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## Rational solutions to the Painlevé II equation from particular polynomials

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**Abstract:** The Painlevé equations were derived by Painlevé and Gambier in 1895–1910. Given a rational function  $R$  in  $w$ ,  $w'$  and analytic in  $z$ , they searched what were the second order ordinary differential equations of the form  $w'' = R(z, w, w')$  with the properties that the singularities other than poles of any solution or this equation depend on the equation only and not of the constants of integration. They proved that there are 50 equations of this type, and the Painlevé II is one of these. Here, we construct solutions to the Painlevé II equation (PII) from particular polynomials. We obtain rational solutions written as a derivative with respect to the variable  $x$  of a logarithm of a quotient of a determinant of order  $n + 1$  by a determinant of order  $n$ . We obtain an infinite hierarchy of rational solutions to the PII equation. We give explicitly the expressions of these solutions solution for the first orders.

**Keywords:** Painlevé equation II; PII; rational solutions; determinants.

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**Biographical notes:** Pierre Gaillard defended his thesis on integrable deformations of the potentials of Darboux Pöschl Teller at the University of Burgundy, Dijon, France. He is an associate researcher at the University of Burgundy in France and has worked on modelling rogue waves within the framework of the nonlinear Schrödinger equation (NLS) as well as for the Kadomtsev Petviashvili equation and the Johnson equation. He defended a second habilitation thesis to direct research on the three preceding themes. His current work focuses on various partial differential equations of mathematical physics: the equations of KdV, mKdV, Burgers, Gardner, Boussinesq, Hirota and recently Painlevé II.

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### 1 Introduction

We consider the Painlevé II equation (PII) which can be written in the form

$$u_{xx} - 2u^3 + 4xu - 4(n + 1) = 0, \quad (1)$$

where the subscript  $x$  denote derivative and  $n$  is an arbitrary integer.

This equation is one of the six famous equations discovered by Painlevé (1900) at the beginning of 20th century.

The first solutions of this equation expressed in a slightly different form

$$u_{xx} - 2u^3 - xu - \alpha = 0, \tag{2}$$

were given by Yablonskii in 1959 and Vorob'ev in 1965, for every  $\alpha$  integer. For this purpose, they constructed polynomials defined by the following recurrence relation

$$Q_{n+1}Q_n = xQ_n^2 - 4[Q_n(Q_n)_{xx} - ((Q_n)_x)^2], \tag{3}$$

which we call today Yablonski-Vorob'ev polynomials.

Other types of rational solutions of the PII equation (2) were constructed by Airault in 1979 and Murata in 1985. A proof of the irreducibility of the PII equation (2) was done by Umemura and Watanabe in 1997. It has been proven that the solutions of the PII equation (2) can be represented by a logarithmic derivative of certain polynomials in Airault (1979) and Clarkson and Mansfield (2003). It has been shown in Lukashевич (1971) that any rational solution of equation (2) can be constructed via Bäcklund transformations. PII equation (2) has solutions which can be expressed in terms of classical special functions, more precisely in terms of Airy functions as these given by Clarkson in Clarkson (2016).

In the following, we consider particular polynomials and we construct rational solutions to the PII equation (1) by using the Hirota bilinear method. Then we obtain rational solutions to the PII equation as a derivative of a logarithm of a quotient of a determinant of order  $N + 1$  by a determinant of order  $N$ . That provides an effective method to construct an infinite hierarchy of rational solutions of order  $N$ . We present rational solutions for the first simplest orders.

## 2 Rational solutions to the PII equation

We consider the polynomials  $p_n$  defined by

$$\left\{ \begin{array}{l} p_n(x) = \sum_{k=0}^n x^k \frac{1}{\left(\frac{n-k}{3}\right)!} \times \left(1 - \left[\frac{1}{2} \left(n - k + 1 - 3 \left[\frac{n-k}{3}\right]\right)\right]\right), \\ \text{for } n \geq 0, \\ p_n(x) = 0, \text{ for } n < 0. \end{array} \right. \tag{4}$$

In the previous definition of  $p_n$ ,  $[x]$  means the largest integer less than or equal to  $x$ . We denote  $A_n$  the determinant defined by

$$A_n = \det(p_{n+1-2i+j}(x))_{\{1 \leq i \leq n, 1 \leq j \leq n\}} \tag{5}$$

With these notations we have the following result:

*Theorem 2.1:* The function  $v_n$  defined by

$$v_n(x) = \partial_x \left( \ln \frac{A_{n+1}}{A_n} \right) \tag{6}$$

is a rational to the PII equation (1)

$$u_{xx} - 2u^3 + 4xu - 4(n+1) = 0.$$

*Proof:* We know that  $v_n(x) = \partial_x \left( \ln \frac{f}{g} \right)$  is a solution to the PII equation if  $f$  and  $g$  verify the following equations:

$$(D_x^2)f \cdot g = 0, \tag{7}$$

$$(D_x^3 + 4xD_x - 4(n+1))f \cdot g = 0, \tag{8}$$

where  $D$  is the bilinear differential operator.

We have to check the relation (7) for  $f = \det(p_{n+2-2i+j})_{\{1 \leq i \leq n+1, 1 \leq j \leq n+1\}}$  and  $g = \det(p_{n+1-2i+j})_{\{1 \leq i \leq n, 1 \leq j \leq n\}}$ . We choose the following notations:  $C_j$  denotes the column  $j$  of  $A_{n+1}$ ,  $1 \leq j \leq n+1$  and  $\tilde{C}_j$  denotes the column  $j$  of  $A_n$ ,  $1 \leq j \leq n$ :

$$C_j = \begin{pmatrix} p_{n+j} \\ p_{n-2+j} \\ \vdots \\ p_{-n+j} \end{pmatrix}, \quad \tilde{C}_j = \begin{pmatrix} p_{n-1+j} \\ p_{n-3+j} \\ \vdots \\ p_{-n+1+j} \end{pmatrix}. \tag{9}$$

Using these notations,  $A_{n+1}(x, t)$  and  $A_n(x, t)$  can be written as:

$$A_{n+1}(x, t) = |C_1, \dots, C_{n+1}|, \quad A_n(x, t) = |\tilde{C}_1, \dots, \tilde{C}_n|.$$

1 We have to prove the relation (7)  $D_x^2 f \cdot g = 0$ .

Let  $G$  be the expression  $G = D_x^2 f \cdot g$ . We evaluate  $G$ .

$G = G_1 + G_2$  can be written as a sum of 6 terms where

$$\begin{aligned} G_1 = & |C_0, C_1, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & + |C_1, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_{-1}, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \\ & - |C_0, C_2, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \end{aligned}$$

and

$$\begin{aligned} G_2 = & |C_{-1}, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & + |C_1, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_1, \tilde{C}_3, \dots, \tilde{C}_n| \\ & - |C_0, C_2, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \end{aligned}$$

We write  $G_1$  and  $G_2$  as the determinants of order  $2n + 1$

$$G_1 = \left| \begin{array}{cccc|cccc} C_1 & C_2 & C_3 & C_4 & \dots & C_{n+1} & C_0 & 0 & \dots & 0 \\ \hline \tilde{C}_0 & \tilde{C}_1 & 0 & 0 & \dots & 0 & \tilde{C}_{-1} & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 & \dots & \tilde{C}_n \end{array} \right| \tag{10}$$

and

$$G_2 = \left| \begin{array}{cccc|cccc} C_{-1} & C_2 & C_3 & C_4 & \dots & C_{n+1} & C_0 & C_1 & \dots & 0 \\ \hline \tilde{C}_0 & 0 & 0 & 0 & \dots & 0 & \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 & \dots & \tilde{C}_n \end{array} \right| \tag{11}$$

$G_1$  can be rewritten as

$$G_1 = \left| \begin{array}{cccc|cccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \dots & p_{2n+1} & p_n & 0 & 0 & \dots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & p_{n-2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+1} & \dots & p_1 & p_{-n} & 0 & 0 & \dots & 0 \\ p_{n-1} & p_n & 0 & 0 & \dots & 0 & p_{n-2} & p_{n+1} & p_{n+2} & \dots & p_{2n-1} \\ p_{n-3} & p_{n-2} & 0 & 0 & \dots & 0 & p_{n-4} & p_{n-1} & p_n & \dots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & p_{-n+2} & 0 & 0 & \dots & 0 & p_{-n} & p_{-n+3} & p_{-n+4} & \dots & p_1 \end{array} \right| \tag{12}$$

Denoting  $\mathcal{L}$  the rows and  $\mathcal{C}$  the columns of this determinant of order  $2n + 1$ , we combine the rows of the previous determinant in the following way:

We replace  $\mathcal{L}_{n+1+j}$  by  $\mathcal{L}_{n+1+j} - \mathcal{L}_{j+1}$  for  $1 \leq j \leq n$ , then we obtain the following determinant

$$G_1 = \left| \begin{array}{cccc|cccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \dots & p_{2n+1} & p_n & 0 & 0 & \dots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+3} & \dots & p_{2n-1} & p_{n-2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+1} & \dots & p_1 & p_{-n} & 0 & 0 & \dots & 0 \\ 0 & 0 & -p_{n+1} & -p_{n+3} & \dots & -p_{2n-1} & 0 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} \\ 0 & 0 & -p_{n-1} & -p_{n+1} & \dots & -p_{2n-3} & 0 & p_{n-1} & p_n & \dots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -p_{-n+3} & -p_{-n+1} & \dots & -p_1 & 0 & p_{-n+3} & p_{-n+4} & \dots & p_1 \end{array} \right| \tag{13}$$

Then replacing  $\mathcal{C}_j$  by  $\mathcal{C}_j + \mathcal{C}_{n+j}$  for  $3 \leq j \leq n$ , we obtain the following determinant

$$G_1 = \left| \begin{array}{cccc|cccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \dots & p_{2n+1} & p_n & 0 & 0 & \dots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+3} & \dots & p_{2n-1} & p_{n-1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+1} & \dots & p_1 & p_{-n} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & p_{n+1} & 0 & \dots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & p_{n-1} & 0 & \dots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & p_{-n+3} & \dots & \dots & p_1 \end{array} \right| \tag{14}$$

It is clear that this last determinant is equal to 0.

For the expression  $G_2$ , we prove by combining rows and columns that it is equal to 0.

$$G_2 = \left| \begin{array}{cccccc|cccc} p_{n-1} & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & p_n & p_{n+1} & 0 & \cdots & 0 \\ p_{n-3} & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} & p_{n-2} & p_{n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n-1} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & p_{-n} & p_{-n+1} & 0 & \cdots & 0 \\ \hline p_{n-1} & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ p_{n-3} & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (15)$$

We combine the rows  $\tilde{\mathcal{L}}$  and the columns  $\tilde{\mathcal{C}}$  in the following way; we replace  $\tilde{\mathcal{L}}_j$  by  $\tilde{\mathcal{L}}_j - \tilde{\mathcal{L}}_{n+1+j}$  for  $1 \leq j \leq n$  and we obtain

$$G_2 = \left| \begin{array}{cccccc|cccc} 0 & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & 0 & 0 & -p_{n+2} & \cdots & -p_{2n+1} \\ 0 & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & 0 & 0 & -p_n & \cdots & -p_{2n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & p_{-n+4} & p_{-n+5} & p_{-n+6} & \cdots & p_3 & 0 & 0 & -p_{-n+4} & \cdots & -p_3 \\ \hline p_{-n-1} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & p_{-n} & p_{-n+1} & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (16)$$

Then we replace  $\tilde{\mathcal{C}}_{n+1+j}$  by  $\tilde{\mathcal{C}}_{n+j+1} + \tilde{\mathcal{C}}_{j-1}$  for  $3 \leq j \leq n$  and we obtain, using that  $p_{-n-2} = 0$ ,  $p_{-n} = 0$  and  $p_{-n+1} = 0$  for  $n \geq 0$

$$G_2 = \left| \begin{array}{cccccc|cccc} 0 & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & p_{-n+4} & p_{-n+5} & p_{-n+6} & \cdots & p_3 & 0 & 0 & -0 & \cdots & 0 \\ \hline 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & 0 & 0 & p_{-n+2} & \cdots & p_1 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (17)$$

It is also clear that this last determinant is equal to 0.

- 2 We denote  $H$  the expression  $H = (D_x^3 + 4xD_x - 4(n + 1))A_{n+1} \cdot A_n$ . We have to evaluate  $H$ .

We can use the same strategy that this used previously. We chose to present another method. For this we consider the following polynomials  $\tilde{p}_n$  defined by

$$\left\{ \begin{array}{l} \tilde{p}_n(x_1, x_2) = \sum_{k=0}^n x_1^k \frac{x_2^{\binom{n-k}{3}}}{\left(\frac{n-k}{3}\right)!} \\ \quad \times \left(1 - \left[\frac{1}{2} \left(n - k + 1 - 3 \left[\frac{n-k}{3}\right]\right)\right]\right), \text{ for } n \geq 0, \\ p_n(x_1, x_2) = 0, \text{ for } n < 0. \end{array} \right. \quad (18)$$

We have proved in a previous paper Gaillard (2021a) concerning the mKdV equation (19)

$$4u_t + 6u^2u_x - u_{xxx} = 0 \quad (19)$$

that  $\tilde{A}_n = \det(\tilde{p}_{n+1-2i+j}(x))_{\{1 \leq i \leq n, 1 \leq j \leq n\}}$  verify  $(D_{x_1}^3 - 4D_{x_2})\tilde{A}_{n+1} \cdot \tilde{A}_n = 0$ .

The polynomial  $\tilde{p}_k$  is an homogeneous polynomial of weight  $k$  in  $x_1$  and  $x_2$ , thus  $\tilde{A}_n$  is an homogeneous polynomial of weight  $\frac{n(n+1)}{2}$  in  $x_1$  and  $x_2$ .

Using the Euler relation for  $\tilde{A}_n$ , we obtain

$$\partial_{x_2} \tilde{A}_n = \frac{1}{3x_2} \left( \frac{n(n+1)}{2} \tilde{A}_n - x_1 \partial_{x_1} \tilde{A}_n \right).$$

The expression  $(D_{x_1}^3 - 4D_{x_2})\tilde{A}_{n+1} \cdot \tilde{A}_n = 0$  can be rewritten as

$$\left( D_{x_1}^3 + \frac{4}{3x_2} x_1 D_{x_1} - \frac{4}{3x_2} (n+1) \right) \tilde{A}_{n+1} \cdot \tilde{A}_n = 0.$$

Then it is sufficient to take  $x_1 = x$  and  $x_2 = \frac{1}{3}$  to get the relation

$$H = (D_x^3 + 4xD_x - 4(n + 1))A_{n+1} \cdot A_n = 0.$$

This proves that

$$H = (D_x^3 + 4xD_x - 4(n + 1))A_{n+1} \cdot A_n = 0.$$

So we get the result,  $v_n(x) = \partial_x \left( \ln \frac{f(x, t)}{g(x, t)} \right)$  is a solution to the PII equation.  $\square$

### 3 Explicit rational solutions to the PII equation for the first orders

Many studies on this equation have been realised but only a few gives explicit solutions, and just for little orders. The efficiency of this method gives a hierarchy of rational solutions and very easily the explicit expressions of the first orders of these solutions. To the best of my knowledge, these explicit rational solutions have never been constructed. We denote  $v_k$  in the following, a rational solution to the PII equation

$$u_{xx} - 2u^3 + 4xu - 4(k + 1) = 0$$

defined by

$$v_k(x, t) = \partial_x \left( \ln \frac{A_{k+1}(x, t)}{A_k(x, t)} \right)$$

These solutions  $v_k$  can be rewritten as

$$v_k(x, t) = \frac{n_k(x, t)}{d_k(x, t)}$$

We present here some explicit examples of rational solutions to the PII equation for the first orders as corollaries of the theorem of the previous section.

For this purpose, we define  $n_k$  and  $d_k$  for  $1 \leq k \leq 3$  as:

$$\mathbf{n}_1(\mathbf{x}, \mathbf{t}) = 2x^3 + 1, \quad \mathbf{d}_1(\mathbf{x}, \mathbf{t}) = x(x^3 - 1)$$

$$\mathbf{n}_2(\mathbf{x}, \mathbf{t}) = 3x^2(x^6 - 2x^3 + 10), \quad \mathbf{d}_2(\mathbf{x}, \mathbf{t}) = (x^6 - 5x^3 - 5)(x^3 - 1)$$

$$\mathbf{n}_3(\mathbf{x}, \mathbf{t}) = 4x^{15} - 50x^{12} + 250x^9 + 1400x^6 - 1750x^3 + 875,$$

$$\mathbf{d}_3(\mathbf{x}, \mathbf{t}) = (x^9 - 15x^6 - 175)(x^6 - 5x^3 - 5)x$$

With the previous notations, we have constructed the explicit solutions to the PII equation for  $k = 1$  to 3 given by

*Corollary 3.1:* The functions  $v_k$  defined by

$$v_k(x) = \frac{n_k(x)}{d_k(x)} \tag{20}$$

are rational solutions to the PII equation (1)

$$u_{xx} - 2u^3 + 4xu - 4(n + 1) = 0.$$

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator includes 59 terms and the degree of the polynomial in  $x$  is equal to 175; the denominator contains 59 terms and its degree in  $x$  is equal to 176. It will be relevant to study in detail the structure of these polynomials.



## 4 Conclusions

This study is a part of a program of research of rational solutions to differential equations. Using particular explicit polynomials as in the case of the mKdV equation (Gaillard, 2021a) or in the KPI equation (Gaillard, 2021b), rational solutions to Painlevé II equation have been constructed. We obtain rational solutions written as a derivative with respect to the variable  $x$  of a logarithm of a quotient of a determinant of order  $N + 1$  by a determinant of order  $N$ , which we call solution of order  $N$ . A complete proof of this result has been given. It will be relevant to study the structure of the polynomials given in these solutions.

Recently, other works has been presented concerning this equation. In Gromak (2020), analytic properties of solutions to equations in the generalised hierarchy of the second Painlevé equation has been studied, in particular, the local properties of solutions, the rational solutions and their representations via generalised Yablonskii-Vorob'ev polynomials.

In Mahmood and Waseem (2021), with the Darboux transformation method, solutions of the PII equation (2) are constructed in terms of wronskians, using the corresponding Lax pair.

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