}{y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{\frac{A}{y_{-2} x_0}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};$$
$$x_6 = \max\left\{\frac{A}{x_{-1} y_{-2}}, x_0\right\} = x_0,\ \ y_6 = \max\left\{\frac{x_0^2 y_{-1} y_{-2}}{A}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = \frac{A}{x_0 y_{-1}};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = x_0,\ \ y_{3n-2} = y_{-2},$$
$$y_{3n-1} = \frac{A}{y_{-2} x_0};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(iii) if $\frac{A}{x_0 y_{-1}} \le x_{-2}$, $\frac{A}{y_0 x_{-1}} \ge y_{-2}$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = x_{-2},\ \ y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_2 = \max\left\{\frac{A}{x_{-2} y_0}, x_{-1}\right\} = x_{-1};\ \ y_2 = \max\left\{\frac{y_0 x_{-1}}{x_0}, y_{-1}\right\} = \frac{y_0 x_{-1}}{x_0}.$$

(a) $x_0 \ge y_0$, we have

$$x_3 = \max\{y_0, x_0\} = x_0,\ \ y_3 = \max\left\{\frac{x_0 A}{y_0 x_{-1} x_{-2}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{A}{y_0 x_{-1}}, x_{-2}\right\} = x_{-2},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\left\{\frac{A}{x_{-2} y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{\frac{y_0 x_{-1}}{x_0}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_6 = \max\{y_0, x_0\} = x_0,\ \ y_6 = \max\left\{\frac{x_0 A}{y_0 x_{-1} x_{-2}}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = x_{-2};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = x_0,\ \ y_{3n-2} = \frac{A}{y_0 x_{-1}},$$
$$y_{3n-1} = \frac{y_0 x_{-1}}{x_0};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(b) $x_0 \le y_0$, we have

$$x_3 = \max\{y_0, x_0\} = y_0,\ \ y_3 = \max\left\{\frac{x_0 A}{y_0 x_{-1} x_{-2}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{x_0 A}{y_0^2 x_{-1}}, x_{-2}\right\} = x_{-2},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, \frac{A}{y_0 x_{-1}}\right\} = \frac{A}{y_0 x_{-1}};$$
$$x_5 = \max\left\{\frac{A}{x_{-2} y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{x_{-1}, \frac{y_0 x_{-1}}{x_0}\right\} = \frac{y_0 x_{-1}}{x_0};$$
$$x_6 = \max\{y_0, y_0\} = y_0,\ \ y_6 = \max\left\{\frac{x_0 A}{y_0 x_{-1} x_{-2}}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = x_{-2};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = y_0,\ \ y_{3n-2} = \frac{A}{y_0 x_{-1}},$$
$$y_{3n-1} = \frac{y_0 x_{-1}}{x_0};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(iv) if $\frac{A}{x_0 y_{-1}} \le x_{-2}$, $\frac{A}{y_0 x_{-1}} \le y_{-2}$, then

$$x_1 = \max\left\{\frac{A}{x_0 y_{-1}}, x_{-2}\right\} = x_{-2},\ \ y_1 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_2 = \max\left\{\frac{A}{x_{-2} y_0}, x_{-1}\right\} = x_{-1};\ \ y_2 = \max\left\{\frac{A}{y_{-2} x_0}, y_{-1}\right\} = \frac{A}{y_{-2} x_0}.$$

(a) $\frac{A}{x_{-1} y_{-2}} \ge x_0$, we have

$$x_3 = \max\left\{\frac{A}{x_{-1} y_{-2}}, x_0\right\} = \frac{A}{x_{-1} y_{-2}},\ \ y_3 = \max\left\{\frac{x_0 y_{-2}}{x_{-2}}, y_0\right\} = y_0;$$
$$x_4 = \max\left\{\frac{x_{-1} y_{-2}^2 x_0}{A}, x_{-2}\right\} = x_{-2},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_5 = \max\left\{\frac{A}{x_{-2} y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{x_{-1}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};$$
$$x_6 = \max\left\{\frac{A}{x_{-1} y_{-2}}, \frac{A}{x_{-1} y_{-2}}\right\} = \frac{A}{x_{-1} y_{-2}},\ \ y_6 = \max\left\{\frac{x_0 y_{-2}}{x_{-2}}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = x_{-2};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = \frac{A}{x_{-1} y_{-2}},\ \ y_{3n-2} = y_{-2},$$
$$y_{3n-1} = \frac{A}{y_{-2} x_0};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

(b) $\frac{A}{x_{-1} y_{-2}} \le x_0$, we have

$$x_3 = \max\left\{\frac{A}{x_{-1} y_{-2}}, x_0\right\} = x_0,\ y_3 = \max\left\{\frac{x_0 y_{-2}}{x_{-2}}, y_0\right\} = y_0;$$
$$x_4 = \max\{y_{-2}, x_{-2}\} = x_{-2},\ \ y_4 = \max\left\{\frac{A}{y_0 x_{-1}}, y_{-2}\right\} = y_{-2};$$
$$x_5 = \max\left\{\frac{A}{x_{-2} y_0}, x_{-1}\right\} = x_{-1},\ \ y_5 = \max\left\{\frac{A}{y_{-2} x_0}, \frac{A}{y_{-2} x_0}\right\} = \frac{A}{y_{-2} x_0};$$
$$x_6 = \max\left\{\frac{A}{x_{-1} y_{-2}}, x_0\right\} = x_0,\ \ y_6 = \max\left\{\frac{x_0 y_{-2}}{x_{-2}}, y_0\right\} = y_0;$$

Hence $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6,\ y_1 = y_4,\ y_2 = y_5,\ y_3 = y_6$, by Lemma 1, the solution is periodic with period three as the following

$$x_{3n-2} = x_{-2};\ \ x_{3n-1} = x_{-1};\ \ x_{3n} = x_0,\ \ y_{3n-2} = y_{-2},$$
$$y_{3n-1} = \frac{A}{y_{-2}x_0};\ \ y_{3n} = y_0,\ \ n = 1, 2, \ldots.$$

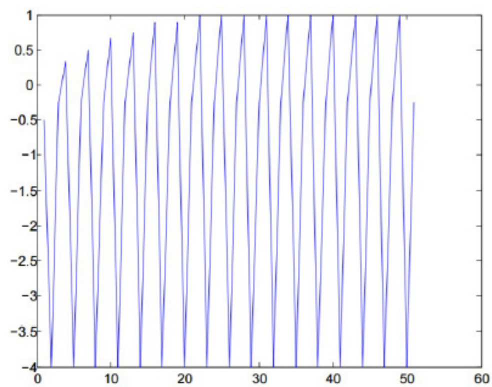
Since there are too many cases according to the signs of initial values and they have the similar proof by induction, we will not list all of them by the limit length of the paper.

Remark 6: $A < 0$ and (H3) imply that (3) have either eventually positive or eventually sign-changing solutions.

5 Examples

Example 1: Let $A = 1,\ x_{-2} = -1/2,\ x_{-1} = -4,\ x_0 = -1/4,\ y_{-2} = -1,\ y_{-1} = -12,\ y_0 = -1/2$. Then, by Theorem 1, (3) has eventually three-periodic solutions described as Figures 1 and 2.

Figure 1 Plot of $x(n)$ (see online version for colours)



Example 2: Let $A = 2/3,\ x_{-2} = 4,\ x_{-1} = 6,\ x_0 = 2,\ y_{-2} = 5/2,\ y_{-1} = 5,\ y_0 = 5/6$. Then, by Theorem 2, (3) has eventually three-periodic solutions described as Figures 3 and 4.

Example 3: Let $A = 1/4,\ x_{-2} = 4,\ x_{-1} = -7/2,\ x_0 = -2/3,\ y_{-2} = -4,\ y_{-1} = -1/2,\ y_0 = 5/3$. Then, by Theorem 3, (3) has eventually three-periodic solutions described as Figures 5 and 6.

Example 3': Let $A = 3,\ x_{-2} = -3/5,\ x_{-1} = -8,\ x_0 = -4/3,\ y_{-2} = 5,\ y_{-1} = 6/7,\ y_0 = 3$. Then, by Theorem 3, (3) has eventually three-periodic solutions described as Figures 7 and 8.

Figure 2 Plot of $y(n)$ (see online version for colours)

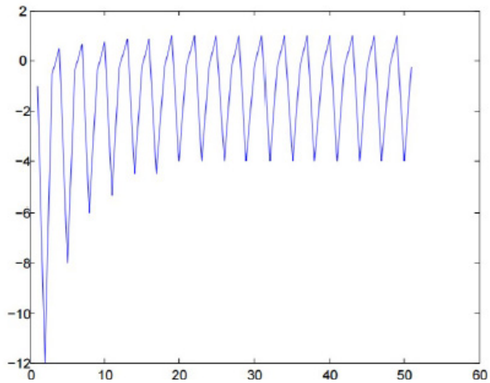


Figure 3 Plot of $x(n)$ (see online version for colours)

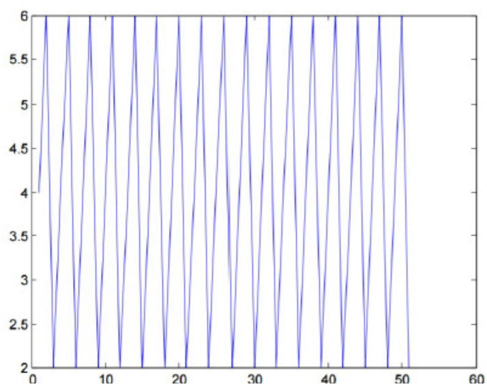


Figure 4 Plot of $y(n)$ (see online version for colours)

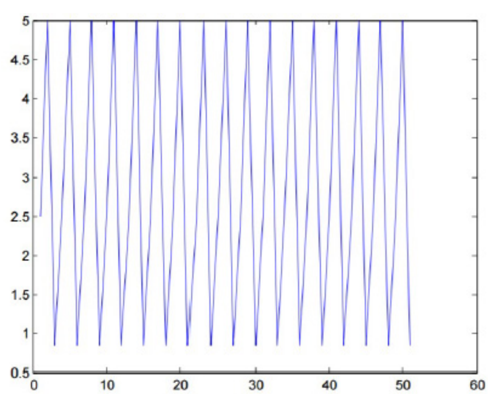


Figure 5 Plot of $x(n)$ (see online version for colours)

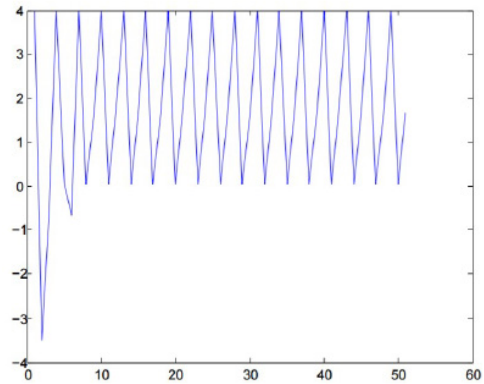


Figure 6 Plot of $y(n)$ (see online version for colours)

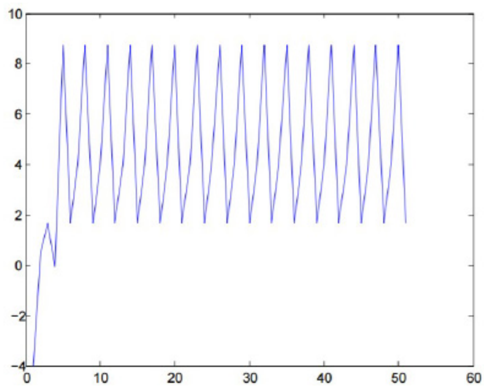
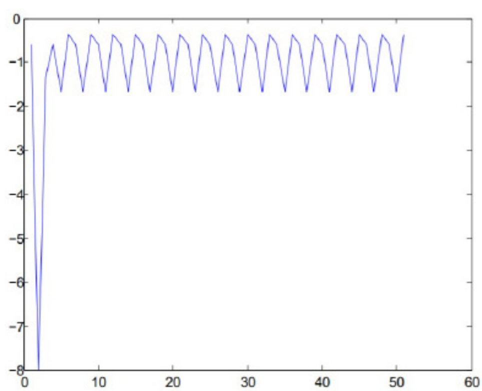


Figure 7 Plot of $x(n)$ (see online version for colours)



Example 3": Let $A = 3$, $x_{-2} = 2$, $x_{-1} = -5$, $x_0 = -6$, $y_{-2} = 1$, $y_{-1} = -4$, $y_0 = -8$. Then, by Theorem 3, (3) has eventually three-periodic solutions described as Figures 9 and 10.

Figure 8 Plot of $y(n)$ (see online version for colours)

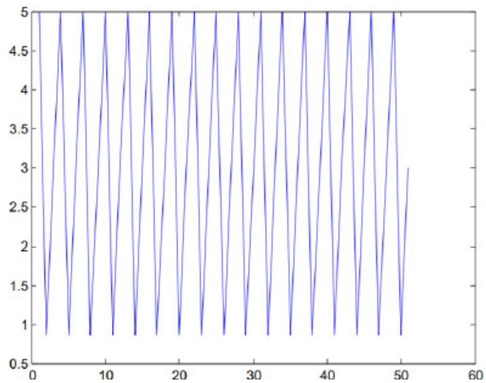


Figure 9 Plot of $x(n)$ (see online version for colours)

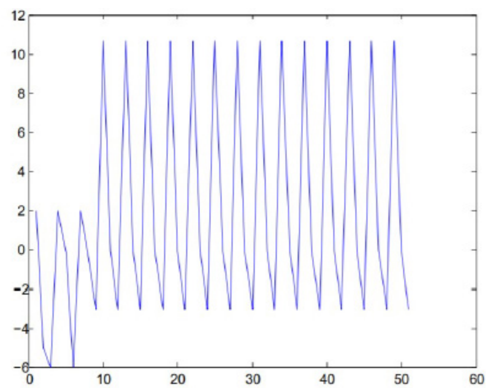
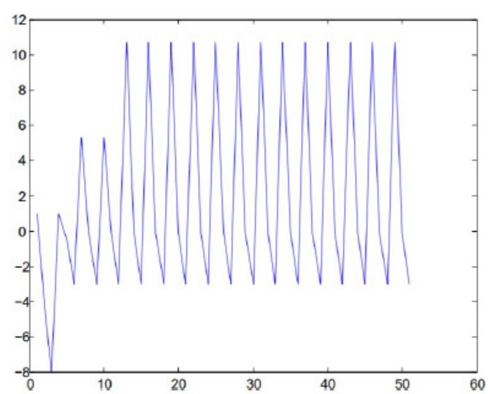


Figure 10 Plot of $y(n)$ (see online version for colours)



Example 4: Let $A = -1/3,\ x_{-2} = -3,\ x_{-1} = -5/2,\ x_0 = -7/2,\ y_{-2} = -2,\ y_{-1} = -6,\ y_0 = -3/2$. Then, by Theorem 4, (3) has eventually three-periodic solutions described as Figures 11 and 12.

Figure 11 Plot of $x(n)$ (see online version for colours)

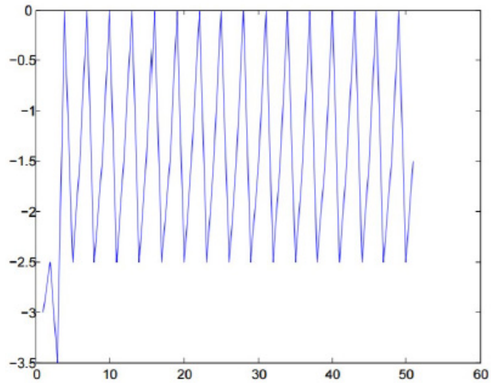
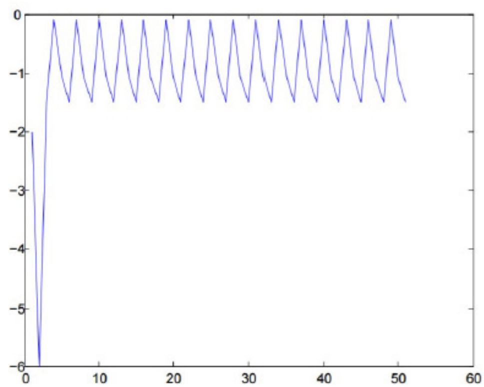


Figure 12 Plot of $y(n)$ (see online version for colours)



Example 5: Let $A = -1/2$, $x_{-2} = 2$, $x_{-1} = 3/2$, $x_0 = 2/3$, $y_{-2} = 1$, $y_{-1} = 7/3$, $y_0 = 4$. Then, by Theorem 5, (3) has eventually three-periodic solutions described as Figures 13 and 14.

Figure 13 Plot of $x(n)$ (see online version for colours)

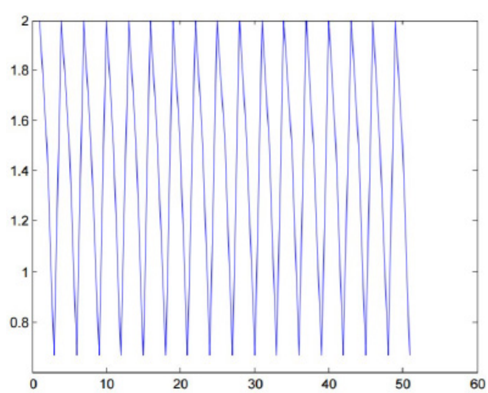
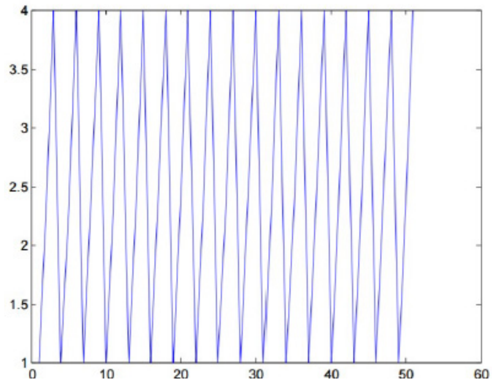
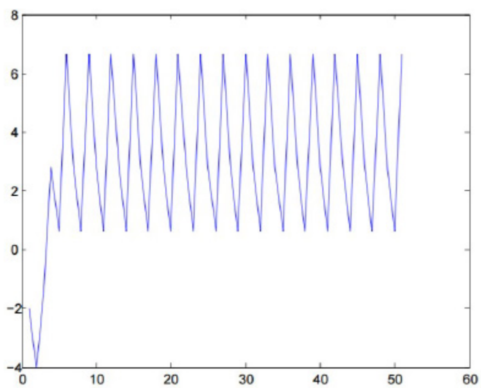


Figure 14 Plot of $y(n)$ (see online version for colours)



Example 6: Let $A = -5/3,\ x_{-2} = -2,\ x_{-1} = -4,\ x_0 = -3/2,\ y_{-2} = -7/5,\ y_{-1} = 2/5,\ y_0 = -1$. Then, by Theorem 6, (3) has eventually three-periodic solutions described as Figures 15 and 16.

Figure 15 Plot of $x(n)$ (see online version for colours)



Example 6': Let $A = -5,\ x_{-2} = 8/3,\ x_{-1} = -5/7,\ x_0 = 2,\ y_{-2} = -3,\ y_{-1} = -4,\ y_0 = 2$. Then, by Theorem 6, (3) has eventually three-periodic solutions described as Figures 17 and 18.

Example 6'': Let $A = -3,\ x_{-2} = -11,\ x_{-1} = -2,\ x_0 = -4,\ y_{-2} = 3,\ y_{-1} = -1/4,\ y_0 = -5$. Then, by Theorem 6, (3) has eventually three-periodic solutions described as Figures 19 and 20.

Figure 16 Plot of $y(n)$ (see online version for colours)

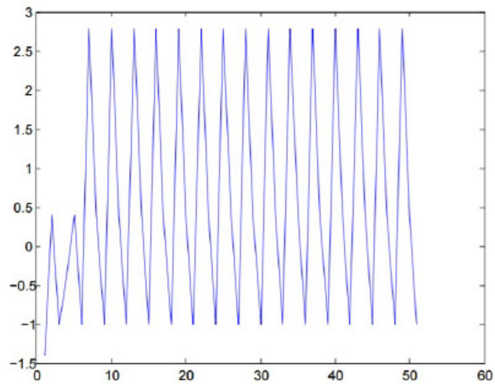


Figure 17 Plot of $x(n)$ (see online version for colours)

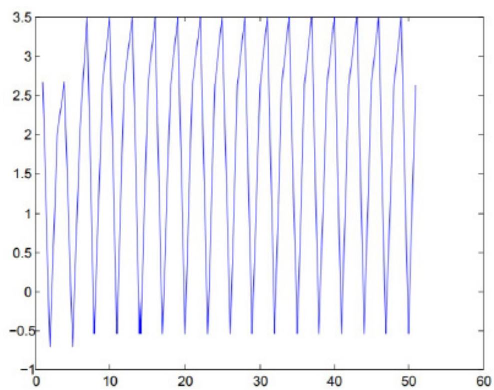


Figure 18 Plot of $y(n)$ (see online version for colours)

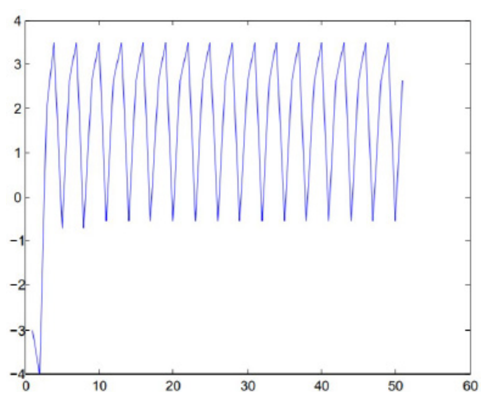


Figure 19 Plot of $x(n)$ (see online version for colours)

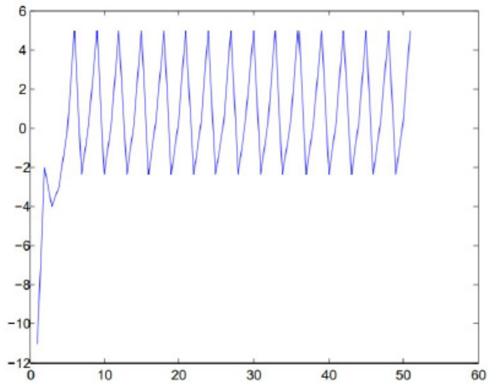
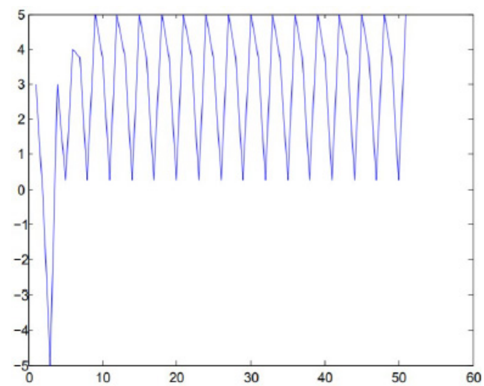


Figure 20 Plot of $y(n)$ (see online version for colours)



Acknowledgements

This paper was funded by the National Natural Science Foundation of China(11861059), Natural Science Foundation of Gansu Province(145RJZA232,145RJYA259) and Promotion Funds for Young Teachers in Northwest Normal University (NWNULKQN-18-16).

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