On the initial value problem of impulsive differential equation involving Caputo-Katugampola fractional derivative of order $q \in (1, 2)$

Xian-Min Zhang

School of Mathematics and Statistics,
Yangtze Normal University,
Fuling, Chongqing, 408100, China
Email: z6x2m@126.com
Email: XianminZhang@126.com

Abstract: This paper mainly focuses on the non-uniqueness of solution to the initial value problem (IVP) of impulsive fractional differential equations (IFrDE) with Caputo-Katugampola derivative (of order $q \in (1, 2)$). The system of impulsive higher order fractional differential equations may involve two different kinds of impulses, and the obtained result shows that its equivalent integral equations include two arbitrary constants, which means that its solution is non-unique. Next, two numerical examples are used to show the non-uniqueness of solution for the IVP of IFrDE.

Keywords: fractional differential equation; IFrDE; impulsive fractional differential equation; impulse; Caputo-Katugampola derivative; differential equations with impulses; initial value problems.

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Biographical notes: Xian-Min Zhang received his PhD from the School of Automation of Chongqing University in 2011. He worked at the Jiujiang University in 2011–2017, and works at the Yangtze Normal University since 2018.

1 Introduction

Since fractional calculus has been put forward, there appeared several fractional derivatives: Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov etc., see Podlubny (1999), Kilbas et al. (2006). And for fractional differential systems, some progresses were obtained in Mellin transformation, numerical calculation, controllability, existence of solution, Chaos synchronisation, stability etc. see Butzer et al. (2002a, 2002b), He (2016), Wu and Baleanu (2014), Kailasavalli et al. (2016), Suganya et al. (2016), Wu and Baleanu (2014), Wu et al. (2016), Ben Makhlouf (2018), Naifar et al. (2019). To unify these fractional derivatives, some generalised fractional operators were presented in Kilbas et al. (2006), Kiryakova (1994). Recently, a new type of fractional operators was defined by generalising both the Riemann-Liouville and Hadamard fractional operators in Katugampola (2011, 2014). Then a Caputo-type fractional derivative was presented for this new fractional operator in Jarad et al. (2017), and some basic properties of the Caputo-type fractional differential systems were studied in Zeng et al. (2017), Almeida et al. (2016), Ben Makhlouf and Nagy (2018), Boucenna et al. (2018).

On the other hand, as a key tool to characterising impulsive effects, the subject of impulsive (fractional) differential equations is getting an enormous amount of attention, see Lakshmikantham et al. (1989), Agarwal et al. (2016), Wang et al. (2016), Zhang et al. (2014), Zhang (2015a, 2015b, 2016), Stamova and Stamov (2014), Abbas and Benchohra (2010), Guo and Zhang (2015) and Fan (2014). Moreover, the impulsive differential equations with Caputo-Katugampola fractional derivative of order $q \in (0,1)$ were recently researched in Zhang (2019). Therefore, we will further consider the initial value problem (IVP) for the system of impulsive fractional differential equations (IFrDE) with Caputo-Katugampola derivative of order $q \in (1,2)$:

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t, x(t)),\ t \in (a,T], t \neq t_i\ (i = 1,\cdots,m) \\ \qquad\qquad\qquad\qquad \text{and}\ t \neq \overline{t}_j\ (j = 1,\cdots,n), \\ \Delta x(t)\big|_{t=t_i} = x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)),\ i = 1,\cdots,m, \\ \Delta x'(t)\big|_{t=\overline{t}_j} = x'(\overline{t}_j^+) - x'(\overline{t}_j^-) = J_j(x(\overline{t}_j^-)),\ j = 1,\cdots,n, \\ x(a) = x_a, x'(a) = \overline{x}_a,\ x_a, \overline{x}_a \in \mathbb{R}. \end{cases} \tag{1.1}$$

where ${}_a^C\mathcal{D}_t^{q,\rho}$ (where $a, \rho > 0$ and $q \in (1,2)$) denotes the left-sided Caputo-Katugampola fractional derivative of order q, $g : [a,T] \times \mathbb{R} \to \mathbb{R}$, $I_i : \mathbb{R} \to \mathbb{R}$ ($i = 1,2,\cdots,m$) and $J_j : \mathbb{R} \to \mathbb{R}$ ($j = 1,2,\cdots,n$) are some appropriate continuous functions, impulsive points satisfy $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ and $a = \overline{t}_0 < \overline{t}_1 < \cdots < \overline{t}_n < \overline{t}_{n+1} = T$. Moreover, $x(t_i^+) = \lim_{\varepsilon \to 0^+} x(t_i + \varepsilon)$ and $x(t_i^-) = \lim_{\varepsilon \to 0^-} x(t_i + \varepsilon)$ represent the right and left limits of $x(t)$ at $t = t_i$, respectively, and $x'(\overline{t}_j^+)$ and $x'(\overline{t}_j^-)$ have similar meaning for $x'(t)$ at $t = \overline{t}_j$.

Suppose that impulsive points $t_1, t_2, \cdots, t_m, \overline{t}_1, \overline{t}_2, \cdots, \overline{t}_n$ satisfy $\{t_1, \cdots, t_m, \overline{t}_1, \cdots, \overline{t}_n\} = \{t_1', t_2', \cdots, t_M'\}$ with $a = t_0' < t_1' < \cdots < t_M' < t_{M+1}' = T$. For each $[a, t_k']$ $(k = 1,2,\cdots,M)$, assume $[a, t_{k_1}] \subseteq [a, t_k'] \subset [a, t_{k_1+1}]$ (where $k_1 \in \{1,2,\cdots,m\}$) and $[a, \overline{t}_{k_2}] \subseteq [a, t_k'] \subset [a, \overline{t}_{k_2+1}]$ (where $k_2 \in \{1,2,\cdots,n\}$), respectively.

In particular, (1.1) can be simplified into the following system:

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t, x(t)),\ t \in (a,T]\ \text{and}\ t \neq t_i\ (i = 1,2,\cdots,m), \\ \Delta x(t)\big|_{t=t_i} = x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)),\ i = 1,2,\cdots,m, \\ \Delta x'(t)\big|_{t=t_i} = x'(t_i^+) - x'(t_i^-) = J_i(x(t_i^-)),\ i = 1,2,\cdots,m, \\ x(a) = x_a, x'(a) = \overline{x}_a,\quad x_a, \overline{x}_a \in \mathbb{R}. \end{cases} \tag{1.2}$$

Furthermore, letting $J_j(x(\bar{t}_j^-)) \to 0$ for all $j \in \{1,2,\cdots,n\}$ and $I_i(x(t_i^-)) \to 0$ for all $i \in \{1,2,\cdots,m\}$ in (1.1), we get three simple systems as

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t,x(t)),\ t \in (a,T] \text{ and } t \neq t_i\ (i=1,2,\cdots,m), \\ \Delta x(t)\big|_{t=t_i} = x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)),\ i=1,2,\cdots,m, \\ x(a) = x_a, x'(a) = \bar{x}_a,\ x_a, \bar{x}_a \in \mathbb{R}. \end{cases} \tag{1.3}$$

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t,x(t)),\ t \in (a,T] \text{ and } t \neq \bar{t}_j\ (j=1,2,\cdots,n), \\ \Delta x'(t)\big|_{t=\bar{t}_j} = x'(\bar{t}_j^+) - x'(\bar{t}_j^-) = J_j(x(\bar{t}_j^-)),\ j=1,2,\cdots,n, \\ x(a) = x_a, x'(a) = \bar{x}_a,\ x_a, \bar{x}_a \in \mathbb{R}. \end{cases} \tag{1.4}$$

and

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t,x(t)),\ t \in (a,T], \\ x(a) = x_a, x'(a) = \bar{x}_a,\ x_a, \bar{x}_a \in \mathbb{R}. \end{cases} \tag{1.5}$$

For the solution of (1.1) and (1.3)-(1.5), there exist some hidden conditions:

(i) $\lim_{J_j(x(\bar{t}_j^-)) \to 0 \text{ for all } j \in \{1,\cdots,n\}} \{\text{the solution of (1.1)}\} = \{\text{the solution of (1.3)}\}$.

(ii) $\lim_{I_i(x(t_i^-)) \to 0 \text{ for all } i \in \{1,\cdots,m\}} \{\text{the solution of (1.1)}\} = \{\text{the solution of (1.4)}\}$.

(iii)
$$\begin{aligned} &\lim_{\substack{I_i(x(t_i^-)) \to 0 \text{ for all } i \in \{1,\cdots,m\} \\ J_j(x(\bar{t}_j^-)) \to 0 \text{ for all } j \in \{1,\cdots,n\}}} \{\text{the solution of (1.1)}\} = \{\text{the solution of (1.5)}\} \\ &= \lim_{J_j(x(\bar{t}_j^-)) \to 0 \text{ for all } j \in \{1,\cdots,n\}} \{\text{the solution of (1.4)}\} \\ &= \lim_{I_i(x(t_i^-)) \to 0 \text{ for all } i \in \{1,\cdots,m\}} \{\text{the solution of (1.3)}\} \\ &\Leftrightarrow x(t) = x_a + a^{1-\rho}\bar{x}_a \frac{t^\rho - a^\rho}{\rho} + \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g\,d\tau}{\tau^{1-\rho}},\ \text{for } t \in [a,T]. \end{aligned}$$

To seek the equivalent integral equation of (1.1), we will firstly consider the transformed impulsive system:

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t,x(t)),\ t \in (a,T], t \neq t_i\ (i=1,\cdots,m) \\ \qquad\qquad \text{and } t \neq \bar{t}_j\ (j=1,\cdots,n), \\ \Delta x(t)\big|_{t=t_i} = x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)),\ i=1,\cdots,m, \\ \Delta \gamma x(t)\big|_{t=\bar{t}_j} = \gamma x(\bar{t}_j^+) - \gamma x(\bar{t}_j^-) = \bar{J}_j(x(\bar{t}_j^-)),\ j=1,\cdots,n, \\ x(a) = x_a, \gamma x(a) = \hat{x}_a,\ x_a, \hat{x}_a \in \mathbb{R}. \end{cases} \tag{1.6}$$

where differential operator $\gamma = t^{1-\rho} \frac{d}{dt}$.

Remark 1.1: Because of $\gamma x(a)=a^{1-\rho}x'(a)$ and $\gamma x(\overline{t}_j^{+})-\gamma x(\overline{t}_j^{-})=(\overline{t}_j)^{1-\rho}\left[x'(\overline{t}_j^{+})-x'(\overline{t}_j^{-})\right]$, (1.6) is identical to (1.1) under $\hat{x}_a=a^{1-\rho}\overline{x}_a$ and $\overline{J}_j(x(\overline{t}_j^{-}))=(\overline{t}_j)^{1-\rho}J_j(x(\overline{t}_j^{-}))$ for all $j\in\{1,2,\cdots,n\}$.

Similarly, letting $\overline{J}_j(x(\overline{t}_j^{-}))\to 0$ for all $j\in\{1,2,\cdots,n\}$ and $I_i(x(t_i^{-}))\to 0$ for all $i\in\{1,2,\cdots,m\}$ in (1.6), three simple systems are gotten as

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho}x(t)=g(t,x(t)),\ t\in(a,T]\,\text{and}\,t\neq t_i\ (i=1,2,\cdots,m), \\ \Delta x(t)\big|_{t=t_i}=x(t_i^{+})-x(t_i^{-})=I_i(x(t_i^{-})),\ i=1,2,\cdots,m, \\ x(a)=x_a,\gamma x(a)=\hat{x}_a,\ x_a,\hat{x}_a\in\mathbb{R}. \end{cases} \tag{1.7}$$

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho}x(t)=g(t,x(t)),\ t\in(a,T]\,\text{and}\ t\neq\overline{t}_j\ (j=1,2,\cdots,n), \\ \Delta\gamma x(t)\big|_{t=\overline{t}_j}=\gamma x(\overline{t}_j^{+})-\gamma x(\overline{t}_j^{-})=\overline{J}_j(x(\overline{t}_j^{-})),\ j=1,2,\cdots,n, \\ x(a)=x_a,\gamma x(a)=\hat{x}_a,\ x_a,\hat{x}_a\in\mathbb{R}. \end{cases} \tag{1.8}$$

and

$$\begin{cases} {}_a^C\mathcal{D}_t^{q,\rho}x(t)=g(t,x(t)),\ t\in(a,T], \\ x(a)=x_a,\gamma x(a)=\hat{x}_a,\ x_a,\hat{x}_a\in\mathbb{R}. \end{cases} \tag{1.9}$$

Thus, some hidden conditions for the solution of systems (1.6)–(1.9) are given as:

(iv) $\lim\limits_{\overline{J}_j(x(\overline{t}_j^{-}))\to 0\text{ for all }j\in\{1,\cdots,n\}}\{\text{the solution of (1.6)}\}=\{\text{the solution of (1.7)}\}$.

(v) $\lim\limits_{I_i(x(t_i^{-}))\to 0\text{ for all }i\in\{1,\cdots,m\}}\{\text{the solution of (1.6)}\}=\{\text{the solution of (1.8)}\}$.

(vi) $$\lim_{\substack{I_i(x(t_i^{-}))\to 0\text{ for all }i\in\{1,\cdots,m\}\\ \overline{J}_j(x(\overline{t}_j^{-}))\to 0\text{ for all }j\in\{1,\cdots,n\}}}\{\text{the solution of (1.6)}\}=\{\text{the solution of (1.9)}\}$$
$$=\lim_{I_i(x(t_i^{-}))\to 0\text{ for all }i\in\{1,\cdots,m\}}\{\text{the solution of (1.7)}\}$$
$$=\lim_{\overline{J}_j(x(\overline{t}_j^{-}))\to 0\text{ for all }j\in\{1,\cdots,n\}}\{\text{the solution of (1.8)}\}$$
$$\Leftrightarrow x(t)=x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}},\ \text{for}\ t\in[a,T].$$

The remainder of this paper is organised as follows: some definitions and conclusions will be introduced on the left-sided Caputo-Katugampola fractional derivative in Section 2, and the equivalent integral equations are presented for the IVP of IFrDE with the Caputo-Katugampola derivative in Section 3. Finally, two numerical examples are provided to show the main results in Section 4.

2 Preliminaries

First we will introduce the definition of left-sided generalised fractional calculus in Katugampola (2011) and Katugampola (2014) (in the sense of Katugampola).

Let $[a,b]$ ($-\infty \le a < b < \infty$) be a finite interval. Define the function space

$$X_c^p(a,b) = \left\{ f : [a,b] \to \mathbb{C} : \|f\|_{X_c^p} < \infty \right\} \; (a < b, c \in \mathbb{R}, 1 \le p \le \infty),$$

endowed with the norm $\|f\|_{X_c^p} = \left(\int_a^b \left| t^c f(t) \right|^p \frac{dt}{t} \right)^{1/p}$ $(1 \le p < \infty)$ and $\|f\|_{X_c^\infty} = \operatorname{ess\,sup}_{t \in [a,b]} \left[t^c \left| f(t) \right| \right]$.

Definition 2.1 (*Katugampola, 2011*): The generalised left fractional integral (in the sense of Katugampola) of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) of function $f \in X_c^p(a,b)$ are defined by

$$\left({}_a\mathcal{I}_t^{\alpha,\rho} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha - 1} f(s) \frac{ds}{s^{1-\rho}} \quad (t > a).$$

Definition 2.2 (*Katugampola, 2014*): The generalised left fractional derivative (in the sense of Katugampola) of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) of function $f \in X_c^p(a,b)$ are defined by

$$\begin{aligned} \left({}_a\mathcal{D}_t^{\alpha,\rho} f \right)(t) &= \gamma^n \left({}_a\mathcal{I}_t^{n-\alpha,\rho} f \right)(t) \\ &= \frac{\gamma^n}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha-1} f(s) \frac{ds}{s^{1-\rho}} \; (\rho > 0, t > a). \end{aligned}$$

Next, in order to give the definition of the Caputo-type generalised fractional derivative in Jarad et al. (2017), we introduce some correlative notations. Let $C[a,b]$ and $AC[a,b]$ denote the space of continuous functions and absolutely continuous functions on $[a,b]$, respectively. Two spaces are given as

$$AC_\gamma^n[a,b] = \left\{ f : [a,b] \to \mathbb{C} \text{ and } \gamma^{n-1} f \in AC[a,b], \gamma = t^{1-\rho} \frac{d}{dt} \right\}$$

with $AC_\gamma^1[a,b] = AC[a,b]$,

and

$$C_\gamma^n[a,b] = \left\{ f : [a,b] \to \mathbb{C} \text{ and } \gamma^{n-1} f \in C[a,b], \gamma^n f \in C[a,b], \gamma = t^{1-\rho} \frac{d}{dt} \right\}.$$

Definition 2.3 (*Jarad et al., 2017*): Let $\mathrm{Re}(\alpha) > 0$ and $n = \mathrm{Re}(\alpha) + 1$. If $f \in AC_\gamma^n[a,b]$, where $0 < a < b < \infty$, the left generalised Caputo fractional derivative of *f* of order α are defined as

$$ {}_{a}^{C}\mathcal{D}_{t}^{\alpha,\rho} f(t) = {}_{a}\mathcal{D}_{t}^{\alpha,\rho}\left[f(s) - \sum_{k=0}^{n-1} \frac{\gamma^k f(a)}{k!}\left(\frac{s^\rho - a^\rho}{\rho}\right)^k \right](t). $$

In case $0 < \mathrm{Re}(\alpha) < 1$, we have ${}_{a}^{C}\mathcal{D}_{t}^{\alpha,\rho} f(x) = {}_{a}\mathcal{D}_{t}^{\alpha,\rho}\left[f(s) - f(a)\right](t)$.

Theorem 2.4 *(Jarad et al., 2017)***:** *Let* $\mathrm{Re}(\alpha) > 0$, $n = \mathrm{Re}(\alpha) + 1$ *and* $f \in AC_\gamma^n[a,b]$, *where* $0 < a < b < \infty$. *Then,*

- if $\alpha \notin \mathbb{N}_0$,

$$ {}_{a}^{C}\mathcal{D}_{t}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1} \frac{(\gamma^n f)(s)ds}{s^{1-\rho}} = {}_{a}\mathcal{I}_{t}^{n-\alpha,\rho}\left(\gamma^n f\right)(t). $$

- if $\alpha \in \mathbb{N}$,

$$ {}_{a}^{C}\mathcal{D}_{t}^{\alpha,\rho} f = \gamma^n f. $$

Theorem 2.5 *(Jarad et al., 2017)***:** *Let* $f \in AC_\gamma^n[a,b]$ *or* $C_\gamma^n[a,b]$ *and* $\alpha \in \mathbb{C}$ *(*$\mathrm{Re}(\alpha) > 0$*). Then*

$$ {}_{a}\mathcal{I}_{t}^{\alpha,\rho}\, {}_{a}^{C}\mathcal{D}_{t}^{\alpha,\rho} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(\gamma^k f)(a)}{k!}\left(\frac{t^\rho - a^\rho}{\rho}\right)^k. $$

In particular, if $0 < \alpha \le 1$, we have ${}_{a}\mathcal{I}_{t}^{\alpha,\rho}\, {}_{a}^{C}\mathcal{D}_{t}^{\alpha,\rho} f(t) = f(t) - f(a)$.

Theorem 2.6 *(Zhang, 2019)***:** *Let* $q \in (0,1)$ *and* $a, \rho > 0$. *The IVP of IFrDE with Caputo-Katugampola derivative*

$$ \begin{cases} {}_{a}^{C}\mathcal{D}_{t}^{q,\rho} z(t) = g(t, z(t)),\ t \in (a,T] \text{ and } t \ne t_i\ (i = 1,2,\cdots,m), \\ \Delta z(t)\big|_{t=t_i} = z(t_i^+) - z(t_i^-) = J_i(z(t_i^-)) \in \mathbb{R},\ i = 1,2,\cdots,m, \\ z(a) = z_a \in \mathbb{R}. \end{cases} \tag{2.1} $$

is equivalent with the integral equation

$$ z(t) = \begin{cases} z_a + \dfrac{1}{\Gamma(q)}\displaystyle\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}}, \ \text{for } t \in [a, t_1], \\ z_a + \displaystyle\sum_{i=1}^{k} J_i(z(t_i^-)) + \frac{1}{\Gamma(q)}\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\ \quad + \xi \displaystyle\sum_{i=1}^{k} \frac{J_i(z(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho}\right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_i}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\ \quad \left. - \displaystyle\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right], \ \text{for } t \in (t_k, t_{k+1}], k = 1,2,\cdots,m, \end{cases} \tag{2.2} $$

provided that the integral in (2.2) exists, where ξ is an arbitrary constant and $g = g(\tau, z(\tau))$.

3 The equivalent integral equation of the VIP of IFrDE

For brevity let $g = g(\tau, x(\tau))$ and $\sum_{i=1}^{0} z_i = 0$, $\ell_0 = [a, t_1]$ and $\ell_i = (t_i, t_{i+1}]$ $(i = 1, 2, \cdots, m)$, $\overline{\ell}_0 = [a, \overline{t}_1]$ and $\overline{\ell}_j = (\overline{t}_j, \overline{t}_{j+1}]$ ($j = 1, 2, \cdots, n$), $\ell'_0 = [a, t'_1]$ and $\ell'_k = (t'_k, t'_{k+1}]$ ($k = 1, 2, \ldots, M$) in this section.

For the IVP of the IFrDE (1.1) (or (1.6)), we define function space:

$$
\begin{aligned}
IC([a,T],\mathbb{R}) := \Big\{ & x : [a,T] \to \mathbb{R} : x \in C_\gamma^2(\ell'_k)\ (\text{here } k = 0, 1, \cdots M), \\
& x(t_i) = x(t_i^-) = \lim_{t \uparrow t_i} x(t) < \infty, x(t_i^+) = \lim_{t \downarrow t_i} x(t) < \infty \\
& (i = 1, \cdots, m), x'(\overline{t}_j) = x'(\overline{t}_j^-) = \lim_{t \uparrow \overline{t}_j} x'(t) < \infty \\
& \text{and } x'(\overline{t}_j^+) = \lim_{t \downarrow \overline{t}_j} x'(t) < \infty \ \ (\text{here } j = 1, 2, \cdots, n) \Big\}.
\end{aligned}
$$

Thus, we give the definition of the solution for the IVP of the IFrDE (1.1) as follows:

Definition 3.1: A function $x(t) \in IC([a,T],\mathbb{R})$ be a solution of (1.1) if $x(a) = x_a$ and $x'(a) = \overline{x}_a$, the equation condition ${}_{a}^{C}\mathcal{D}_t^{q,\rho} x(t) = g(t, x(t))$ for each $t \in \ell'_k$ (where $k = 0, 1, 2, \cdots, M$) is verified, the impulsive conditions $\Delta x|_{t=t_i} = I_i(x(t_i^-))$ ($i = 1, 2, \cdots, m$) and $\Delta x'(t)|_{t=\overline{t}_j} = J_j(x(\overline{t}_j^-))$ ($j = 1, 2, \cdots, n$) are satisfied, and the hidden conditions (i)-(iii) hold.

Remark 3.2: Just as Definition 3.1, the definition of solution can be presented for the others VIP of IFrDE in the section of introduction.

To seek the equivalent integral equations for all IFrDEs in the section of introduction, we will first consider the simplest IFrDE (1.7) and define a piecewise integral function as

$$
\tilde{x}(t) = \begin{cases} x_a + \gamma x(a) \dfrac{t^\rho - a^\rho}{\rho} + \dfrac{1}{\Gamma(q)} \displaystyle\int_a^t \left(\dfrac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \dfrac{g d\tau}{\tau^{1-\rho}}, & \text{for } t \in \ell_0, \\ x(t_k^+) + \gamma x(t_k^+) \dfrac{t^\rho - (t_k)^\rho}{\rho} + \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_k}^t \left(\dfrac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \dfrac{g d\tau}{\tau^{1-\rho}}, & \text{for } t \in \ell_k, \end{cases} \tag{3.1}
$$

with $x(t_k^+) = x(t_k^-) + I_k(x(t_k^-))$ and $\gamma x(t_k^+) = \gamma x(t_k^-)$ (where $k = 1, 2, \ldots, m$).

It is sure that (3.1) satisfies three kinds of conditions (initial value, impulses and fractional derivative conditions) in the impulsive system (1.7). However, (3.1) dissatisfies the condition (vi), which means that (3.1) is not a solution of (1.7). Thus, (3.1) will be only considered as *the approximate solution* to seek the exact solution of (1.7).

Lemma 3.3: *Let* $q\in(1,2)$ *and* $a,\rho>0$. *Let* $g:[a,T]\times\mathbb{R}\to\mathbb{R}$ *be a function such that* $g((\cdot),x(\cdot))\in AC(\ell_i)$ *for any* $x(\cdot)\in AC(\ell_i)$ *(where* $i=0,1,\cdots,m$ *). If* $x(t)\in IC([a,T],\mathbb{R})$ *is a solution of (1.7) if and only if* $x(t)$ *satisfies the integral equations*

$$\begin{aligned} x(t)=&x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+\sum_{i=1}^{k}I_i(x(t_i^-))+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\\ &+\xi\sum_{i=1}^{k}\frac{I_i(x(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.\\ &\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_i)^\rho]}{\rho}\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],\\ &\text{for } t\in\ell_k, k=0,1,\cdots,m, \end{aligned} \tag{3.2}$$

where ξ *is an arbitrary constant.*

Proof. First we prove the sufficiency by mathematical induction. By applying Theorem 2.5, the solution of (1.7) as $t\in\ell_0$ satisfies

$$x(t)=x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}},\ \text{for } t\in\ell_0. \tag{3.3}$$

Then by (3.3) we have

$$\begin{aligned} x(t_1^+)&=x(t_1^-)+I_1(x(t_1^-))\\ &=x_a+\hat{x}_a\frac{(t_1)^\rho-a^\rho}{\rho}+I_1(x(t_1^-))+\frac{1}{\Gamma(q)}\int_a^{t_1}\left(\frac{(t_1)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}, \end{aligned}$$

and

$$\begin{aligned} \gamma x(t_1^+)&=\gamma x(t_1^-)\\ &=\hat{x}_a+\frac{1}{\Gamma(q-1)}\int_a^{t_1}\left(\frac{(t_1)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}. \end{aligned}$$

Therefore the approximate solution $\tilde{x}(t)$ as $t\in\ell_1$ can be calculated as

$$\begin{aligned}\tilde{x}(t) &= x(t_1^+) + \gamma x(t_1^+)\frac{t^\rho - (t_1)^\rho}{\rho} + \frac{1}{\Gamma(q)}\int_{t_1}^{t}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}} \\ &= x_a + \hat{x}_a\frac{t^\rho - a^\rho}{\rho} + I_1(x(t_1^-)) \\ &\quad + \frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\frac{(t_1)^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}} + \int_{t_1}^{t}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}}\right] \\ &\quad + \frac{t^\rho - (t_1)^\rho}{\rho\Gamma(q-1)}\int_a^{t_1}\left(\frac{(t_1)^\rho - \tau^\rho}{\rho}\right)^{q-2}\frac{g d\tau}{\tau^{1-\rho}}, \text{ for } t \in \ell_1.\end{aligned} \tag{3.4}$$

Define $e_k(t) = x(t) - \tilde{x}(t)$ for $t \in \ell_k$ (where $k = 1, 2, \ldots, m$), which denote the error between the approximate solution $\tilde{x}(t)$ and the exact solution of (1.7) as $t \in \ell_k$.

By (3.3), the exact solution of (1.7) as $t \in \ell_1$ satisfies

$$\lim_{I_1(x(t_1^-))\to 0} x(t) = x_a + \gamma x(a)\frac{t^\rho - a^\rho}{\rho} + \frac{1}{\Gamma(q)}\int_a^{t}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}}, \text{ for } t \in \ell_1. \tag{3.5}$$

By (3.4) and (3.5) we get

$$\begin{aligned}\lim_{I_1(x(t_1^-))\to 0} e_1(t) &= \lim_{I_1(x(t_1^-))\to 0}\{x(t) - \tilde{x}(t)\} \\ &= \frac{1}{\Gamma(q)}\left[\int_a^{t}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}} - \int_a^{t_1}\left(\frac{(t_1)^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}}\right. \\ &\quad \left. - \int_{t_1}^{t}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}} - \frac{(q-1)[t^\rho - (t_1)^\rho]}{\rho}\int_a^{t_1}\left(\frac{(t_1)^\rho - \tau^\rho}{\rho}\right)^{q-2}\frac{g d\tau}{\tau^{1-\rho}}\right].\end{aligned} \tag{3.6}$$

Because $e_1(t)$ is connected with $I_1(x(t_1^-))$ and $\lim_{I_1(x(t_1^-))\to 0} e_1(t)$ from (3.6), we make an assumption that

$$\begin{aligned}e_1(t) &= \kappa(I_1(x(t_1^-)))\lim_{I_1(x(t_1^-))\to 0} e_1(t) \\ &= \frac{\kappa(I_1(x(t_1^-)))}{\Gamma(q)}\left[\int_a^{t}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}} - \int_a^{t_1}\left(\frac{(t_1)^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}}\right. \\ &\quad \left. - \int_{t_1}^{t}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{q-1}\frac{g d\tau}{\tau^{1-\rho}} - \frac{(q-1)[t^\rho - (t_1)^\rho]}{\rho}\int_a^{t_1}\left(\frac{(t_1)^\rho - \tau^\rho}{\rho}\right)^{q-2}\frac{g d\tau}{\tau^{1-\rho}}\right],\end{aligned} \tag{3.7}$$

where $\kappa(\cdot)$ is an undetermined function with $\kappa(0) = 1$. According to (3.4) and (3.7), we get

$$
\begin{aligned}
x(t) &= \tilde{x}(t) + e_1(t) \\
&= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&+ \frac{1 - \kappa(I_1(x(t_1^-)))}{\Gamma(q)} \left[\int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_1}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&- \left. \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_1)^\rho]}{\rho} \int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right], \\
&\text{for } t \in \ell_1.
\end{aligned} \tag{3.8}
$$

Then by (3.8) we obtain

$$
\begin{aligned}
x(t_2^+) &= x(t_2^-) + I_2(x(t_2^-)) \\
&= x_a + \hat{x}_a \frac{(t_2)^\rho - a^\rho}{\rho} + \sum_{i=1}^{2} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&+ \frac{1 - \kappa(I_1(x(t_1^-)))}{\Gamma(q)} \left[\int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_1}^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&- \left. \int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[(t_2)^\rho - (t_1)^\rho]}{\rho} \int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right],
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma x(t_2^+) &= \gamma x(t_2^-) \\
&= \hat{x}_a + \frac{1}{\Gamma(q-1)} \int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} + \frac{1 - \kappa(I_1(x(t_1^-)))}{\Gamma(q-1)} \\
&\cdot \left[\int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_1}^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} - \int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right].
\end{aligned}
$$

Thus the approximate solution $\tilde{x}(t)$ as $t \in \ell_2$ is given as

$$
\begin{aligned}
\tilde{x}(t) &= x(t_2^+) + \gamma x(t_2^+) \frac{t^\rho - (t_2)^\rho}{\rho} + \frac{1}{\Gamma(q)} \int_{t_2}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{i=1}^{2} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \left[\int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&+ \left. \int_{t_2}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_2)^\rho]}{\rho} \int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right]
\end{aligned}
$$

$$+\frac{1-\kappa(I_1(x(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_1}\left(\frac{(t_1)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_1}^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.-\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[(t_2)^\rho-(t_1)^\rho]}{\rho}\int_a^{t_1}\left(\frac{(t_1)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right] \quad (3.9)$$
$$+\frac{[1-\kappa(I_1(x(t_1^-)))][t^\rho-(t_2)^\rho]}{\rho\Gamma(q-1)}\left[\int_a^{t_1}\left(\frac{(t_1)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.+\int_{t_1}^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],\quad \text{for } t\in \ell_2.$$

On the other hand, by (3.8) the exact solution $x(t)$ of (1.7) as $t\in \ell_2$ satisfies two conditions:

$$\lim_{I_1(x(t_1^-))\to 0} x(t)$$
$$=x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+I_2(x(t_2^-))+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\frac{1-\kappa(I_2(x(t_2^-)))}{\Gamma(q)}\left[\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_2}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right. \quad (3.10)$$
$$\left.-\int_a^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_2)^\rho]}{\rho}\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$
for $t\in \ell_2$,

and

$$\lim_{I_2(x(t_2^-))\to 0} x(t)$$
$$=x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+I_1(x(t_1^-))+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\frac{1-\kappa(I_1(x(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_1}\left(\frac{(t_1)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_1}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right. \quad (3.11)$$
$$\left.-\int_a^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_1)^\rho]}{\rho}\int_a^{t_1}\left(\frac{(t_1)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$
for $t\in \ell_2$.

Thus, applying (3.9)–(3.11), the error $e_2(t)$ satisfies the following two conditions:

$$\lim_{I_1(x(t_1^-))\to 0} e_2(t) = \lim_{I_1(x(t_1^-))\to 0} \{x(t)-\tilde{x}(t)\}$$

$$= -\frac{\kappa(I_2(x(t_2^-)))}{\Gamma(q)}\left[\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_2}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right. \tag{3.12}$$

$$\left.-\int_a^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_2)^\rho]}{\rho}\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$

and

$$\lim_{I_2(x(t_2^-))\to 0} e_2(t) = \lim_{I_2(x(t_2^-))\to 0} \{x(t)-\tilde{x}(t)\}$$

$$= \frac{-\kappa(I_1(x(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_2}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$

$$\left.-\int_a^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_2)^\rho]}{\rho}\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right] \tag{3.13}$$

$$-\frac{1-\kappa(I_1(x(t_1^-)))}{\Gamma(q)}\left[\int_{t_1}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_1}^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$

$$\left.-\int_{t_2}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_2)^\rho]}{\rho}\int_{t_1}^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right].$$

Therefore, by (3.12)-(3.13), we have

$$e_2(t) = \frac{1-\kappa(I_1(x(t_1^-)))-\kappa(I_2(x(t_2^-)))}{\Gamma(q)}$$

$$\cdot\left[\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_2}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$

$$\left.-\int_a^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_2)^\rho]}{\rho}\int_a^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right] \tag{3.14}$$

$$-\frac{1-\kappa(I_1(x(t_1^-)))}{\Gamma(q)}\left[\int_{t_1}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_1}^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$

$$\left.-\int_{t_2}^{t}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_2)^\rho]}{\rho}\int_{t_1}^{t_2}\left(\frac{(t_2)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$

for $t\in \ell_2$.

Thus, using (3.9) and (3.14) we obtain

$$\begin{aligned} x(t) &= \tilde{x}(t) + e_2(t) \\ &= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{i=1}^{2} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\ &\quad + \frac{1-\kappa(I_1(x(t_1^-)))}{\Gamma(q)} \left[\int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_1}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\ &\quad \left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_1)^\rho]}{\rho} \int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right] \\ &\quad + \frac{1-\kappa(I_2(x(t_2^-)))}{\Gamma(q)} \left[\int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_2}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\ &\quad \left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_2)^\rho]}{\rho} \int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right], \\ &\quad \text{for } t \in \ell_2. \end{aligned} \tag{3.15}$$

To determine function $\kappa(\cdot)$ in (3.8) and (3.15), consider a special case of (1.7):

$$\begin{aligned} \lim_{t_2 \to t_1} \begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t, x(t)),\ t \in (a, T], t \neq t_1, t_2, \\ \Delta x(t)\big|_{t=t_i} = I_i(x(t_i^-)),\ i = 1, 2, \\ x(a) = x_a, \gamma x(a) = \hat{x}_a. \end{cases} \\ = \begin{cases} {}_a^C\mathcal{D}_t^{q,\rho} x(t) = g(t, x(t)),\ t \in (a, T], t \neq t_1, \\ \Delta x(t)\big|_{t=t_1} = I_1(x(t_1^-)) + I_2(x(t_1^-)), \\ x(a) = x_a, \gamma x(a) = \hat{x}_a. \end{cases} \end{aligned} \tag{3.16}$$

Using (3.8) and (3.15) to both sides of (3.16), respectively, we have

$$1 - \kappa\left[I_1(x(t_1^-)) + I_2(x(t_1^-)) \right] = 1 - \kappa\left[I_1(x(t_1^-)) \right] + 1 - \kappa\left[I_2(x(t_1^-)) \right] \tag{3.17}$$

Then $\kappa(\cdot)$ satisfies $1-\kappa(z) = \xi z$ for $\forall z \in \mathbb{R}$, where ξ is an arbitrary constant. Thus we rewrite (3.8) and (3.15) into

$$\begin{aligned} x(t) &= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\ &\quad + \frac{\xi I_1(x(t_1^-))}{\Gamma(q)} \left[\int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_1}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\ &\quad \left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_1)^\rho]}{\rho} \int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right], \\ &\quad \text{for } t \in \ell_1, \end{aligned} \tag{3.18}$$

and

$$
\begin{aligned}
x(t) &= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{i=1}^{2} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&+ \frac{\xi I_1(x(t_1^-))}{\Gamma(q)} \left[\int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_1}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_1)^\rho]}{\rho} \int_a^{t_1} \left(\frac{(t_1)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right] \\
&+ \frac{\xi I_2(x(t_2^-))}{\Gamma(q)} \left[\int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_2}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_2)^\rho]}{\rho} \int_a^{t_2} \left(\frac{(t_2)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right], \\
&\text{for } t \in \ell_2.
\end{aligned} \tag{3.19}
$$

Next, for $t \in \ell_k$ we suppose that the solution of (1.7) satisfies

$$
\begin{aligned}
x(t) &= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&+ \xi \sum_{i=1}^{k} \frac{I_i(x(t_i^-))}{\Gamma(q)} \left[\int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&+ \int_{t_i}^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&\left. + \frac{(q-1)[t^\rho - (t_i)^\rho]}{\rho} \int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right], \text{ for } t \in \ell_k.
\end{aligned} \tag{3.20}
$$

Then applying (3.20) we get

$$
\begin{aligned}
x(t_{k+1}^+) &= x(t_{k+1}^-) + I_{k+1}(x(t_{k+1}^-)) = x_a + \hat{x}_a \frac{(t_{k+1})^\rho - a^\rho}{\rho} \\
&+ \sum_{i=1}^{k+1} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^{t_{k+1}} \left(\frac{(t_{k+1})^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&+ \xi \sum_{i=1}^{k} \frac{I_i(x(t_i^-))}{\Gamma(q)} \left[\int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&+ \int_{t_i}^{t_{k+1}} \left(\frac{(t_{k+1})^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} - \int_a^{t_{k+1}} \left(\frac{(t_{k+1})^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&\left. + \frac{(q-1)[(t_{k+1})^\rho - (t_i)^\rho]}{\rho} \int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right],
\end{aligned}
$$

and

$$\gamma x(t_{k+1}^{+})=\gamma x(t_{k+1}^{-})=\hat{x}_a+\frac{1}{\Gamma(q-1)}\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\xi\sum_{i=1}^{k}\frac{I_i(x(t_i^{-}))}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\frac{(t_i)^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.+\int_{t_i}^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right].$$

Thus the approximate solution $\tilde{x}(t)$ as $t\in \ell_{k+1}$ can be gotten as

$$\tilde{x}(t)=x(t_{k+1}^{+})+\gamma x(t_{k+1}^{+})\frac{t^{\rho}-(t_{k+1})^{\rho}}{\rho}+\frac{1}{\Gamma(q)}\int_{t_{k+1}}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}$$
$$=x_a+\hat{x}_a\frac{t^{\rho}-a^{\rho}}{\rho}+\sum_{i=1}^{k+1}I_i(x(t_i^{-}))+\frac{1}{\Gamma(q)}\left[\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.+\int_{t_{k+1}}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^{\rho}-(t_{k+1})^{\rho}]}{\rho}\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right]$$
$$+\xi\sum_{i=1}^{k}\frac{I_i(x(t_i^{-}))}{\Gamma(q)}\left[\int_{t_i}^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.\tag{3.21}$$
$$\left.+\int_a^{t_i}\left(\frac{(t_i)^{\rho}-\tau^{\rho}}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[(t_{k+1})^{\rho}-(t_i)^{\rho}]}{\rho}\int_a^{t_i}\left(\frac{(t_i)^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right]$$
$$+\frac{t^{\rho}-(t_{k+1})^{\rho}}{\rho}\sum_{i=1}^{k}\frac{\xi I_i(x(t_i^{-}))}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\frac{(t_i)^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.+\int_{t_i}^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^{\rho}-\tau^{\rho}}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right].$$

Moreover, using (3.20), we know that the exact solution of (1.7) as $t\in \ell_{k+1}$ satisfies

$$\lim_{I_r(x(t_r^{-}))\to 0\text{ for all }r\in\{1,\cdots,k+1\}}x(t)$$
$$=x_a+\hat{x}_a\frac{t^{\rho}-a^{\rho}}{\rho}+\frac{1}{\Gamma(q)}\int_a^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}},\ \text{for } t\in \ell_{k+1},\tag{3.22}$$

and

$$\lim_{I_r(x(t_r^-))\to 0 \text{ for } r\in\{1,2,\cdots,k+1\}} x(t)$$
$$= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{\substack{1\le i\le k+1\\ \text{and } i\neq r}} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)}\int_a^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\xi \sum_{\substack{1\le i\le k+1\\ \text{and } i\neq r}} \frac{I_i(x(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} + \int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right. \tag{3.23}$$
$$\left. -\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho-(t_i)^\rho]}{\rho}\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$
for $t\in \ell_{k+1}$.

Using (3.21)–(3.23), the error $e_{k+1}(t)$ satisfies the following conditions:

$$\lim_{\substack{I_r(x(t_r^-))\to 0 \text{ for}\\ \text{all } r\in\{1,2,\cdots,k+1\}}} e_{k+1}(t) = \lim_{\substack{I_r(x(t_r^-))\to 0 \text{ for}\\ \text{all } r\in\{1,2,\cdots,k+1\}}} \{x(t)-\tilde{x}(t)\}$$
$$=\frac{1}{\Gamma(q)}\left[\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} - \int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right. \tag{3.24}$$
$$\left. -\int_{t_{k+1}}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} - \frac{(q-1)[t^\rho-(t_{k+1})^\rho]}{\rho}\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$

and

$$\lim_{\substack{I_r(x(t_r^-))\to 0 \text{ for}\\ r\in\{1,2,\cdots,k+1\}}} e_{k+1}(t) = \lim_{\substack{I_r(x(t_r^-))\to 0 \text{ for}\\ r\in\{1,2,\cdots,k+1\}}} \{x(t)-\tilde{x}(t)\}$$
$$=\xi \sum_{\substack{1\le i\le k+1\\ \text{and } i\neq r}} \frac{I_i(x(t_i^-))}{\Gamma(q)}\left[\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} - \int_{t_i}^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho-(t_{k+1})^\rho]}{\rho}\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}$$
$$\left.+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} - \frac{(q-1)[t^\rho-(t_{k+1})^\rho]}{\rho}\int_{t_i}^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right] \tag{3.25}$$
$$-\frac{1}{\Gamma(q)}\left[\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} + \int_{t_{k+1}}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho-(t_{k+1})^\rho]}{\rho}\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right].$$

By (3.24) and (3.25), we have

$$
\begin{aligned}
&e_{k+1}(t)\\
&=\xi\sum_{i=1}^{k+1}\frac{I_i(x(t_i^-))}{\Gamma(q)}\Bigg[\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}-\int_{t_i}^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\\
&\quad+\frac{(q-1)[t^\rho-(t_{k+1})^\rho]}{\rho}\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\\
&\quad+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}-\frac{(q-1)[t^\rho-(t_{k+1})^\rho]}{\rho}\int_{t_i}^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\Bigg] \qquad (3.26)\\
&\quad-\frac{1}{\Gamma(q)}\Bigg[\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_{k+1}}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\\
&\quad-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_{k+1})^\rho]}{\rho}\int_a^{t_{k+1}}\left(\frac{(t_{k+1})^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\Bigg].
\end{aligned}
$$

Therefore by (3.21) and (3.26), we obtain

$$
\begin{aligned}
&x(t)=\tilde{x}(t)+e_{k+1}(t)\\
&=x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+\sum_{i=1}^{k+1}I_i(x(t_i^-))+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\\
&\quad+\xi\sum_{i=1}^{k+1}\frac{I_i(x(t_i^-))}{\Gamma(q)}\Bigg[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\\
&\quad-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(t_i)^\rho]}{\rho}\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\Bigg],
\end{aligned}
$$

for $t\in \ell_{k+1}$.

Thus the solution of (1.7) satisfies (3.2) as $t\in \ell_{k+1}$, and the sufficiency is proved.

Now we prove the necessity. Letting $I_i(x(t_i^-))\to 0$ for all $i\in\{1,2,\cdots,m\}$ in (3.2), we get

$$
\begin{aligned}
&\lim_{I_i(x(t_i^-))\to 0 \text{ for all } i\in\{1,2,\cdots,m\}}\{\text{equations (3.2)}\}\\
&\Leftrightarrow x(t)=x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}},\ \text{for } t\in[a,T].\\
&\Leftrightarrow \lim_{I_i(x(t_i^-))\to 0 \text{ for all } i\in\{1,2,\cdots,m\}}\{\text{system (1.7)}\}.
\end{aligned}
$$

Next, for (3.2) as $t\in(t_k,t_{k+1}]$, we have

$$
\begin{aligned}
&{}_a^C\mathcal{D}_t^{q,\rho}\left\{x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+\sum_{i=1}^{k}I_i(x(t_i^-))+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.\\
&\quad+\xi\sum_{i=1}^{k}\frac{I_i(x(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.\\
&\quad\left.\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)\left[t^\rho-(t_i)^\rho\right]}{\rho}\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right]\right\}_{t\in\ell_k}\\
&=\left\{{}_a^C\mathcal{D}_t^{q,\rho}\left[\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right]+\xi\sum_{i=1}^{k}\frac{I_i(x(t_i^-))}{\Gamma(q)}\right.\\
&\quad\left.\cdot\left\{{}_{t_i}^C\mathcal{D}_t^{q,\rho}\left[\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right]-{}_a^C\mathcal{D}_t^{q,\rho}\left[\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right]\right\}\right\}_{t\in\ell_k}\\
&=\left\{g(t,x(t))\big|_{t>a}+\xi\sum_{i=1}^{k}I_i(x(t_i^-))\left[g(t,x(t))\big|_{t>t_i}-g(t,x(t))\big|_{t>a}\right]\right\}_{t\in\ell_k}\\
&=g(t,x(t))\big|_{t\in\ell_k}.
\end{aligned}
$$

Thus, (3.2) satisfies the fractional derivative condition in system (1.7).

Finally, we have $x(t_i^+)-x(t_i^-)=\lim_{t\to t_i^+}x(t)-x(t_i)=I_i(x(t_i^-))$ for $\forall i\in\{1,2,\cdots,m\}$ in (3.2). Thus, (3.2) satisfies all conditions of (1.7). As a result, the system (1.7) is equivalent to the integral equations (3.2). The proof is completed.

Similarly, we define the *approximate solution* of (1.8) as follows:

$$
\tilde{\tilde{x}}(t)=\begin{cases}x_a+\gamma x(a)\dfrac{t^\rho-a^\rho}{\rho}+\dfrac{1}{\Gamma(q)}\displaystyle\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}, & \text{for } t\in\overline{\ell}_0,\\[2ex] x(\overline{t}_l^{\,+})+\gamma x(\overline{t}_l^{\,+})\dfrac{t^\rho-(\overline{t}_l)^\rho}{\rho}+\dfrac{1}{\Gamma(q)}\displaystyle\int_{\overline{t}_l}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}, & \text{for } t\in\overline{\ell}_l,\end{cases}\tag{3.27}
$$

with $\gamma x(t_l^+)=\gamma x(t_l^-)+\overline{J}_l(x(t_l^-))$ and $x(\overline{t}_l^{\,+})=x(\overline{t}_l^{\,-})$, where $l=1,2,\cdots,n$.

Also, using the thought of Lemma 3.3, we can draw the following conclusion about the equivalent integral equations of (1.8).

Lemma 3.4: *Let* $q\in(1,2)$ *and* $a,\rho>0$. *Let* $g:[a,T]\times\mathbb{R}\to\mathbb{R}$ *be a function such that* $g((\cdot),x(\cdot))\in AC(\overline{\ell}_j)$ *for any* $x(\cdot)\in AC(\overline{\ell}_j)$ *(where* $j=0,1,\cdots,n$*). If* $x(t)\in IC([a,T],\mathbb{R})$ *is a solution of (1.8) if and only if* $x(t)$ *satisfies the integral equations*

$$x(t)=x_a+\hat{x}_a\frac{t^\rho-a^\rho}{\rho}+\sum_{j=1}^{l}\bar{J}_j(x(\bar{t}_j^-))\frac{t^\rho-(\bar{t}_j)^\rho}{\rho}+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\eta\sum_{j=1}^{l}\frac{\bar{J}_j(x(\bar{t}_j^-))}{\Gamma(q)}\left[\int_a^{\bar{t}_j}\left(\frac{(\bar{t}_j)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{\bar{t}_j}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.\tag{3.28}$$
$$\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\frac{(q-1)[t^\rho-(\bar{t}_j)^\rho]}{\rho}\int_a^{\bar{t}_j}\left(\frac{(\bar{t}_j)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$
for $t\in\bar{\ell}_l, l=0,1,\cdots,n,$

where η is an arbitrary constant.

Next applying Lemmas 3.3 and 3.4, we can draw the following two conclusions (Corollaries 3.5 and 3.6).

Corollary 3.5: *Let $q\in(1,2)$ and $a,\rho>0$. Let $g:[a,T]\times\mathbb{R}\to\mathbb{R}$ be a function such that $g((\cdot),x(\cdot))\in AC(\ell_i)$ for any $x(\cdot)\in AC(\ell_i)$ (where $i=0,1,\cdots,m$). If $x(t)\in IC([a,T],\mathbb{R})$ is a solution of (1.7), then*

$$\gamma x(t)=\hat{x}_a+\frac{1}{\Gamma(q-1)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\xi\sum_{i=1}^{k}\frac{I_i(x(t_i^-))}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.\tag{3.29}$$
$$\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],\ \text{for } t\in\ell_k, k=0,1,\cdots,m,$$

where ξ is an arbitrary constant.

Corollary 3.6: *Let $q\in(1,2)$ and $a,\rho>0$. Let $g:[a,T]\times\mathbb{R}\to\mathbb{R}$ be a function such that $g((\cdot),x(\cdot))\in AC(\bar{\ell}_j)$ for any $x(\cdot)\in AC(\bar{\ell}_j)$ (where $j=0,1,\ldots,n$). If $x(t)\in IC([a,T],\mathbb{R})$ is a solution of (1.8), then*

$$\gamma x(t)=\hat{x}_a+\sum_{j=1}^{l}\bar{J}_j(x(\bar{t}_j^-))+\frac{1}{\Gamma(q-1)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\eta\sum_{j=1}^{l}\frac{\bar{J}_j(x(\bar{t}_j^-))}{\Gamma(q-1)}\left[\int_a^{\bar{t}_j}\left(\frac{(\bar{t}_j)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}+\int_{\bar{t}_j}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.\tag{3.30}$$
$$\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],\ \text{for } t\in\bar{\ell}_l, l=0,1,\cdots,n,$$

where η is an arbitrary constant.

Remark 3.7: Corollaries 3.5 and 3.6 show that two inhomogeneous impulses $\Delta x|_{t=t_i}$ ($i=1,2,\ldots,m$) and $\Delta\gamma x(t)|_{t=\overline{t}_j}=\overline{J}_j(x(\overline{t}_j^-))$ ($j=1,2,\ldots,n$) have similar effect on $\gamma x(t)$ of (1.6).

Lemma 3.8: *Let $q\in(1,2)$ and $a,\rho>0$. Let $g:[a,T]\times\mathbb{R}\to\mathbb{R}$ be a function such that $g((\cdot),x(\cdot))\in AC(\ell_i')$ for any $x(\cdot)\in AC(\ell_i')$ (where $i=0,1,\cdots,M$). If $x(t)\in IC([a,T],\mathbb{R})$ is a solution of (1.6), then*

$$\begin{aligned}\gamma x(t)=&\hat{x}_a+\sum_{j=1}^{k_2}\overline{J}_j(x(\overline{t}_j^-))+\frac{1}{\Gamma(q-1)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\\&+\xi\sum_{i=1}^{k_1}\frac{I_i(x(t_i^-))}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.\\&\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right]+\eta\sum_{j=1}^{k_2}\frac{\overline{J}_j(x(\overline{t}_j^-))}{\Gamma(q-1)}\left[\int_a^{\overline{t}_j}\left(\frac{(\overline{t}_j)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.\\&\left.+\int_{\overline{t}_j}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],\\&\text{for } t\in\ell_k',k=0,1,\cdots,M,\end{aligned}\tag{3.31}$$

where ξ and η are two arbitrary constants.

Proof. For $t\in\ell_0'$, by Corollaries 3.4 and 3.5, we have

$$\gamma x(t)=\hat{x}_a+\frac{1}{\Gamma(q-1)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}},\ \text{for } t\in\ell_0'.$$

By Theorem 2.4, we have

system (1.6)

$$=\begin{cases}{}_a^C\mathcal{D}_t^{q-1,\rho}\left[\gamma\left(x(t)\right)\right]=g(t,x(t)),\ t\in(a,T],t\neq t_i\ (i=1,\cdots,m)\\ \qquad\qquad\text{and } t\neq\overline{t}_j\ (j=1,\cdots,n),\\ \Delta x(t)|_{t=t_i}=x(t_i^+)-x(t_i^-)=I_i(x(t_i^-)),\ i=1,2,\cdots,m,\\ \Delta\gamma x(t)|_{t=\overline{t}_j}=\gamma x(\overline{t}_j^+)-\gamma x(\overline{t}_j^-)=\overline{J}_j(x(\overline{t}_j^-)),\ j=1,2,\cdots,n,\\ x(a)=x_a,\gamma x(a)=\hat{x}_a,\quad x_a,\hat{x}_a\in\mathbb{R}.\end{cases}\tag{3.32}$$

In addition, impulses $\Delta x|_{t=t_i}$ ($i=1,2,\cdots,m$) are considered as a special impulses of impulses $\Delta\gamma x(t)|_{t=\overline{t}_j}=\overline{J}_j(x(\overline{t}_j^-))$ ($j=1,2,\cdots,n$) on $\gamma x(t)$ of (1.6) by Remark 3.7. Thus, using Lemma 2.6 to (3.32) as $t\in\ell_k'$ (where $k=0,1,2,\cdots,M$), we have

$$\gamma x(t)=\hat{x}_a+\sum_{j=1}^{k_2}\bar{J}_j(x(\bar{t}_j^-))+\frac{1}{\Gamma(q-1)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\sum_{i=1}^{k_1}\frac{\xi_i I_i(x(t_i^-))}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right]+\sum_{j=1}^{k_2}\frac{\eta_j\bar{J}_j(x(\bar{t}_j^-))}{\Gamma(q-1)}\left[\int_a^{\bar{t}_j}\left(\frac{(\bar{t}_j)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.+\int_{\bar{t}_j}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$
$$\text{for } t\in\ell_k',\ k=0,1,\cdots,M,$$

where ξ_i and η_j are some undetermined constants. Letting $\bar{J}_j(x(\bar{t}_j^-))=0$ (for all $j\in\{1,2,\cdots,n\}$) and $I_i(x(t_i^-))=0$ (for all $i\in\{1,2,\cdots,m\}$) in (1.6), respectively, we get $\xi_i=\xi$ (for all $i\in\{1,2,\cdots,m\}$) and $\eta_j=\eta$ (for all $j\in\{1,2,\cdots,n\}$) by Corollaries 3.5 and 3.6. Thus

$$\gamma x(t)=\hat{x}_a+\sum_{j=1}^{k_2}\bar{J}_j(x(\bar{t}_j^-))+\frac{1}{\Gamma(q-1)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}$$
$$+\xi\sum_{i=1}^{k_1}\frac{I_i(x(t_i^-))}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right]+\eta\sum_{j=1}^{k_2}\frac{\bar{J}_j(x(\bar{t}_j^-))}{\Gamma(q-1)}\left[\int_a^{\bar{t}_j}\left(\frac{(\bar{t}_j)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right.$$
$$\left.+\int_{\bar{t}_j}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],$$
$$\text{for } t\in\ell_k',\ k=0,1,\cdots,M.$$

The proof is completed.

Using the inverse operation of $\gamma x(t)$ and Lemmas 3.3–3.4 for the integral equations (3.31), we can draw the following conclusion on the solution of (1.6).

Theorem 3.9: *Let $q\in(1,2)$ and $a,\rho>0$. Let $g:[a,T]\times\mathbb{R}\to\mathbb{R}$ be a function such that $g((\cdot),x(\cdot))\in AC(\ell_i')$ for any $x(\cdot)\in AC(\ell_i')$ (where $i=0,1,\cdots,M$). If $x(t)\in IC([a,T],\mathbb{R})$ is a solution of (1.6) if and only if $x(t)$ satisfies the integral equations*

$$
\begin{aligned}
x(t) &= x_a + \hat{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{i=1}^{k_1} I_i(x(t_i^-)) + \sum_{j=1}^{k_2} \overline{J}_j(x(\overline{t}_j^-)) \frac{t^\rho - (\overline{t}_j)^\rho}{\rho} \\
&+ \xi \sum_{i=1}^{k_1} \frac{I_i(x(t_i^-))}{\Gamma(q)} \left[\int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_i}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)\left[t^\rho - (t_i)^\rho\right]}{\rho} \int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right] \\
&+ \eta \sum_{j=1}^{k_2} \frac{\overline{J}_j(x(\overline{t}_j^-))}{\Gamma(q)} \left[\int_a^{\overline{t}_j} \left(\frac{(\overline{t}_j)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{\overline{t}_j}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)\left[t^\rho - (\overline{t}_j)^\rho\right]}{\rho} \int_a^{\overline{t}_j} \left(\frac{(\overline{t}_j)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right] \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}}, \quad \text{for } t \in \ell'_k, k = 0, 1, \cdots, M,
\end{aligned} \tag{3.33}
$$

where ξ and η are two arbitrary constants.

Using Theorem 3.9 and Remark 1.1, we come to the following conclusion on (1.1).

Corollary 3.10: Let $q \in (1,2)$ and $a, \rho > 0$. Let $g : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $g((\cdot), x(\cdot)) \in AC(\ell'_i)$ for any $x(\cdot) \in AC(\ell'_i)$ (where $i = 0, 1, \cdots, M$). If $x(t) \in IC([a,T], \mathbb{R})$ is a solution of (1.1) if and only if $x(t)$ satisfies the integral equations

$$
\begin{aligned}
x(t) &= x_a + a^{1-\rho} \overline{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{i=1}^{k_1} I_i(x(t_i^-)) + \sum_{j=1}^{k_2} (\overline{t}_j)^{1-\rho} J_j(x(\overline{t}_j^-)) \frac{t^\rho - (\overline{t}_j)^\rho}{\rho} \\
&+ \xi \sum_{i=1}^{k_1} \frac{I_i(x(t_i^-))}{\Gamma(q)} \left[\int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_i}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_i)^\rho]}{\rho} \int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right] \\
&+ \eta \sum_{j=1}^{k_2} \frac{(\overline{t}_j)^{1-\rho} J_j(x(\overline{t}_j^-))}{\Gamma(q)} \left[\int_a^{\overline{t}_j} \left(\frac{(\overline{t}_j)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{\overline{t}_j}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (\overline{t}_j)^\rho]}{\rho} \int_a^{\overline{t}_j} \left(\frac{(\overline{t}_j)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right], \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}}, \quad \text{for } t \in \ell'_k, k = 0, 1, \cdots, M,
\end{aligned} \tag{3.34}
$$

where ξ and η are two arbitrary constants.

Finally, letting $J_j(x(\bar{t}_j^-)) \to 0$ *for all* $j \in \{1,2,\cdots,n\}$ *and* $I_i(x(t_i^-)) \to 0$ *for all* $i \in \{1,2,\cdots,m\}$ *in the system (1.1) and the integral equations (3.34), respectively, we draw the following conclusions.*

Corollary 3.11: Let $q \in (1,2)$ and $a, \rho > 0$. Let $g : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $g((\cdot), x(\cdot)) \in AC(\ell_i)$ for any $x(\cdot) \in AC(\ell_i)$ (where $i = 0,1,\cdots,m$). If $x(t) \in IC([a,T],\mathbb{R})$ is a solution of (1.3) if and only if $x(t)$ satisfies the integral equations

$$
\begin{aligned}
x(t) &= x_a + a^{1-\rho}\bar{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \\
&+ \xi \sum_{i=1}^{k} \frac{I_i(x(t_i^-))}{\Gamma(q)} \left[\int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{t_i}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (t_i)^\rho]}{\rho} \int_a^{t_i} \left(\frac{(t_i)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right], \\
&\text{for } t \in \ell_k, k = 0,1,\cdots,m,
\end{aligned} \tag{3.35}
$$

where ξ *is an arbitrary constant.*

Corollary 3.12: Let $q \in (1,2)$ and $a, \rho > 0$. Let $g : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $g((\cdot), x(\cdot)) \in AC(\bar{\ell}_j)$ for any $x(\cdot) \in AC(\bar{\ell}_j)$ (where $j = 0,1,\cdots,n$). If $x(t) \in IC([a,T],\mathbb{R})$ is a solution of (1.4) if and only if $x(t)$ satisfies the integral equations

$$
\begin{aligned}
x(t) &= x_a + a^{1-\rho}\bar{x}_a \frac{t^\rho - a^\rho}{\rho} + \sum_{j=1}^{l} (\bar{t}_j)^{1-\rho} J_j(x(\bar{t}_j^-)) \frac{t^\rho - (\bar{t}_j)^\rho}{\rho} \\
&+ \eta \sum_{j=1}^{l} \frac{(\bar{t}_j)^{1-\rho} J_j(x(\bar{t}_j^-))}{\Gamma(q)} \left[\int_a^{\bar{t}_j} \left(\frac{(\bar{t}_j)^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \int_{\bar{t}_j}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} \right. \\
&\left. - \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}} + \frac{(q-1)[t^\rho - (\bar{t}_j)^\rho]}{\rho} \int_a^{\bar{t}_j} \left(\frac{(\bar{t}_j)^\rho - \tau^\rho}{\rho} \right)^{q-2} \frac{g d\tau}{\tau^{1-\rho}} \right] \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{q-1} \frac{g d\tau}{\tau^{1-\rho}}, \text{ for } t \in \bar{\ell}_l, l = 0,1,\cdots,n,
\end{aligned} \tag{3.36}
$$

where η *is an arbitrary constant.*

Corollary 3.13: Let $q \in (1,2)$ and $a, \rho > 0$. Let $g : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $g((\cdot), x(\cdot)) \in AC(\ell_i)$ for any $x(\cdot) \in AC(\ell_i)$ (where $i = 0,1,\cdots,m$). If $x(t) \in IC([a,T],\mathbb{R})$ is a solution of (1.2) if and only if $x(t)$ satisfies the integral equations

$$x(t)=x_a+a^{1-\rho}\overline{x}_a\frac{t^\rho-a^\rho}{\rho}+\sum_{i=1}^{k}\left[I_i(x(t_i^-))+(t_i)^{1-\rho}J_i(x(t_i^-))\frac{t^\rho-(\overline{t}_i)^\rho}{\rho}\right]$$
$$+\frac{1}{\Gamma(q)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\sum_{i=1}^{k}\frac{\xi I_i(x(t_i^-))+\eta(\overline{t}_i)^{1-\rho}J_i(x(\overline{t}_i^-))}{\Gamma(q)}$$
$$\cdot\left[\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}+\int_{t_i}^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}-\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}\frac{gd\tau}{\tau^{1-\rho}}\right.\tag{3.37}$$
$$\left.+\frac{(q-1)[t^\rho-(t_i)^\rho]}{\rho}\int_a^{t_i}\left(\frac{(t_i)^\rho-\tau^\rho}{\rho}\right)^{q-2}\frac{gd\tau}{\tau^{1-\rho}}\right],\text{ for } t\in\ell_k, k=0,1,\cdots,m,$$

where ξ and η are two arbitrary constants.

4 Applications

Example 4.1: Consider the following impulsive fractional differential equation

$$\begin{cases} {}_1^C\mathcal{D}_t^{\frac{5}{4},\rho}x(t)=t, \quad t\in(1,4], t\neq 2 \text{ and } t\neq 3, \\ \Delta x(t)\big|_{t=2}=x(2^+)-x(2^-)=1, \\ \Delta x'(t)\big|_{t=3}=x'(3^+)-x'(3^-)=1, \\ x(1)=0, x'(1)=1. \end{cases}\tag{4.1}$$

According to Corollary 3.10, (4.1) is equivalent to the following integral equations

$$x(t)=\frac{t^\rho-1}{\rho}+\frac{1}{\Gamma(\frac{5}{4})}\int_1^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}},\ \text{for } t\in[1,2],\tag{4.2a}$$

$$x(t)=\frac{t^\rho-1}{\rho}+1+\frac{1}{\Gamma(\frac{5}{4})}\int_1^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}+\frac{\xi}{\Gamma(\frac{5}{4})}$$
$$\cdot\left\{\int_1^2\left(\frac{2^\rho-\tau^\rho}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}+\int_2^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}-\int_1^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}\right.\tag{4.2b}$$
$$\left.+\frac{t^\rho-2^\rho}{4\rho}\int_1^2\left(\frac{2^\rho-\tau^\rho}{\rho}\right)^{-\frac{3}{4}}\frac{d\tau}{\tau^{-\rho}}\right\},\ \text{for } t\in(2,3],$$

$$x(t)=\frac{t^{\rho}-1}{\rho}+1+3^{1-\rho}\frac{t^{\rho}-3^{\rho}}{\rho}+\frac{1}{\Gamma(\frac{5}{4})}\int_1^t\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}$$
$$+\frac{\xi}{\Gamma(\frac{5}{4})}\left\{\int_1^2\left(\frac{2^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}+\int_2^t\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}\right.$$
$$\left.-\int_1^t\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}+\frac{t^{\rho}-2^{\rho}}{4\rho}\int_1^2\left(\frac{2^{\rho}-\tau^{\rho}}{\rho}\right)^{-\frac{3}{4}}\frac{d\tau}{\tau^{-\rho}}\right\} \tag{4.2c}$$
$$+\frac{\eta 3^{1-\rho}}{\Gamma(\frac{5}{4})}\left\{\int_1^3\left(\frac{3^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}+\int_3^t\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}\right.$$
$$\left.-\int_1^t\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{1}{4}}\frac{d\tau}{\tau^{-\rho}}+\frac{t^{\rho}-3^{\rho}}{4\rho}\int_1^3\left(\frac{3^{\rho}-\tau^{\rho}}{\rho}\right)^{-\frac{3}{4}}\frac{d\tau}{\tau^{-\rho}}\right\},\quad \text{for } t\in(3,4],$$

where ξ and η are two arbitrary constants.

From (4.2a–c), we find that there are many solutions for (4.1) due to taking some different values of ξ and η. Next we will apply the integral equations (4.2a–c) and the numerical simulation method to show some solution trajectories of (4.1) with some different values of ρ.

Figures 1–4 denote some solution trajectories of (4.1) with $\rho\to 0+$, $\rho=0.5$, $\rho=1$ and $\rho=1.5$, respectively. In each figure, these curves $\text{xi}=0,\text{elta}=0;\text{xi}=0,\text{elta}=1;\text{xi}=1,\text{elta}=0$ and $\text{xi}=1,\text{elta}=1$ denote four solution trajectories of (4.1) with the corresponding ρ ($\rho=0,0.5,1,1.5$), which are drawn by the numerical simulation of (4.2a-c) with the corresponding ρ ($\rho=0,0.5,1,1.5$) and $\xi=0,\eta=0$; $\xi=0,\eta=1$; $\xi=1,\eta=0$ and $\xi=1,\eta=1$, respectively.

Figure 1 The solution trajectory of (4.1) with $\rho\to 0+$ (see online version for colours)

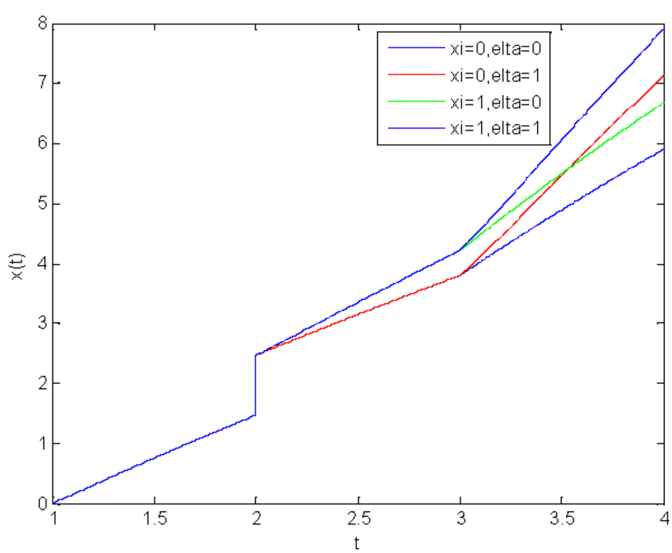


Figure 2 The solution trajectory of (4.1) with ρ = 0.5 (see online version for colours)

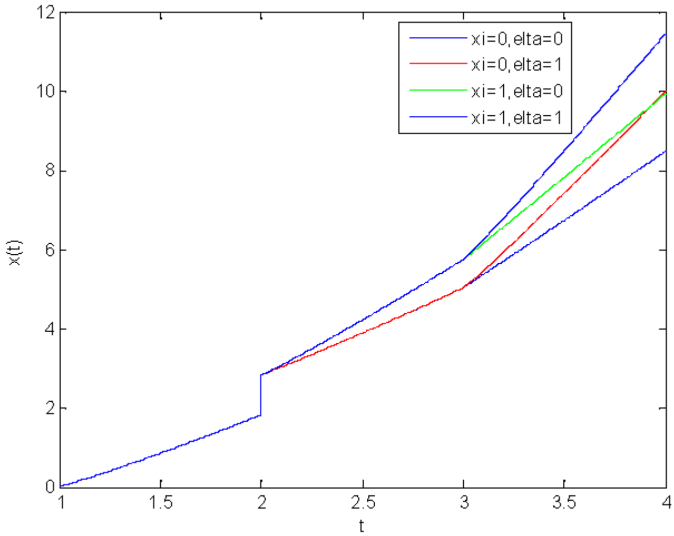
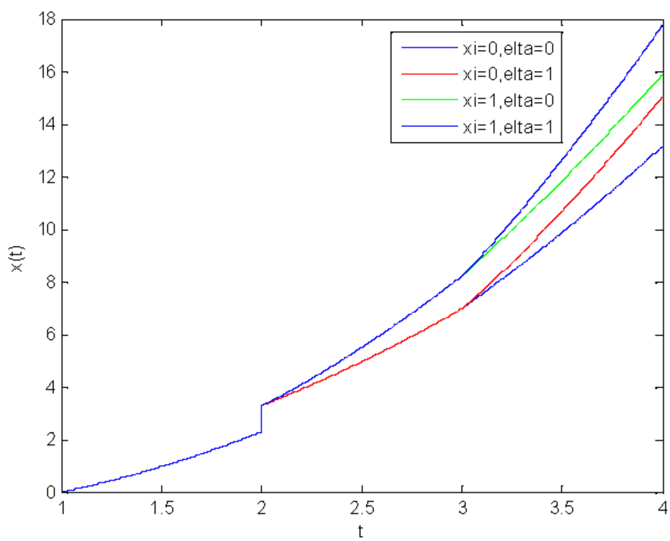


Figure 3 The solution trajectory of (4.1) with ρ = 1 (see online version for colours)



Example 4.2: Consider the fractional order linear system with an impulse

$$\begin{cases} {}_1^C\mathcal{D}_t^{\frac{5}{4},\rho} x(t) = x(t), \quad t \in (1,3] \text{ and } t \neq 2, \\ \Delta x(t)\big|_{t=2} = x(2^+) - x(2^-) = 1, \\ x(1) = 1, x'(1) = 1. \end{cases} \tag{4.3}$$

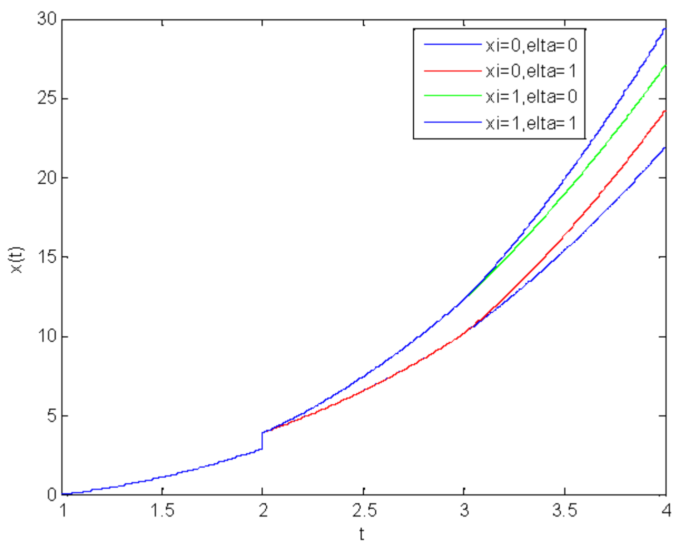
Using Corollary 3.11, (4.3) is equivalent to the following integral equations

$$x(t) = 1 + \frac{t^\rho - 1}{\rho} + \frac{1}{\Gamma(\frac{5}{4})}\int_1^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\frac{1}{4}} \frac{x(\tau)d\tau}{\tau^{1-\rho}}, \text{ for } t \in [1,2], \tag{4.4a}$$

$$\begin{aligned} x(t) = 2 + \frac{t^\rho - 1}{\rho} + \frac{1}{\Gamma(\frac{5}{4})}\int_1^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\frac{1}{4}} \frac{x(\tau)d\tau}{\tau^{1-\rho}} \\ + \frac{\xi}{\Gamma(\frac{5}{4})}\left\{\int_1^2 \left(\frac{2^\rho - \tau^\rho}{\rho}\right)^{\frac{1}{4}} \frac{x(\tau)d\tau}{\tau^{1-\rho}} + \int_2^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\frac{1}{4}} \frac{x(\tau)d\tau}{\tau^{1-\rho}}\right. \\ \left. - \int_1^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\frac{1}{4}} \frac{x(\tau)d\tau}{\tau^{1-\rho}} + \frac{[t^\rho - 2^\rho]}{4\rho}\int_1^2 \left(\frac{2^\rho - \tau^\rho}{\rho}\right)^{-\frac{3}{4}} \frac{x(\tau)d\tau}{\tau^{1-\rho}}\right\}, \\ \text{for } t \in (2,3], \end{aligned} \tag{4.4b}$$

where ξ is an arbitrary constants.

Figure 4 The solution trajectory of (4.1) with ρ = 1.5 (see online version for colours)



From (4.4a–b), the solutions of (4.3) are non-unique owing to taking some different values of ξ, which will be shown by using the numerical simulation of (4.4a–b).

Figures 5–8 denote some solution trajectories of (4.3) with $\rho = 0.1$, $\rho = 0.5$, $\rho = 1$ and $\rho = 1.5$, respectively. Moreover, in each figure, these curves $\text{xi} = 0, \text{xi} = -1$ and $\text{xi} = 1$ represent three solution trajectories of (4.3) with the corresponding ρ

($\rho = 0.1, 0.5, 1, 1.5$), which are drawn by the numerical simulation of (4.4a-b) with the corresponding ρ and $\xi = 0, \xi = -1$ and $\xi = 1$, respectively.

Figure 5 The solution trajectory of (4.3) with ρ = 0.1 (see online version for colours)

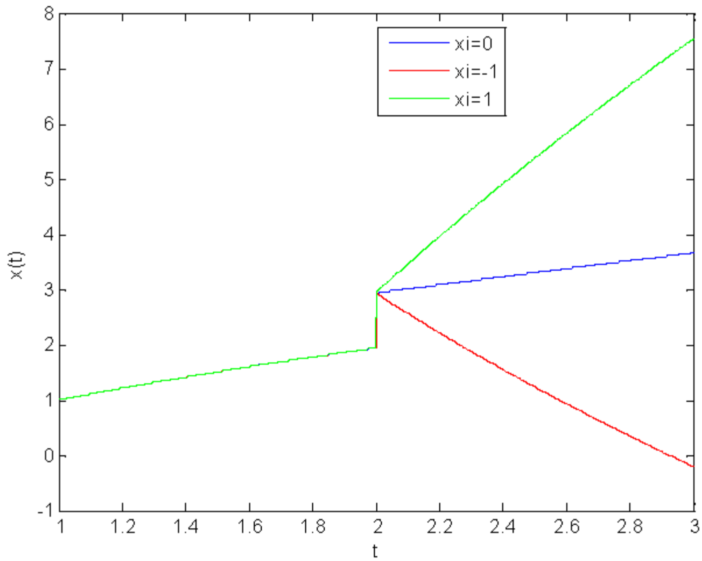


Figure 6 The solution trajectory of (4.3) with ρ = 0.5 (see online version for colours)

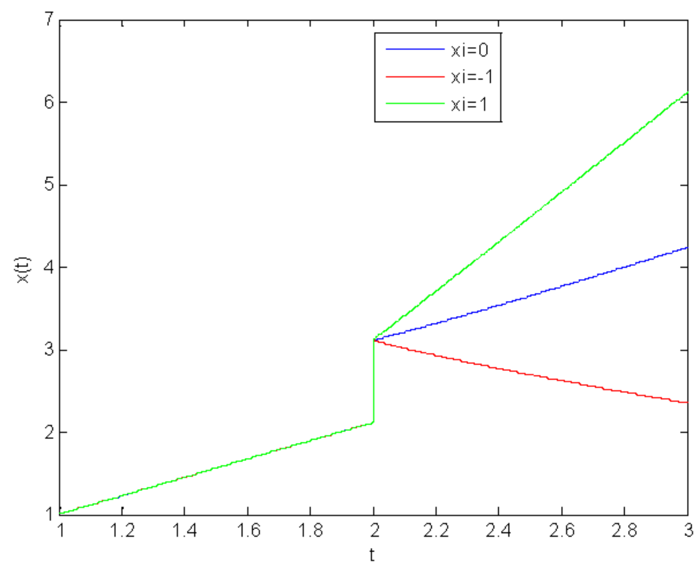


Figure 7 The solution trajectory of (4.3) with $\rho = 1$ (see online version for colours)

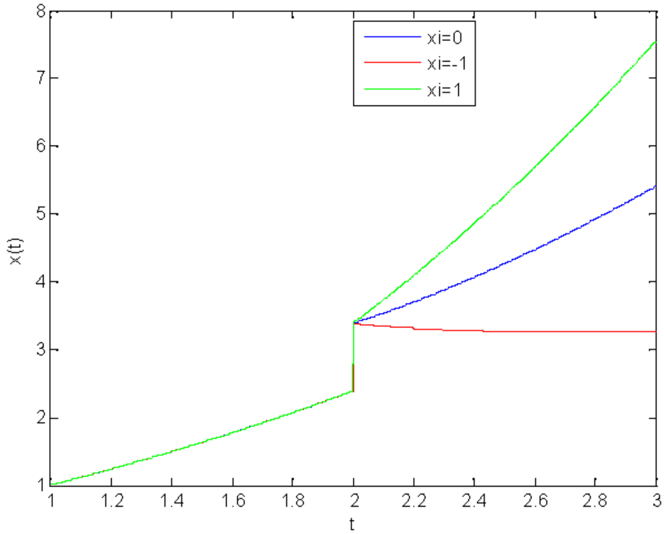
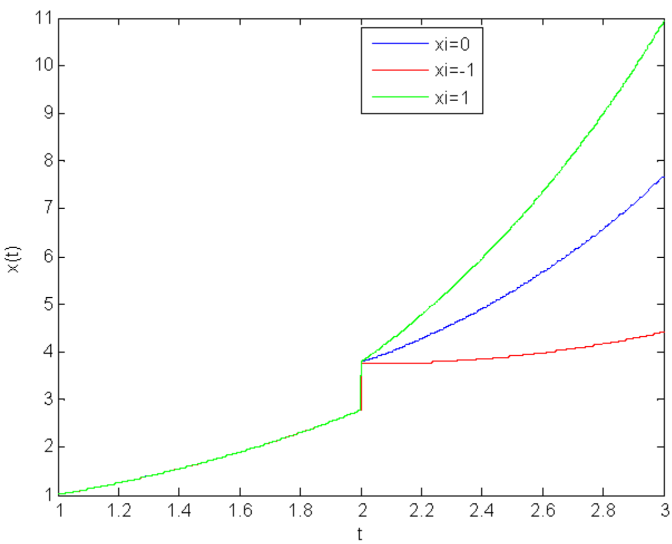


Figure 8 The solution trajectory of (4.3) with $\rho = 1.5$ (see online version for colours)



5 Conclusions

Because impulsive higher order fractional differential equations may involve some inhomogeneous impulses (such as two types of impulses $I_i(x(t_i^-))$ and $J_j(x(\bar{t}_j^-))$ of (1.1)), their equivalent integral equations could involve two arbitrary constants, which show that the solutions of impulsive higher order fractional differential equations are non-unique.

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