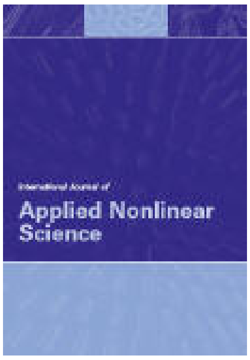

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An analogue of Nadler's result in Hardy-Rogers type iterated multifunction system

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Abstract: The iterated multifunction system (IMS) provides a primary way of deriving a class of set-valued functions constructed on a complete metric space. An exemplification of the generalisation of single-valued function to set-valued function is the Hardy-Rogers type iterated function system (HR-IFS). In this regard, set-valued functions show their efficiency in various domains like robotics, preventive maintenance, control systems, and energy. We extend the notion of HR-IFS to a class of Hardy-Roger type iterated multifunction system (HR-IMS). Moreover, concentrating on tremendous applicability of fixed point theory in real-life scenario, we have obtained a fixed point of the newly constructed IMS with the aid of the Hutchinson-Barnsley theory. In the main result, the attractor of the HR-IMS is constructed in an unconventional way. Consequently, a common idea of the Banach contraction principle, known as Nadler's type result, is gleaned for our HR-IMS.

Keywords: iterated function system; IFS; iterated multifunction system; IMS; Hardy-Rogers type iterated multifunction system; HR-IMS; Nadler's result; attractor; fractal.

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1 Introduction

An amalgam of topology and geometry produces a dazzling track in mathematics, known as fixed point theory. The most rudimentary property of a norm is convexity, and it is used for the earliest development of metric fixed point theory. One of the most wanted environments in the fixed point theory field is to get a solution of a system of equations, and a little bit of explanation about attaining the solution is given here.

Suppose we have a finite system of equations $\breve{\psi}_i(x) = 0, i = 1, 2, \ldots, n$ with unknowns that is equal in the number of equations, in which each equation is a continuous real-valued function of real variables, in the n-dimensional Euclidean space $\mathbb{R}^n$. If we take any element x in $\mathbb{R}^n$ so that

$$\breve{\phi}_i(x) = \breve{\psi}_i(x) + x_i, i = 1, 2, \ldots, n$$

Then the entry x is a solution to the system.

A careful inspection, in the cases of the calculus of variations, partial differential equations, optimal control, and inverse problems, reveals the usage of fixed point theory. Moreover, fixed point theory has some deep roots in applied mathematics, physics, chemistry, and other branches of sciences. Mathematics has a very effective branch known as ‘fixed point theory’ in all aspects. The applicability of this field to real scenarios is not bounded, specifically in nonlinear optimisation problem, economics, geometry and topology manifolds, game theory, etc. Thus our paper initially moves on the fixed point theorem in a complete metric space having a generalised contraction called Hardy-Rogers type contraction. Several researchers get the inspiration to show some extention and generalisation of the fixed point results in a variety of ways. The works of Hussain et al. (2020) and Hammad et al. (2021) provides the evolution of the fixed point theory with some novel non-expansive mapping and homotopy theory. Recently, Goyal et al. (2021) have discussed the Hardy-Rogers type iterated function system (HR-IFS) and showed the attractor for the same in metric space. Whenever we hit with the word ‘fixed point theorem’, we undoubtedly get an idea of a peculiar invariant point of a contraction defined on a complete metric space and it is identified as Banach contraction principle.

A relaxation of the assumptions and the generalisation of the classical Banach contraction principle is termed Nadler’s fixed point theorem in which a set-valued contraction is used instead of point-valued mapping. A further generalisation of this particular Banach contraction principle were commenced by Nadler (1969) and he widens this concept from functions of one variable to set-valued contractive operations. Also, several authors gave a generalisation of this result, see Du (2012) and Ćirić (2009). Moreover, several self-similar phenomena are described by the term iterated function system (IFS) that was introduced by Hutchinson (1981) and a deepening this notion is known as an IMS. Construction of fractal sets using contractive maps either in deterministic nature or in probabilistic environment is IFSs. In general, iteration theory shows its efficiency in computational mathematics, especially by imposing the preceding output in the current iteration and a class of problems can be fathomed by a sequence of approximate solutions with an initial value. In this direction, the theory of dynamical systems makes use of the iterative methods in a wide sense for their valuable application of the system's reliability and stability. Moreover, the attractors gained by considering an IFS have a computational aspect in probabilistic and geometric-measure theoretic developments. A primary initiation of the existence of solutions for fixed point equations is introducing the use of multifunction in fractal analysis. This was started by Andres et al. (2005) and Andres and Fišer (2004). The idea of extending IFSs to IMSs is raised in the field of image analysis. For reducing the issues of the inverse problem for image approximation so many fixed point results were proved to convey the importance of supplementation of IFS to IMS for instance, see Kunze et al. (2007).

In addition to this work, Petruşel et al. (2015) developed some fixed point theorems for non-self-multivalued generalised contractions and the fixed sets. Further, some properties given by them enrich the significance of IMS. Further, in some point of view of IMS, Kunze et al. (2007) had implemented a continuity theorem for fixed point sets as well as a generalised collage theorem for contractive multifunctions. Moreover, as an application side, they provided the notion of the IMS through their result. The problems of applicable mathematics and applied physics can be easily handled with the background of IMSs besides the results proved for IFSs (Singh et al., 2009). A tactful application of IFS theory can be encountered in economics and finance. Moreover, it is meticulously applicable in fractal simulations of Brownian motions, fractal approximations of distribution and density function, and stochastic processes. As a general case of IFS, one can find the usage of IMS in the inverse problem of approximation.

The enhancement of HR-type fixed point theorem is shown as follows to recognise the history and development of this specific notion. A very well-known fixed point theorem defined on a complete metric space $(\mathfrak{P}, p)$ with usual contraction is named as Banach contraction mapping principle. Whereas for self-maps like $\breve{\phi} : \mathfrak{P} \to \mathfrak{P}$, a contractive condition given by

$$p(\breve{\phi}(x), \breve{\phi}(y)) \leq \lambda[p(x, \breve{\phi}(x)) + p(y, \breve{\phi}(y))] \tag{1}$$

for every $x, y \in \mathfrak{P}$ and $\lambda \in [0, 1/2)$ is used to prove the existence of fixed point theorem by Kannan (1968). Futher, the restriction on contractive condition is intircated by a little amount such as

$$p(\breve{\phi}(x), \breve{\phi}(y)) \leq \alpha p(x, y) + \beta p(x, \breve{\phi}(x)) + \gamma p(x, \breve{\phi}(y)) \tag{2}$$

for each $x, y \in \mathfrak{P}$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma = 1$. This elaboration was done by Reich (1971) and an exemplification was achieved by him to show that it was a proper generalisation of Banach and Kannan fixed point theorem. Some more additional works were established by several mathematicians (Chatterjea, 1972; Ciric, 1971) with imposing the constraints on contractive conditions as follows:

$$p(\breve{\phi}(x), \breve{\phi}(y)) \leq \lambda[p(x, \breve{\phi}(y)) + p(y, \breve{\phi}(x))] \tag{3}$$

for all $x, y \in \mathfrak{P}$ and $\lambda \in [0, 1/2)$ and

$$\begin{aligned} p(\breve{\phi}(x), \breve{\phi}(y)) \leq {} & \alpha p(x, y) + \beta p(x, \breve{\phi}(x)) + \gamma p(y, \breve{\phi}(y)) \\ & + \delta[p(x, \breve{\phi}(y)) + p(y, \breve{\phi}(x))] \end{aligned} \tag{4}$$

for any $x, y \in \mathfrak{P}$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$. The mapping given in (4) is called generalised contraction. In this order, a more general contractive condition of all the preceding conditions was made by Hardy and Rogers (1973) and proved fixed point theorem. Moreover, it is given by

$$\begin{aligned} p(\breve{\phi}(x), \breve{\phi}(y)) \leq {} & \alpha p(x, y) + \beta p(x, \breve{\phi}(x)) + \gamma p(y, \breve{\phi}(y)) \\ & + \delta p(x, \breve{\phi}(y)) + \mu p(y, \breve{\phi}(x)) \end{aligned} \tag{5}$$

for every $x, y \in \mathfrak{P}$ and $\alpha, \beta, \gamma, \delta, \mu$ are any non-negative reals with the condition that $\alpha + \beta + \gamma + \delta + \mu < 1$.

Several predecessors analysed so many circumstances of metric and topological fractals in the early enhancement of multivalued fractals. Also, this is considered as the very beginning stage of the classical point-to-point to set-valued contractive mappings. Further extensions of the standard IFS to IMS were covered in Kunze et al. (2007). A few works that are made on the HR-type space is explored as below. Singh et al. (2009) obtained a Hardy-Rogers type fixed point theorem in cone 2-metric spaces over Banach algebras for a family of self-maps and a corollary was gained by Wang et al. (2015) using the result obtained by the same authors above mentioned. Shukla et al. (2013) enriched this field by giving some generalisations of Presic type contractions and also exhausted a fixed point theorem for Presic-Hardy-Rogers type contractive conditions in metric spaces.

In a 0-complete partially ordered partial metric space under HR-type contractive condition, Nashine et al. (2012) proved a fixed point theorem for a monotone self-map. This establishment enhances some results which are derived using weaker conditions. Moreover, this work is considered as an extension and strengthening of a few results in standard ordered metric spaces. Additionally, Arshad et al. (2015) developed some fixed point results in the sense of Hardy-Rogers type condition in a complete metric space for α-η-GF contraction.

Moreover, scholars are interested in doing research on different kinds of Hardy-Rogers contractive conditions to gain novel fixed point theorems. One of the works is done by Barman et al. (2020). In their article, they proved some common fixed point theorems using T-Hardy-Rogers type contraction condition and F-contraction on a complete 2-metric space. Also, Karapınar et al. (2019) used an interpolative approach to recognise the Hardy-Rogers fixed point theorem in the class of metric spaces. However, they gave a partial metric case, according to the result they obtained. In the recent days, Georgescu et al. (2020) initiated the notion of HR-IFS.

By considering all the above works and in view of application of fixed point results in various kinds of metric spaces, we focus on the HR-IMS. Moreover, we exhibited with this notion for proving Nadler's type result especially. Also, the elaboration of IFS to IMS in Hardy-Rogers type space is done for the first time in this paper and as a benefit of this result, we made the Nadler's fixed point theorem for set-valued contractive mappings.

So we emphasise the concept of HR-IMS to retrieve the Nadler's result. The present paper is regularised as follows: Section 2 emphasises the basic ideas and results for construction of our primary result. Also, it gives the perception of shift space. A key finding of this article is given in Section 3. That is, it explores derivation of attractor of HR-IMS. Finally, Section 4 exposes several consequences of the key result proved in Section 3. At last, by considering a particular case of the main result, we give the Nadler's type result for HR-IMS.

2 Preludes and preparations

This section gives the notion of HR-type IMS and some results that support the existence of Nadler's result. We have made the following notations for the simplicity of some long expansions. $(\mathfrak{P}, p)$ is always referred as a metric space. The set of all closed and bounded subsets of the metric space $(\mathfrak{P}, p)$ is marked by $CLB(\mathfrak{P})$. The collection of all closed subsets of $(\mathfrak{P}, p)$ are given by $CL(\mathfrak{P})$. $COM(\mathfrak{P})$ denotes the class of all compact subsets of the metric space $(\mathfrak{P}, p)$. $\mathbb{N} \cup \{0\}$ and $\mathbb{R}^+$ are the set of all positive integers together with zero and the set of all positive real numbers, respectively. The Hausdorff distance is defined as below.

Suppose $(\mathfrak{P}, p)$ is a metric space and $CLB(\mathfrak{P})$ is the set of all closed and bounded subsets of the metric space $(\mathfrak{P}, p)$. Then the Hausdorff distance (denoted as $\mathcal{H}_{\mathfrak{P}}$) between two elements $A, B \in CLB(\mathfrak{P})$ is a mapping from $CLB(\mathfrak{P}) \times CLB(\mathfrak{P})$ to $[0, \infty)$ given by

$$\mathcal{H}_{\mathfrak{P}}(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} p(a, b), \sup_{b\in B}\inf_{a\in A} p(b, a)\right\}$$

for every $A, B \in CLB(\mathfrak{P})$. Also, the distance between a point a and a subset B in the metric space $(\mathfrak{P}, p)$ is defined by

$$p'(a, B) = \inf\{p(a, b) | b \in B\}$$

The upcoming theorem shows the common form of the Banach contraction principle.

Theorem 2.1 (Pitchaimani and Kumar, 2017): A mapping $\breve{\phi}$ defined on a complete ultrametric space $(\mathfrak{P}, p)$ and having the set of all closed and bounded subsets of $\mathfrak{P}$ as the codomain, satisfying the inequality $\mathcal{H}_{\mathfrak{P}}(\breve{\phi}(a), \breve{\phi}(b)) \leq \alpha\, p(a, b)$ for every $a, b \in \mathfrak{P}$ and $0 \leq \alpha < 1$, has a unique fixed point in $\mathfrak{P}$.

Definition 2.1 (Miculescu and Mihail, 2019): Given a function $\breve{\phi}$ defined over a metric space $(\mathfrak{P}_1, p_1)$ with a possible codomain $CLB(\mathfrak{P}_2)$ (where $(\mathfrak{P}_2, p_2)$ is a metric space), we let

$$lip(\breve{\phi}) = \sup_{a,b\in\mathfrak{P}_1,\ a\neq b} \frac{\mathcal{H}_{\mathfrak{P}_2}(\breve{\phi}(a), \breve{\phi}(b))}{p_1(a, b)} \in [0, \infty]. \tag{6}$$

Such a mapping $\breve{\phi}$ is known as Lipschitz provided that $lip(\breve{\phi}) < \infty$ and it is said to be a contraction if $lip(\breve{\phi}) < 1$.

Definition 2.2 (Miculescu and Mihail, 2019): Suppose we have a complete metric space $(\mathfrak{P}, p)$ and a finite number of contractive mappings $\breve{\phi}_i : \mathfrak{P} \to CLB(\mathfrak{P}), i = 1, 2, \ldots, n$. An iterated multifunction system, simply called as IMS, is a duplet $\mathcal{S} = (\mathfrak{P}, (\breve{\phi}_i)_{i \in \{1,2,\ldots,n\}})$

Definition 2.3 (Georgescu et al., 2020): Take a complete metric space $(\mathfrak{P}, p)$ and consider a family of continuous functions $(\breve{\phi}_i)_{i \in I}, \breve{\phi}_i : \mathfrak{P} \to \mathfrak{P}$. Then a HR-IFS is defined by:

1 There exists $u, v, w \geq 0$ assures the following two properties:

  a $u + v + w < 1$

  b $p(\breve{\phi}_i(a), \breve{\phi}_i(b)) \leq u\, p(a, b) + vM_i(a, b) + wN_i(a, b)$,

  for each $i \in I$ and every $a, b \in \mathfrak{P}$, where $M_i(a, b) = p(a, \breve{\phi}_i(a)) + p(b, \breve{\phi}_i(a))$ and $N_i(a, b) = p(a, \breve{\phi}_i(b)) + p(b, \breve{\phi}_i(a))$.

2 There exists $a_1, b_1, b_2, c_1, c_2 \geq 0$ satisfying a couple of conditions:

  a $a_1 + b_1 + b_2 + c_1 + c_2 < 1/2$

  b $p(\breve{\phi}_i(a), (\breve{\phi}_i \circ \breve{\phi}_j)(b)) < a_1\, p(a, b) + b_1 M_j(a, b) + b_2 M_i(a, b) + c_1 N_j(a, b) + c_2 N_i(a, b)$,

  for every $i, j \in I$ and every $a, b \in \mathfrak{P}$.

Focusing on our primary aim and based on Definitions 2.2 and 2.3, we give the idea of HR-type IMS as below:

Definition 2.4: Let N be any positive integer. A finite collection of contractions $\{\breve{\phi}_n\}_{n=1}^N$ of the form $\breve{\phi}_n : \mathfrak{P} \to CLB(\mathfrak{P})$ defined on a complete metric space $(\mathfrak{P}, p)$ is said to construct a Hardy-Rogers type iterated multifunction system (HR-IMS) if

1 There are some $\sigma_1, \sigma_2, \sigma_3 \geq 0$ such that for each $n \in \{1, 2, \ldots, N\}$

  a $\sigma_1 + \sigma_2 + \sigma_3 < 1$

  b $\mathcal{H}_{\mathfrak{P}}(\breve{\phi}_n(a), \breve{\phi}_n(b)) \leq \sigma_1 p(a, b) + \sigma_2[p'(a, \breve{\phi}_n(a)) + p'(b, \breve{\phi}_n(b))] + \sigma_3[p'(a, \breve{\phi}_n(b)) + p'(b, \breve{\phi}_n(a))]$.

2 The following two conditions are satisfied by arbitrarily chosen $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \geq 0$

  c $\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 < 1/2$

  d $\mathcal{H}_{\mathfrak{P}}(\breve{\phi}_n(a), (\breve{\phi}_n \circ \breve{\phi}_m)(b)) \leq \delta_1 p(a, b) + \delta_2[p'(a, \breve{\phi}_m(a)) + p'(b, \breve{\phi}_m(b))] + \delta_3[p'(a, \breve{\phi}_n(a)) + p'(b, \breve{\phi}_n(b))] + \delta_4[p'(a, \breve{\phi}_m(b)) + p'(b, \breve{\phi}_m(a))] + \delta_5[p'(a, \breve{\phi}_n(b)) + p'(b, \breve{\phi}_n(a))]$

  for each $n, m \in \{1, 2, \ldots, N\}$ and every $a, b \in \mathfrak{P}$.

Such a system is denoted by $(HR-IMS(\mathfrak{P}), (\breve{\phi}_n)_{n\in\{1,2,\ldots,N\}})$.

Definition 2.5: Given a HR-IMS($\mathfrak{P}$) we implement a function $\Phi : \mathfrak{P} \to CLB(\mathfrak{P})$ defined by

$$\Phi(a) = \bigcup_{i=1}^{n} \breve{\phi}_i(a)$$

for each $a \in \mathfrak{P}$. Such a mapping Φ is said to be the Hutchinson-Barnsley operator generated by $\breve{\phi}_i$s.

Remark 2.1 (Miculescu and Mihail, 2019): The couple of metric spaces $(CLB(\mathfrak{P}), \mathcal{H}_{\mathfrak{P}}), (COM(\mathfrak{P}), \mathcal{H}_{\mathfrak{P}})$ are complete while the metric space $(\mathfrak{P}, p)$ is complete.

Remark 2.2 (Miculescu and Mihail, 2019): If a sequence of sets $\{P_n\}_{n\in\mathbb{N}\cup\{0\}}$ taken from $CLB(\mathfrak{P})$ converges with respect to $\mathfrak{P}$, where $(\mathfrak{P}, p)$ is a complete metric space, then $\lim\limits_{n\to\infty} P_n = \{m \in \mathfrak{P}|$ for each $n \in \mathbb{N}\cup\{0\}$, there is some $m_n \in P_n$ such that $\lim\limits_{n\to\infty} m_n = m\}$.

2.1 The shift space

Suppose $\mathfrak{Y}$ be a non-empty set. By $\Lambda(\mathfrak{Y})$ we mean the set of infinite words $\hat{\omega} = \hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n\hat{\omega}_{n+1}\ldots$ having each alphabet from $\mathfrak{Y}$. The set $\mathfrak{Y}^{\{1,2,\ldots,n\}}$ can be denoted by $\Lambda_n(\mathfrak{Y})$ for each $n \in \mathbb{N}$. i.e., any word with finite length $\hat{\omega} = \hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n$ belong to $\Lambda_n(\mathfrak{Y})$. The length of any word is notated by $\bar{\mathfrak{L}}(\hat{\omega})$. The set having a single word, particularly the empty word $\hat{\lambda}$ is given by $\Lambda_0(\mathfrak{Y})$. The countable union of $\Lambda_n(\mathfrak{Y})$ will be seen by

$$\Lambda^c(\mathfrak{Y}) = \bigcup_{n\in\mathbb{N}\cup\{0\}} \Lambda_n(\mathfrak{Y}).$$

Representation of the set of words consisting at most n letters with letters from the alphabet $\mathfrak{Y}$ is given by

$$ATMOST_n(\mathfrak{Y}) = \bigcup_{i\in\{1,2,\ldots,n\}} \Lambda_i(\mathfrak{Y})$$

For any two integers $n, m \in \mathbb{N}$ and arbitrary words $\hat{\theta} = \hat{\theta}_1\hat{\theta}_2\ldots\hat{\theta}_n \in \Lambda_n(\mathfrak{Y})$ and $\hat{\eta} = \hat{\eta}_1\hat{\eta}_2\ldots\hat{\eta}_m \in \Lambda_m(\mathfrak{Y})$ or $\hat{\eta} = \hat{\eta}_1\hat{\eta}_2\ldots\hat{\eta}_m\hat{\eta}_{m+1}\ldots \in \Lambda(\mathfrak{Y})$, the concatenation of the words $\hat{\theta}$ and $\hat{\eta}$, denoted by $\hat{\theta}\hat{\eta}$, is taken as

$$\hat{\theta}\hat{\eta} = \hat{\theta}_1\hat{\theta}_2\ldots\hat{\theta}_n\hat{\eta}_1\hat{\eta}_2\ldots\hat{\eta}_m$$

and respectively

$$\hat{\theta}_1\hat{\theta}_2\ldots\hat{\theta}_n\hat{\eta}_1\hat{\eta}_2\ldots\hat{\eta}_m\hat{\eta}_{m+1}\ldots$$

A truncation of an infinite word $\hat{\omega} = \hat{\omega}_1\hat{\omega}_2...\hat{\omega}_n\hat{\omega}_{n+1}... \in \Lambda(\mathfrak{Y})$ upto n is

$$\hat{\omega}|_n = \hat{\omega}_1\hat{\omega}_2...\hat{\omega}_n$$

For any $\hat{\omega} \in \Lambda(\{1,2,...,n\}) - \{\hat{\lambda}\}$, by $\hat{\omega}$, we let $\hat{\omega}\hat{\omega}\hat{\omega}...\hat{\omega}...$ Now define

$$p_\Lambda(\hat{\omega},\hat{\theta}) = \begin{cases} 0 & \hat{\omega} = \hat{\theta} \\ 2^{\frac{1}{\min\{i\in\mathbb{N}|\hat{\omega}_i \neq \hat{\omega}_i\}}} & \hat{\omega} \neq \hat{\theta} \end{cases}$$

where $\hat{\omega} = \hat{\omega}_1\hat{\omega}_2...\hat{\omega}_n\hat{\omega}_{n+1}...$ and $\hat{\theta} = \hat{\theta}_1\hat{\theta}_2...\hat{\theta}_n\hat{\theta}_{n+1}... \in \Lambda(\mathfrak{Y})$. Then p_Λ is a metric on $\Lambda(\mathfrak{Y})$ and $(\Lambda(\mathfrak{Y}), p_\Lambda))$ is a metric space.

Remark 2.3 (Miculescu and Mihail, 2019):

1 The convergence property in $(\Lambda(\mathfrak{Y}), p_\Lambda)$ is a metric space.

2 $(\Lambda(\mathfrak{Y}), p_\Lambda)$ is complete.

3 Suppose $\mathfrak{Y}$ is finite. Then $(\Lambda(\mathfrak{Y}), p_\Lambda)$ is compact.

1 and 2 are consequences of the convention that the Tyhonoff product topology is induced by the metric p_Λ.

Remark 2.4 (Miculescu and Mihail, 2019): Consider $m \in \mathbb{N}, (\hat{\omega}_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq \Lambda(\{1,2,...,m\})$ and $\hat{\omega} \in \Lambda(\{1,2,...,m\})$ such that $\lim_{n\to\infty} \hat{\omega}_n = \hat{\omega}$. Then for any $s \in \mathbb{N}$, we can choose a positive integer n_s with the termination that $\hat{\omega}_n|_s = \hat{\omega}|_s$ for all $n \in \mathbb{N}, n \geq n_s$

Theorem 2.2: For each HR-type IMS $(HR\text{-}IMS(\mathfrak{P}), (\breve{\phi}_n)_{n\in\{1,2,\ldots,N\}})$, we can find a unique $C_\mathfrak{P} \in CLB(\mathfrak{P})$ with $\Phi(C_\mathfrak{P}) = C_\mathfrak{P}$. Further, $\lim_{n\to\infty} \Phi^{[n]}(C) = C_\mathfrak{P}$ for all $C \in CLB(\mathfrak{P})$.

Proof: Our aim is to prove that the HR operator Φ admits a unique fixed point $C_\mathfrak{P} \in CLB(\mathfrak{P})$ with the aid of Theorem 2.3 and Definition 2.6. The ultimate result accompanies from Theorem 2.4 if we will show that the function Φ should follow:

$$\mathcal{H}_\mathfrak{P}(\Phi(a), \Phi(b)) \leq \breve{\phi}(p(a,b)) \quad \text{for each } a, b \in \mathfrak{P}$$

and for a suitable function $\breve{\phi}$. Certainly, we have

$$\begin{aligned} \mathcal{H}_\mathfrak{P}(\Phi(a), \Phi(b)) &= \mathcal{H}_\mathfrak{P}\left(\bigcup_{i=1}^{n} \Phi_i(a), \bigcup_{i=1}^{n} \Phi_i(b)\right) \\ &\leq \max\{\mathcal{H}_\mathfrak{P}(\Phi(a), \Phi(b))|i = 1,2,...,n\} \\ &\leq \max\{\breve{\phi}_i(p(a,b))|i = 1,2,...,n\} \\ &:= \breve{\phi}(p(a,b)) \end{aligned}$$

Here, the function $\breve{\phi}(s) := \max\{\breve{\phi}_i(s)|i = 1,2,...,n\},\ s \in \mathbb{R}^+$, is increasing, right semi-continuous and $\breve{\phi}(0) = 0,\ \breve{\phi}(s) < s$, for every $s > 0$. Hence, the result arrives at its conclusion from Theorem 2.4.

Theorem 2.3: Suppose $(HR\text{-}IMS(\mathfrak{P}), (\breve{\phi}_n)_{1,2,\ldots,N})$ is an HR-type IMS. Then the following results hold:

1 If each $\breve{\phi}_n$ is usc (respectively lsc) for each n, then Φ is usc (lsc).

2 $\overline{\bigcup_{i=1}^{n} \breve{\phi}_i(E)} = \overline{\bigcup_{e \in E} \bigcup_{i=1}^{n} \breve{\phi}_i(e)}$ for every $E \in CL(\mathfrak{P})$.

Proof:

1 The result follows from the conventional fact that finite union of usc (respectively lsc) multivalued operators is an usc (respectively lsc) multivalued operator.

2 The relation follows from the uncomplicated expressions given below:

$$\overline{\Phi(E)} = \overline{\bigcup_{e \in E} \Phi(e)} = \overline{\bigcup_{e \in E} \bigcup_{i}^{n} \breve{\phi}_i(e)} = \overline{\bigcup_{i=1}^{n} \breve{\phi}_i(E)}.$$

Definition 2.6: Let $(\mathfrak{P}, p)$ be an metric space and $(HR\text{-}IMS(\mathfrak{P}), (\breve{\phi}_n)_{1,2,\ldots,N})$ be an HR-type IMS. Then a multivalued fractal of the HR-IMS system is the fixed point for the HR operator Φ.

Theorem 2.4: Presume that $(\mathfrak{P}, p)$ is a complete metric space and $\hat{\zeta} : \mathfrak{P} \to CL(\mathfrak{P})$ is any function. If we have a special mapping $\breve{\phi} : \mathbb{R}^+ \to \mathbb{R}^+$ which is increasing and usc from the right with the property that $\breve{\phi}(0) = 0$, $\breve{\phi}(s) < s$, for every $s > 0$ and $\mathcal{H}_{\mathfrak{P}}(\hat{\zeta}(a), \hat{\zeta}(b)) \leq \breve{\phi}(p(a, b))$, for each $a, b \in \mathfrak{P}$, then

1 The self-mapping $\hat{\zeta}$ defined on $CL(\mathfrak{P})$ assures the inequality

$$\mathcal{H}_{\mathfrak{P}}\ (\hat{\zeta}(A_1), \hat{\zeta}(A_2)) \leq \breve{\phi}(\mathcal{H}_{\mathfrak{P}}(A_1, A_2)) \tag{7}$$

for every $A_1, A_2 \in CL(\mathfrak{P})$

2 Suppose additionally that $\hat{\zeta} : \mathfrak{P} \to CLB(\mathfrak{P})$. Then we have

$\hat{\rho} : CLB(\mathfrak{P}) \to CLB(\mathfrak{P})$ and $Fix(\hat{\rho}) = \{A^*\}$

3 Moreover if $\hat{\rho} : \mathfrak{P} \to COM(\mathfrak{P})$ or suppose the inequality in (7) is strict, then $Fix(\hat{\rho}) \neq \emptyset$.

Proof:

Claim 1: First we want to show that $\hat{\rho} : CLB(\mathfrak{P}) \to CLB(\mathfrak{P})$. Certainly it is evident that $\hat{\rho}(A) \in CL(\mathfrak{P})$. Take $A \in CLB(\mathfrak{P})$, $a, a_0 \in A$ and $b \in \hat{\zeta}(a), b_0 \in \hat{\zeta}(a_0)$. Then

$$\begin{aligned} \mathcal{H}_{\mathfrak{P}}(\hat{\zeta}(a), \hat{\zeta}(a_0)) &\leq \breve{\phi}(p(a, a_0)) \leq p(a, a_0) \\ &\leq \sup\{p(a, a_0) | a, a_0 \in A\} \end{aligned} \tag{8}$$

Then

$$\begin{aligned} p(b, b_0) &\leq \mathcal{H}_{\mathfrak{P}}(\hat{\zeta}(a), \hat{\zeta}(a_0)) + \{\sup \hat{\zeta}(a_0) | a_0 \in A\} \\ &\leq \sup\{A\} + \sup\{\hat{\zeta}(a_0)\} \\ &:= \mathfrak{P} < +\infty \end{aligned}$$

Now, if we take $c, d \in \hat{\rho}(A)$, then for an arbitrary $\hat{\varepsilon} > 0$ we can choose $b_1, b_2 \in \hat{\zeta}(A)$ such that $p(c, b_1) \leq \hat{\varepsilon},\ p(d, b_2) \leq \hat{\varepsilon}$ and $p(b_1, b_0) \leq M,\ p(b_2, b_0) \leq M$, for some constant M. Then, we obtain $p(c, d) \leq 2M + 2\hat{\varepsilon}$ for each $\hat{\varepsilon} > 0$. Also, this can be $\sup\{\hat{\zeta}(A)\} \leq 2M$ and our Claim 1 follows. Now, we shall show that

$$\mathcal{H}_{\mathfrak{P}}(\hat{\zeta}(A_1), \hat{\zeta}(A_2)) \leq \breve{\phi}(\mathcal{H}_{\mathfrak{P}}(A_1, A_2))$$

for each $A_1, A_2 \in CLB(\mathfrak{P})$.

Let us take $A_1, A_2 \in CLB(\mathfrak{P}),\ a_1 \in A_1,\ a_2 \in A_2$ and $c \in \hat{\zeta}(b_1)$. Then

$$\begin{aligned} \inf\{p(c, c_1) | c_1 \in \hat{\rho}(A_2)\} &\leq \inf\{p(c, c_1) | c_1 \in \hat{\zeta}(a_2)\} \\ &\leq \mathcal{H}_{\mathfrak{P}}(\hat{\zeta}(a_1), \hat{\zeta}(a_2)) \\ &\leq \breve{\phi}(p(a_1, a_2)) \end{aligned}$$

Consequently,

$$\inf\{p(c, c_1) | c_1 \in \hat{\rho}(A_2)\} \leq \inf_{a_2 \in A_2} \breve{\phi}(p(a_1, a_2))$$

Assume that $r := \inf\{p(a_1, a_2) | a_2 \in A_2\}$. We can detect a sequence (a_n) in A_2 having the effect that the sequence $(p(a_1, a_2))$ reduces to r. Because $\breve{\phi}$ is a special function having its own property, we sequentially get:

$$\breve{\phi}(r) \geq \limsup_{n \to \infty} \breve{\phi}(p(a_1, a_n)) \geq \inf_{a_2 \in A_2} \breve{\phi}(p(a_1, a_2))$$

with the aid of increasing property of $\breve{\phi}$, we can see

$$\begin{aligned} \inf\{p(c, c_1) | c_1 \in \hat{\rho}(A_2)\} &\leq \breve{\phi}(\inf\{p(a_1, a_2) | a_2 \in A_2\}) \\ &\leq \breve{\phi}(\mathcal{H}_{\mathfrak{P}}(A_1, A_2)) \end{aligned}$$

$$\sup_{c \in \hat{\zeta}(a_1)} \{\inf\{p(c, c_1) | c_1 \in \hat{\zeta}(A_2)\}\} \leq \breve{\phi}(\mathcal{H}_{\mathfrak{P}}(a_1, A_2))$$

$$\sup_{c \in \hat{\zeta}(A_1)} \inf_{c_1 \in \hat{\rho}(A_2)} p(c, c_1) \leq \breve{\phi}(\mathcal{H}_{\mathfrak{P}}(A_1, A_2))$$

We can obtain an inequality of the last expression in a symmetric sense. That is

$$\sup_{c_1 \in \hat{\rho}(A_2)} \inf_{c \in \hat{\zeta}(A_1)} p(c_1, c) \leq \breve{\phi}(\mathcal{H}_{\mathfrak{P}}(A_1, A_2))$$

$$\max\left\{ \sup_{c \in \hat{\zeta}(A_1)} \inf_{c_1 \in \hat{\rho}(A_2)} p(c, c_1) \right\} \leq \breve{\phi}(\mathcal{H}_{\mathfrak{P}}(A_1, A_2))$$

for all $A_1, A_2 \in CLB(\mathfrak{P})$.

Next our intend is to verify that $Fix(\hat{\rho}) = \{A^*\}$. It is a consequence of the Boyd-Wong fixed point theorem. Finally, we need to check (3).

Suppose $\hat{\zeta}(m)$ is an arbitrary element in $COM(\mathfrak{P})$ for each $m \in \mathfrak{P}$. From the upper semi continuity property of $\hat{\zeta}$, we can have that for every $A \in COM(\mathfrak{P})$, $\hat{\zeta}(A) \in COM(\mathfrak{P})$. This will imply that $\hat{\zeta}$ is a self-map on $COM(\mathfrak{P})$ and $\hat{\zeta}(A) = \underset{a\in A}{\cup}\hat{\zeta}(a)$.

Similar to the above part we can have a unique fixed point A^* in $COM(\mathfrak{P})$ of $\hat{\zeta}$. A consequent result of this and from Smithson (1971), we can show a fixed point $a^* = \hat{\zeta}(a^*)$ for the contractive mapping $\hat{\zeta}|_{A^*} : A^* \to COM(A^*)$.

From Wegrzyk's theorem (Alsulami, 2013), we can see the same condition for the other case.

Definition 2.7: The set $C_{\mathfrak{P}}$ obtained from Theorem 2.2 is known as the HR-IMS attractor of the HR-type IMS $(HR\text{-}IMS(\mathfrak{P}), (\breve{\phi}_n)_{n\in\{1,2,\ldots,N\}})$.

3 Main outcomes

Theorem 3.1: Suppose we have a HR-type IMS $(HR\text{-}IMS(\mathfrak{P}), (\breve{\phi}_n)_n \in \{1, 2, ..., N\})$, $u_0 \in \mathfrak{P}$, $u_1 \in \breve{\phi}_1(u_0)$, ..., $u_m \in \breve{\phi}_m(u_0)$ and $q \in (p, 1)$, where $\max\limits_{1\leq n\leq m} lip(\breve{\phi}_n) \overset{not}{=} p$. Then, we can select a family $(x_{\hat{\omega}})_{\hat{\omega}\in\Lambda^C(\{1,2,\ldots,m\})}$ which are entries from $\mathfrak{P}$ such that

1 $x_\lambda = u_0, x_1 = u_1, \ldots, x_m = u_m$.

2 $x_{i\hat{\omega}} \in \breve{\phi}_i(x_{\hat{\omega}}) \forall i \in \{1, 2, \ldots, m\}$ and $\hat{\omega} \in \Lambda^C(\{1, 2, \ldots, m\})$.

3 For each $n \in \mathbb{N}$ and every $\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_m \in \{1, 2, \ldots, m\}$

$$p\left(x_{\hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n\hat{\omega}_{n+1}}, x_{\hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n}\right) \leq qp\left(x_{\hat{\omega}_2\ldots\hat{\omega}_{n+1}}, x_{\hat{\omega}_2\ldots\hat{\omega}_n}\right)$$

Moreover, we have

4 For each $\hat{\omega} \in \Lambda(\{1, 2, \ldots, m\})$, there is $\underset{n\to\infty}{lim} x_{\hat{\omega}|_n}$ and it is notated by $x_{\hat{\omega}}$.
With the aid of the notation $R_n = \{x_{\hat{\omega}} | \hat{\omega} \in \Lambda_n(\{1, 2, \ldots, m\})\}$, we obtain

5 $\mathcal{H}_{\mathfrak{P}}(R_n, R_{n+1}) \geq q^n \mathcal{H}_{\mathfrak{P}}(R_0, R_1) \forall n \in \mathbb{N} \cup \{0\}$.

6 There survives $A \in COM(\mathfrak{P})$ so that $\lim\limits_{n\to\infty} R_n = A$.

7 $R_{n+1} \subseteq \Phi(R_n) \forall n \in \mathbb{N} \cup \{0\}$.

8 $A = \{x_{\hat{\omega}} | \hat{\omega} \in \Lambda(\{1, 2, \ldots, m\})\}$.

9 A specific function $\diamond : \Lambda(\{1, 2, \ldots, m\}) \to A$ identified by
$\diamond(\hat{\omega}) = x_{\hat{\omega}} \forall \hat{\omega} \in \Lambda(\{1, 2, \ldots, m\})$, is a surjective continuous mapping and
$\diamond(i\hat{\omega}) \in \breve{\phi}_i(\diamond(\hat{\omega})) \forall i \in \{1, 2, \ldots, m\}$ and $\hat{\omega} \in \Lambda(\{1, 2, \ldots, m\})$.

10 $A \subseteq A^*$, where A^* is the attractor of HR-type IMS $(HR\text{-}IMS(\mathfrak{P})$,
$(\breve{\phi}_n)_n \in \{1, 2, ..., n\})$.

Proof:

1–3 For the existence of a collection $(x_{\hat{\omega}})_{\hat{\omega}\in\Lambda^C(\{1,2,\ldots,m\})}$ of components from $\mathfrak{P}$ gratifying the facts (1), (2) and (3), we will make use of the method of mathematical induction on $n = \bar{\mathfrak{L}}(\hat{\omega})$.

For $n = 0$, we regard $x_\lambda = u_0$ and in the case $n = 1$, we select $x_1 = u_1, \ldots, x_m$. Thus (1) is true.

Suppose, as an induction hypothesis, that for any $n \in \mathbb{N}$, we established a family $(x_{\hat{\omega}})_{\hat{\omega}} \in ATMOST_n(\{1, 2, \ldots, m\})$ satisfies (1), (2) and (3).

Now we select $\hat{\omega} = \hat{\omega}_1\hat{\omega}_2 \ldots \hat{\omega}_n\hat{\omega}_{n+1} \in ATMOST_{n+1}(\{1, 2, \ldots, m\})$. Suppose $x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n} \neq x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_{n+1}}$, as $\hat{\omega}_1 \ldots \hat{\omega}_n \in \Lambda_n(\{1, 2, \ldots, m\})$ we have $x_{\hat{\omega}_1\ldots\hat{\omega}_n} \in \breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n})$, and so

$$\begin{aligned}
&p(x_{\hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n}, \breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n\hat{\omega}_{n+1}})) \\
&\leq \sup_{x\in\breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n})} p(x, \breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n\hat{\omega}_{n+1})}) \\
&\leq \mathcal{H}_{\mathfrak{P}}(\breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n}), \breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n\hat{\omega}_{n+1}})) \\
&< qp(x_{\hat{\omega}_2\ldots\hat{\omega}_n}, x_{\hat{\omega}_2\ldots\hat{\omega}_n\hat{\omega}_{n+1}})
\end{aligned}$$

Thus we have $p(x_{\hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n}, x_{\hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n\hat{\omega}_{n+1}}) \leq qp(x_{\hat{\omega}_2\ldots\hat{\omega}_n}, x_{\hat{\omega}_2\ldots\hat{\omega}_n\hat{\omega}_{n+1}})$ for some $x_{\hat{\omega}_1\ldots\hat{\omega}_n\hat{\omega}_{n+1}} \in \breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n\hat{\omega}_{n+1}})$. On the other hand, i.e., if $x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n} = x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_{n+1}}$, we get $x_{\hat{\omega}_1\ldots\hat{\omega}_n} \in \breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n})$ $= \breve{\phi}_{\hat{\omega}_1}(x_{\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n\hat{\omega}_{n+1}})$ and we can select $x_{\hat{\omega}_1\hat{\omega}_2\ldots\hat{\omega}_n\hat{\omega}_{n+1}} = x_{\hat{\omega}_1\hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_n}$. In the ecbatic, the notation $S \overset{not}{=} \max\{p(u_0, u_1), \ldots, p(u_0, u_m)\}$ is followed.

4 Let $\hat{\omega} \in \Lambda(\{1, \ldots, m\})$. Then, with the reference of (3), we see

$$p(x_{\hat{\omega}|_n}, x_{\hat{\omega}}|_{n+1}) \leq q^n S, \text{ for each } n \in \mathbb{N}. \tag{9}$$

$\therefore (x_{\hat{\omega}|_n})_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence and because $(\mathfrak{P}, p)$ is complete, $\lim\limits_{n\to\infty} x_{\hat{\omega}|_n}$ exists.

5 Consider

$$\begin{aligned}
&\mathcal{H}_{\mathfrak{P}}(R_n, R_{n+1}) \\
&= \mathcal{H}_{\mathfrak{P}}\left(\{x_{\hat{\omega}}|\hat{\omega} \in \Lambda_n(\{1, 2, \ldots, m\})\}, \{x_{\hat{\omega}}|\hat{\omega} \in \Lambda_{n+1}(\{1, 2, \ldots, m\})\}\right) \\
&= \mathcal{H}_{\mathfrak{P}}(\{x_{\hat{\omega}}|\hat{\omega} \in \Lambda_n \left(\{1, 2, \ldots, m\})\}, \bigcup_{i=1}^{m}\{x_{\hat{\omega}_i}|\hat{\omega} \in \Lambda_n(\{1, 2, \ldots, m\})\}\right) \\
&\leq \max_{\hat{\omega}\in\Lambda_n(\{1,\ldots,m\})} \mathcal{H}_{\mathfrak{P}}\left(\{x_{\hat{\omega}}\}, \{x_{\hat{\omega}_i}|i \in \{1, 2, \ldots, m\}\}\right) \\
&\leq \max_{\hat{\omega}\in\Lambda_n(\{1,\ldots,m\})} p(x_{\hat{\omega}}, x_{\hat{\omega}_i}),\ i \in \{1, 2, \ldots, m\} \\
&\leq q^n S, \text{ by (9)} \\
&= q^n \mathcal{H}_{\mathfrak{P}}(R_0, R_1) \forall n \in \mathbb{N}.
\end{aligned} \tag{10}$$

6 Because R_n is finite for each $n \in \mathbb{N} \cup \{0\}$, we see that $R_n \in COM(\mathfrak{P})$ for every $n \in \mathbb{N} \cup \{0\}$. Also, the sequence $(R_n)_{n\in\mathbb{N}\cup\{0\}}$ is Cauchy and it is convergent from part (5) of this theorem and Remark 2.1, i.e.,

$$\lim_{n\to\infty} R_n = A \text{ for some } A \in COM(\mathfrak{P})$$

7 Now

$$\begin{aligned}
R_{n+1} &= \{x_{i\hat{\omega}} | i \in \{1,\ldots,m\} \text{ and } \hat{\omega} \in \Lambda_n(\{1,2,\ldots,m\})\} \\
&= \bigcup_{i=1}^{m} \{x_{i\hat{\omega}} | \hat{\omega} \in \Lambda_n(\{1,2,\ldots,m\})\} \\
&\subseteq \bigcup_{i=1}^{m} \bigcup_{\hat{\omega}\in\Lambda_n(\{1,\ldots,m\})} \breve{\phi}_i(x_{\hat{\omega}}) \\
&= \bigcup_{i=1}^{m} \breve{\phi}_i(R_n) \\
&\subseteq \breve{\phi}(R_n)\ \forall n \in \mathbb{N} \cup \{0\}
\end{aligned}$$

8 Observe that

$$A \supseteq \{x_{\hat{\omega}} | \hat{\omega} \in \Lambda(\{1,\ldots,m\})\} \tag{11}$$

Also, for every $\hat{\omega} \in \Lambda(\{1,\ldots,m\})$, we obtain $\lim\limits_{n\to\infty} x_{\hat{\omega}}|_n = x_{\hat{\omega}}$ and $x_{\hat{\omega}}|_n \in R_n \forall n \in \mathbb{N} \cup \{0\}$. Further, from Remark 2.2, we ave $x_{\hat{\omega}} \in A$. But

$$A \subseteq \{x_{\hat{\omega}} | \hat{\omega} \in \Lambda(\{1,\ldots,m\})\} \tag{12}$$

Certainly, suppose $x \in A = \lim\limits_{n\to\infty} R_n$, then we can see $\hat{\omega}_n \in \Lambda_n(\{1,\ldots,m\})$ with $\lim\limits_{n\to\infty} x_{\hat{\omega}_n} = x$ (by Remark 2.2). Also from Remark 2.3 (3), that $\Lambda(\{1,\ldots,m\})$ is compact, we can take out a convergent subsequence from the sequence $(\hat{\omega}_n\rho_0)_{n\in\mathbb{N}\cup\{0\}} \subset \Lambda(\{1,\ldots,m\})$, where $\rho_0 \in \Lambda(\{1,\ldots,m\})$ is fixed, so we shall have $\hat{\omega} \in \Lambda(\{1,\ldots,m\})$ and $(j_n)_{n\in\mathbb{N}\cup\{0\}}$ a strictly increasing subsequence of $\mathbb{N}$ so that $\lim\limits_{n\to\infty} \hat{\omega}_{j_n}\rho_0 = \hat{\omega}$.

Our next intension is to prove that $x = x_{\hat{\omega}}$.

Claim 1: $p(x_{\hat{\omega}}, x_{\hat{\omega}\hat{\omega}_1}) \leq \frac{q^{\bar{\mathfrak{L}}(\hat{\omega})}}{1-q} S$ for every $\hat{\omega} \in \Lambda^C(\{1,\ldots,m\})$ and $\hat{\omega}_1 \in \Lambda(\{1,\ldots,m\})$.

Proof: Consider

$$\begin{aligned}
p(x_{\hat{\omega}}, x_{\hat{\omega}\hat{\omega}_1}) &\leq p(x_{\hat{\omega}\hat{\omega}_1}, x_{(\hat{\omega}\hat{\omega}_1)|_n}) + p(x_{(\hat{\omega}\hat{\omega}_1)|_n}, x_{\hat{\omega}}) \\
&\leq p(x_{\hat{\omega}\hat{\omega}_1}, x_{(\hat{\omega}\hat{\omega}_1)|_n}) \\
&\quad + p(x_{\hat{\omega}}, x_{(\hat{\omega}\hat{\omega}_1)|_{\bar{\mathfrak{L}}(\hat{\omega})+1}}) + p(x_{(\hat{\omega}\hat{\omega}_1)|_{\bar{\mathfrak{L}}(\hat{\omega})+1}}, x_{(\hat{\omega}\hat{\omega}_1)|_{\bar{\mathfrak{L}}(\hat{\omega})+2}}) \\
&\quad + \cdots + p(x_{(\hat{\omega}\hat{\omega}_1)|_{n-1}}, x_{(\hat{\omega}\hat{\omega}_1)|_n}) \\
&\leq p(x_{\hat{\omega}\hat{\omega}_1}, x_{(\hat{\omega}\hat{\omega}_1)|_n}) \\
&\quad + q^{\bar{\mathfrak{L}}(\hat{\omega})} S(1 + q + \ldots + q^{n-1-\bar{\mathfrak{L}}(\hat{\omega})}) \text{ by part (3)} \\
&\leq p(x_{\hat{\omega}\hat{\omega}_1}, x_{(\hat{\omega}\hat{\omega}_1)|_n}) + \frac{q^{\bar{\mathfrak{L}}(\hat{\omega})}}{1-q} S
\end{aligned}$$

for each $n \in \mathbb{N} \cup \{0\}, n > \bar{\mathfrak{L}}(\hat{\omega})$. Allowing $n \to \infty$, we have Claim 1.

Claim 2: $p(x_{\hat{\omega}\hat{\omega}_1}, x_{\hat{\omega}\hat{\omega}_2}) \leq \frac{2q^{\bar{\mathfrak{L}}(\hat{\omega})}}{1-q} S$ for all $\hat{\omega} \in \Lambda^C(\{1, \ldots, m\})$ and $\hat{\omega}_1, \hat{\omega}_2 \in \Lambda(\{1, \ldots, m\})$.

Proof: By considering the triangle inequality for the three elements $x_{\hat{\omega}\hat{\omega}_1}, x_{\hat{\omega}}, x_{\hat{\omega}\hat{\omega}_2}$ we have Claim 2. Choose an $\check{\epsilon} > 0$ arbitrarily, but fixed. Consider $r_0 \in \mathbb{N} \cup \{0\}$ such that $\frac{2q^{r_0}}{1-q} S < \frac{\check{\epsilon}}{3}$. With the reference of Remark 2.4 and since $\lim_{n \to \infty} \hat{\omega}_{j_n \rho_0} = \hat{\omega}$, an element $n_0 \in \mathbb{N} \cup \{0\}$ can be selected with the fact that

$$\hat{\omega}_{j_n \rho_0}|_{r_0} = \hat{\omega}|_{r_0} \ \forall n \in \mathbb{N} \cup \{0\}, \ n \geq n_0$$

$$\therefore p\left(x_{\hat{\omega}}, x_{\hat{\omega}_{j_n \rho_0}}\right) \leq \frac{2q^{r_0}}{1-q} S < \frac{\check{\epsilon}}{3}, \quad \text{by Claim 2} \tag{13}$$

for each $n_0 \in \mathbb{N} \cup \{0\}, n \geq n_0$.

Further, since $\lim_{n \to \infty} \frac{2q^{j_n}}{1-q} S = 0$, there is some $n_1 \in \mathbb{N} \cup \{0\}$ such that $\frac{2q^{j_n}}{1-q} S < \frac{\check{\epsilon}}{3}$, thus

$$p\left(x_{\hat{\omega}_{j_n \rho_0}}, x_{\hat{\omega}_{j_n}}\right) \leq \frac{2q^{r_0}}{1-q} S < \frac{\check{\epsilon}}{3}, \quad \text{by Claim 2} \tag{14}$$

for every $n \in \mathbb{N} \cup \{0\}, n \geq n_1$.

Through the convergent property of the sequence $(x_{\hat{\omega}_{j_n}})$, we will have $n_2 \in \mathbb{N} \cup \{0\}$ with

$$p\left(x_{\hat{\omega}_{j_n}}, x\right) < \frac{\check{\epsilon}}{3} \tag{15}$$

for all $n \in \mathbb{N} \cup \{0\}, n \geq n_2$. Let us take $n \geq \max\{n_0, n_1, n_2\}$ and we have

$$\begin{aligned} p(x, x_{\hat{\omega}}) &\leq p(x, x_{\hat{\omega}_{j_n}}) + p(x_{\hat{\omega}_{j_n}}, x_{\hat{\omega}_{j_n \rho_0}}) + p(x_{\hat{\omega}_{j_n \rho_0}}, x_{\hat{\omega}}) \\ &\leq \check{\epsilon} \ \text{ by (13), (14), (15)} \end{aligned}$$

Because $\check{\epsilon}$ is arbitrary, we finalise that $x = x_{\hat{\omega}}$.

9 A restatement of (8) as $A = \diamond(\Lambda(\{1, \ldots, m\}))$ shows that $\diamond$ is onto. To prove continuity of $\diamond$, let us choose $(\hat{\omega}_n)_{n \in \mathbb{N} \cup \{0\}} \subseteq \Lambda(\{1, \ldots, m\})$ and $\hat{\omega} \in \Lambda(\{1, \ldots, m\})$ such that $\lim_{n \to \infty} \hat{\omega}_n = \hat{\omega}$. Given $\check{\epsilon} > 0$ and belongs to symbol $r_0 \mathbb{N} \cup \{0\}$ such that $\frac{2q^{r_0}}{1-q} S < \check{\epsilon}$. Similar to the above case we can have

$$p(x_{\hat{\omega}_n}, x_{\hat{\omega}}) \leq \frac{2q^{r_0}}{1-q} S < \check{\epsilon}$$

with the aid of Remark 2.4 for every $n \in \mathbb{N}, n \geq n_0$, so $\lim_{n \to \infty} x_{\hat{\omega}_n} = x_{\hat{\omega}}$, i.e.,

$$\lim_{n \to \infty} \diamond(\hat{\omega}_n) = \diamond(\hat{\omega})$$

Subsequently, $\diamond$ is continuous. To show the third part of (9):

In view of (2) and (4) of this theorem, for each $i \in \{1, \dots, m\}$ and every $\hat{\omega} \in \Lambda(\{1, \dots, m\})$, we get

$$x_{i\hat{\omega}|_n} \in \breve{\phi}_i(x_{\hat{\omega}|_n}) \forall n \in \mathbb{N} \cup \{0\}$$

and

$$\lim_{n\to\infty} x_{i\hat{\omega}|_n} = x_{i\hat{\omega}}.$$

By Remark 2.4, we predict that $x_{i\hat{\omega}} \in \lim\limits_{n\to\infty} \breve{\phi}_i(x_{\hat{\omega}|_n})$. Since $\breve{\phi}_i$ is continuous and $\lim\limits_{n\to\infty} x_{\hat{\omega}|_n} = x_{\hat{\omega}}$ [by (4)], we have

$$\lim_{n\to\infty} \breve{\phi}(x_{\hat{\omega}|_n}) = \breve{\phi}(x_{\hat{\omega}})$$

As the above fact, we see $x_{i\hat{\omega}} \in \breve{\phi}_i(x_{\hat{\omega}})$, i.e., $\Diamond(i\hat{\omega}) \in \breve{\phi}_i(\Diamond_{\hat{\omega}})$.

10 Using the third part of (9), we will have

$$A \subseteq \Phi(A), \text{ so } A \subseteq \Phi^{[n]}(A) \tag{16}$$

for every $n \in \mathbb{N}$. Take $x \in A$. Then $x = \lim\limits_{n\to\infty} x_n$, where $x_n = x \in \Phi^{[n]}(A)$ for all $n \in \mathbb{N}$, so by the usage of Remark 2.4, gives $x \in \lim\limits_{n\to\infty} \Phi^{[n]}(A) = A^*$. Thus $A \subseteq A^*$.

4 Final remarks

Remark 4.1: Due to our construction of main result, we obtain

$$x_{\hat{\omega}} = \Diamond(\hat{\omega}) \in \bigcap_{n\in\mathbb{N}} \overline{(A^*)_{\hat{\omega}|_n}},$$

for every $\hat{\omega} \in \Lambda(\{1, 2, ..., m\})$. Subsequently, we have

$$\Diamond(\hat{\omega}) \in (A)_{\hat{\omega}|_n} \subseteq (A^*)_{\hat{\omega}|_n} \subseteq \overline{(A^*)_{\hat{\omega}|_n}}$$

for every $n \in \mathbb{N}$, so we arrive that $\Diamond(\hat{\omega}) \in \bigcap\limits_{n\in\mathbb{N}} \overline{(A^*)_{\hat{\omega}|_n}}$. Also, the following inclusions hold

$$A^* \supseteq \overline{(A^*)_{\hat{\omega}|_1}} \supseteq \overline{(A^*)_{\hat{\omega}|_2}} \supseteq ... \supseteq \overline{(A^*)_{\hat{\omega}|_n}} \supseteq \overline{(A^*)_{\hat{\omega}|_{n+1}}} \supseteq ...$$

for every $n \in \mathbb{N}$.

Remark 4.2: Suppose the space $\Lambda(\{1, 2, ..., m\})$ is imposed with the metric described by $\Diamond_q(\hat{\omega}, \hat{\omega}_1) = 0$ for $\hat{\omega} = \hat{\omega}_1$ and

$$\Diamond_q(\hat{\omega}, \hat{\omega}_1) = q^n$$

for $\hat{\omega} \neq \hat{\omega}_1,\ \hat{\omega}|_n = \hat{\omega}_1|_n\ and\ \hat{\omega}|_{n+1} = \hat{\omega}_1|_{n+1}$

then the mapping $\diamond : \Lambda(\{1, 2, ..., m\}) \to A$ is Lipschitz. Therefore, we have

$$p(\diamond(\hat{\omega}), \diamond(\hat{\omega}_1)) \leq \frac{2\hat{s}}{1-q} p_q(\hat{\omega}, \hat{\omega}_1),$$

from the proof of Claim 2 of (8), for each $\hat{\omega}, \hat{\omega}_1 \in \Lambda\{1, 2, ..., m\}$, so $\diamond$ is Lipschitz with $lip(\diamond) \leq \frac{2\hat{s}}{1-q}$.

Remark 4.3: Notice that in the construction of main result, in particular by part (9), we get

$$\diamond(\hat{\omega}') \in \breve{\phi}_{\hat{\omega}_1}(\breve{\phi}_{\hat{\omega}_2}(...(\breve{\phi}_{\hat{\omega}_n}(\diamond(\hat{\omega}')))))$$

for every $\hat{\omega} = \hat{\omega}_1\hat{\omega}_2...\hat{\omega}_n \in \Lambda_n(\{1, 2, ..., m\})$, where $n \in \mathbb{N}$. This interpretation shows that $\diamond(\hat{\omega}')$ is a fixed point of $\breve{\phi}_{\hat{\omega}}$ for every $\hat{\omega} = \Lambda^C(\{1, 2, ..., m\})\backslash\{\hat{\lambda}\}$.

Definition 4.1: Let $\hat{C}$ be a compact Hausdorff topological space. Suppose a finite set of continuous functions $\breve{\phi}_1, \breve{\phi}_2, ..., \breve{\phi}_m : \hat{C} \to \hat{C}$, where $m \in \mathbb{N}$, and a surjective continuous function $\diamond : \Lambda\{1, 2, ..., m\} \to \hat{C}$ such that the picture

$$\begin{array}{ccc} \Lambda(\{1,2,...,m\}) & \xrightarrow{\tau_i} & \Lambda(\{1,2,...,m\}) \\ \diamond\downarrow & & \diamond\downarrow \\ \hat{C} & \xrightarrow{\breve{\phi}_i} & \hat{C} \end{array}$$

commutes for every $i \in \{1, 2, ..., m\}$, where

$$\tau_i(\hat{\omega}_1\hat{\omega}_2...\hat{\omega}_n\hat{\omega}_{n+1}...) = i\ \hat{\omega}_1\hat{\omega}_2...\hat{\omega}_n\hat{\omega}_{n+1}...$$

for each $\hat{\omega}_1\hat{\omega}_2...\hat{\omega}_n\hat{\omega}_{n+1}... \in \Lambda(\{1, 2, ..., m\})$. Then the set $\hat{C}$ is called a topological self-similar set. The pair $(\hat{C}, \{\breve{\phi}_i\}_{i\in\{1,2,...,m\}})$ is said to be a topological self-similar system.

Taking as a prelude this definition, we make the following notation:

Definition 4.2: Consider an HR-type IMS $(HR\text{-}IMS(\mathfrak{P}), (\breve{\phi}_n)_{n\in\{1,2,...,m\}})$. A mapping $\hat{r} : \Lambda(\{\{1, 2, ..., m\}\}) \to A^*$ is known as a self-similar section of HR-type IMS if $\hat{r}$ assures the following conditions:

1 $\hat{r}$ is continuous

2 $\hat{r}(i\ \hat{\omega}) \in \breve{\phi}_i(\hat{r}(\hat{\omega}))\ \forall\ i \in Y$ and every $\hat{\omega} \in \Lambda(\{\{1, 2, ..., m\}\})$.

From the subdivisions (ix) and (x) of the main result, we see that $\diamond$ is a self-similar section of $(HR\text{-}IMS(\mathfrak{P}), (\breve{\phi}_n)_{n\in\{1,2,...,m\}})$. Further, suppose the continuous contractions $\breve{\phi}_i$ in the $\mathfrak{P}$ containing precisely one element (i.e., $HR\text{-}IMS(\mathfrak{P})$, is an IFS), then the couple $(A, \{\breve{\phi}_i\}_{i\in\{1,2,...,m\}})$ is a topological self-similar system.

Remark 4.4: A generalisation of Nadler's theorem in Hardy-Rogers space can be stated as follows and it takes the value $m = 1$ in our main result:

If $(\mathfrak{P}, p)$ is a complete metric space, and for a mapping $\breve{\phi} : \mathfrak{P} \to CLB(\mathfrak{P})$ we can see $m \in [0, 1)$ so that $\mathcal{H}_{\mathfrak{P}}(\breve{\phi}(r), \breve{\phi}(s)) \leq ap(r, s)$ for each $r, s \in \mathfrak{P},\ b \in (a, 1),\ u_0 \in \mathfrak{P}$ and $u_1 \in \breve{\phi}(u_0)$.

Then there is some sequence of elements $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ from $\mathfrak{P}$ which is convergent and its limit is denoted as x so that

1 $x_0 = u_0;\ x_1 = u_1$

2 $x_{n+1} \in \breve{\phi}(x_n)$

3 $p(x_{n+1}, x_n) \leq b^n p(x_1, x_0)\ \forall n \in \mathbb{N} \cup \{0\}$

4 $x \in \breve{\phi}(x)$.

5 Conclusions

Uneven physical and natural scenarios can be regularised by a mathematical potential called fractal. One of the most common ways of generating fractals is the IFS. Since a sufficient framework for classifying and describing fractals is IFS, several mathematicians enlarged this theory to a more general case, known as IMS. As an impact of this situation, we have defined the notion of HR-IMS and the corresponding Hutchinson-Barnsley operator for the IMS. By proving the existence of a unique fixed point of the HB operator, we assured the survival of the attractor of the HR-type IMS. Moreover, several remarks are given based on the main result. The attractor of this IMS is constructed in an unconventional method that is non-identical with the existing procedures. Utilisation of this significant outcome conveys some crucial facts and one of the particular consequences is Nadler's result.

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