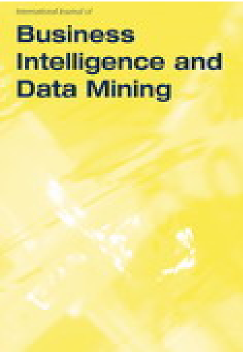

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Forecasting with information extracted from the residuals of ARIMA in financial time series using continuous wavelet transform

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Abstract: Time series of financial or economic data are often considered to have certain trends and patterns. It is believed that the study of historical patterns helps in the forecasting into the future. ARIMA model is one of the popular models for the task. However, long-term forecasting with ARIMA often appears as a straight line. This is due to ARIMA's dependency on previous values and its tendency to omit the outliers that lie outside of the captured general trend. This paper sought to capture useful outlier information from the residual of ARIMA modelling by using continuous wavelet transform (CWT). The CWT captured information was then added to the ARIMA forecasted values to form non-homogenous long-term forecasting. The results were encouraging. It was also found that choices of certain CWT related parameters have positive or negative effect to the forecasting outcomes.

Keywords: wavelet; forecasting; autoregressive integrated moving average; ARIMA; time series; continuous wavelet transform; CWT.

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1 Introduction

Mathematically, time series is a series of time-indexed data points that occur in successive order over certain period of time. Some financial market data do form time series. Given by the microstructure of the financial market, financial time series are often influenced by multiple factors. Although some of these factors are systematics and show seasonal or cyclical patterns, the interplay among them made these pseudo-periodic patterns difficult to detect. On top of that, one of the common features of financial time series is high frequency of individual values, this results in high volatility that usually changes through time. Conventional statistical time series analysis methods have inherent limitations as they often are confined to time-domain.

One of the popular statistical time series analysis tools is autoregressive integrated moving average (ARIMA) modelling. ARIMA is a form of regression analysis that examine the differences between values in the time series. With the assumption that the future will resemble the past, future direction of time series is thus predicted. This approach may work well for forecasting of a few points but often produces homogenous forecasting values in long-term forecasting. The limitation of ARIMA forecasting is further magnified in some financial time series that exhibit frequent market turbulence. In time-domain analysis, the sudden spikes and dips are generally treated as outliers and be smoothen out in the analysis process. There is therefore a need to perform analysis in frequency domain in order to recover additional information that can be used to compliment information captured in time domain.

With its unique ability to toggle signal, which is indeed time series, between time domain and frequency domain, Fourier transforms is able to reveal the set of frequencies hidden in any signal. Dominant frequencies with predetermined criteria can thus be identified. However, Fourier transform has a major restriction that it transforms signals into series of sinusoidal waves that continue until infinitive. This works well with signals that have unchanged frequency components that last throughout the time frame being analysed. However, financial time series often have different statistical properties at different time frames of the time series. This may translate to varying frequency contents along the time axis. In addition, changes in values are often transient in nature.

These properties have prohibited the applications of conventional Fourier transform in the analysis of financial time series. In financial time series analysis, it is often important to identify the exact point of time when noise occurs or when changes in characteristics occur; this may signify of a sudden change in the frequency contents of time series.

Fourier transform's inability to pinpoint the exact time when a particular frequency component exists is a major hindrance for its practical use in typically transient and non-stationary financial time series. The preferred frequency analysing method obviously must be able to provide both time and frequency information simultaneously.

Wavelet transform, particularly continuous wavelet transform (CWT) that provides higher resolution of time series information comes into picture. Through continuous scaling and shifting, as well as scales to frequencies mapping, CWT has the ability to approximate all significant sinusoid components of time series at any point of time. By performing CWT to the residual of ARIMA modelling, additional information in the form of the sum of all significant sinusoid components can be obtained. This paper seeks to improve on the long-term forecasting performance of ARIMA by aggregating ARIMA's homogenous forecasting points with the wave-liked additional information captured by CWT from the residual of ARIMA modelling. The forecasting performance of ARIMA and Hybrid ARIMA+CWT will be evaluated using mean absolute percentage error (MAPE).

The paper is structured as follow: Section 1 provides some basic background, Section 2 introduces theoretical framework and discusses some related works, Section 3 elaborates the research methods used in this paper and discloses the data being analysed, Section 4 presents and discusses the simulation results, and lastly Section 5 concludes and proposes possible extension of the research works.

2 Literature and related works

Accurate prediction of financial market direction always raises interest of researchers and statisticians as it serves as an early recommendation system for investors. Many market prediction studies require some macroeconomic data which takes some effort to obtain. Prediction methods based purely on price data have thus been extensively researched and did show promising results.

Wang (2014) reported accurate prediction of the movements of individual constituents in the Korean Composite Stock Price Index and Hang Seng Index. Some approaches use various techniques trying to look for repeated sub-sequences which are very similar to each other in a long time series (Truong and Anh, 2019). Constantino et al. (2021) proposed an associative classification model based on three technical indicators and claimed an 88.77% accuracy in the forecasting of stock market trends based on analysis of ten stocks and 12-year time series.

More sophisticated techniques in the realm of fuzzy logic and artificial intelligence have also been studied. Artificial neural network (ANN) that was trained with the historical data has been shown to be effective in the prediction of the future values of various stock prices (Yadav et al., 2021). Fuzzy time series (FTS) forecasting methods (Sridevi et al., 2021) have been studied and claimed to perform much better in terms of precision than other existing models.

Among the forecasting techniques, ARIMA remains one of the most widely used models due to its simplicity and its ability to generalise for non-stationary series. However, ARIMA tends to omit outliers as the process smoothen the value fluctuation in the time series. This may have caused the elimination of some useful information. This paper seeks to recover additional valid information by analysing the residual of ARIMA modelling in the frequency domain.

2.1 Time-frequency analysis of financial data

Various adaptations on traditional Fourier transform have been proposed to overcome the limitations of Fourier transform. Works from Haar (1910), Gabor (1946) and Levy and Morlet (1984, 1985, 1986) eventually leaded to the development of modern-day wavelet transform.

The precursor of wavelet transform is short-time Fourier transform (STFT), which is indeed a windowed Fourier transform. STFT tries to overcome the limitations of Fourier transform by segmenting time series into 'windows" to perform analysis separately. However, STFT lacks flexibility as the windows size is fixed throughout the analysis. Wavelet transform takes STFT one step further by introducing 'scale' that serves as a mediator to frequency. Unlike the fixed windows in STFT, wavelet transform performs multiple pass to the time series being analysed using various scales. With these scales, it becomes possible to identify the approximate frequency that occurs in approximate time of the time series.

Fourier transform decomposes signals into the complex exponential basis functions which is made up of infinite length of sines and cosines waves, and thus losing all time localisation information. The source function in wavelet transform is called mother wavelet, which can be any time-limited oscillatory function that is continuous in both time and frequency. From this mother wavelet, daughter wavelets, which are nothing more than scaled and shifted mother wavelet are formed. Wavelet transform decomposes signals into series of daughter wavelets which are time-localised, and thus preserves both time and frequency localisation information.

2.2 Variance of wavelet transform

Wavelet transform involves comparison of time series being analysed with various daughter wavelets. There are generally two forms of implementation, CWT and discrete wavelet transform (DWT). Fundamental differences between CWT and DWT are that, in CWT the scaling and shifting are done in continuous steps, whereas in DWT the scaling and shifting are done in discrete steps. However, in computerised implementation of CWT, continuous scaling and shifting is impossible; CWT has to be adapted to use more finely discretised scales and shifting steps, compared to DWT.

2.3 Scale to frequency conversion in CWT

Mathematically, wavelet transform is expressed as

$$W(u, s) = \int_{-\infty}^{\infty} f(t) \cdot \psi_{u,s}(t) \quad \mathrm{dt}$$

where $W(u, s)$ is the wavelet coefficients, which is a function of scales, s and positions, u; $f(t)$ is the time series being analysed and $\psi_{u,s}(t)$ is the wavelet functions. Wavelet coefficients $W(u, s)$ are the sum over all time of the time series, f(t) multiplied by scaled, shifted versions of wavelet functions, $\psi_{u,s}(t)$ (Chan and Bates, 1996). In fact, $W(u, s)$ indicates the similarity between $f(t)$ and $\psi_{u,s}(t)$.

In computerised implementation of wavelet transform, $W(u, s)$ is a matrix of $s \times u$ dimension. Value of coefficient $W(u, s)$ in row s and column u, indicates the similarity between a daughter wavelet $\psi_{u,s}(t)$ of scale s and the signal $f(t)$ at position u; larger value signifies closer matching. By looking into this $W(u, s)$ matrix, the dominant wavelet function of scale s at position u of the time series can be determined. As long as the wavelet function is known, the dominant sinusoid frequency may be 'estimated'. Unlike Fourier transform that produces precise sinusoid functions, wavelet transform can only provide information of the wavelet functions that do not have a precise sinusoid frequency. Hence the sinusoid frequency can only be estimated, instead of calculated precisely.

Scale of wavelet function is inversely proportional to frequency. There is no precise relationship between scale and frequency, therefore, the scale to frequency conversion can only be done in general sense. This is due to firstly Heisenberg's uncertainty principle and secondly the irregular shape of wavelet functions that do not have dominant sinusoid component to allow a meaningful definition or calculation of centre frequency (Aguiar-Conraria and Soares, 2013).

According to Lilly and Olhede (2009), there is more than one valid interpretation in the assignment of scale to frequency. In most cases, the mapping between scale and frequency is merely an interpretation, there is no universal and precise mapping. Lilly and Olhede (2009) discussed about the notion of wavelet frequency ω_s, which could be represented by the wavelet's central frequency, ω_c, at which the maximum of the wavelet's Fourier transform magnitude, $|\hat{\psi}(\omega)|$ occurs. ω_s may be obtained with the formula $\omega_s = \omega_c / s$ after adjustment with the scale. This is one of the elementary and the most intuitive methods to define the frequency.

2.4 Generalised morse wavelet

For the purpose of analysis in this paper, generalised morse wavelet (GMW), which is a particularly important family of analytic wavelets has been chosen. GMW offers better time localisation and frequency localisation and is more flexible due to its complex value in time domain and positive side only spectra (Conraria and Soares, 2014).

Lilly and Olhede (2012) reported the application of GMW with vanishing support on negative frequencies provided the basis for a powerful analysis of oscillatory signals. In Zerouali's work (2014) and another two recent papers (Munia and Aviyente, 2020; Nakhnikian et al., 2016), GMW has been used to extract the amplitude and phase components from signals.

CWT implementation with GMW is a good combination for time-frequency analysis of financial time series with time-varying amplitude and frequency characteristic. The wavelet coefficients in complex values provide both magnitude and phase information, which is useful for analysing localised discontinuities in the time series.

2.5 ARIMA model and hybrid ARIMA+CWT

Since 1970, ARIMA model has been widely used to perform modelling and prediction for time series data due to its robustness and ease of implementation. ARIMA is one of the more popular statistical models used in forecasting. However, there is a prerequisite that the time series must be stationary to yield reliable results. For financial time series, which is often non-stationary, differencing has to be performed to make the time series stationary.

Non-seasonal ARIMA takes the form ARIMA (*p*, *d*, *q*) where p is the order of the auto regression (AR), d is the order of the differencing that make the series stationery and q is the order of the moving average (MA).

In general, a typical ARIMA model can be expressed mathematically as:

$$Y_t = \omega_t + \alpha_1 Y_{t-1} + \ldots + \alpha_p Y_{t-p} + \beta_1 e_{t-1} + \ldots + \beta_q e_{t-q}$$

where ω is a constant representing the mean of the stochastic time series, α is the coefficient of the AR parameter, β is the coefficient of the MA parameter, Y_t is the predicted value of time series at time *t*, Y_{t-i} is the value of the time series at time $(t-i)$ and e_{t-j} is the error in the predicted value of the time series as compared to actual time series at time $(t-j)$. In actual implementation, ARIMA modelling is an iterative process in order to obtain the best fit parameters (Hillmer and Wei, 1991).

In order to get a parsimonious model, the order of the ARIMA model shall be kept small. However, depending upon the stochastic nature of time series, which is typical in financial market, there is a possibility that the model order may be large. The hybrid application of wavelets and ARIMA in forecasting the non-stationary time series is expected to be able to overcome the large order issue in ARIMA and enhance the accuracy of the prediction.

2.6 Related works

There have been interest in the application of CWT in financial time series analysis. Adjepong et al. (2019) has adopted CWT and wavelet coherence approach to explore co-movement and directional inter-linkage properties among internationally traded financial asset markets.

In the study of time-frequency dynamic co-movement, CWT approach has been used successfully to identify the co-variations over time and across frequencies between the gold and oil prices with BRICS stock markets (Mensi et al., 2018). In another work (Uddin et al., 2016), CWT has been applied successfully to illustrate the changes of correlation and the lead–lag structure between variables over timescales. In has been reported that wavelet-based estimators significantly improved the modelling and forecasting of exchange rate volatility in time-frequency domain (Barunik et al., 2016).

In order to overcome the limitations of ARIMA, hybrid wavelet-ARIMA models were widely used for forecasting in past research. Results were often better than pure ARIMA only approach. A hybrid wavelet ARIMA approach that make use of Haar wavelet has been proven to be more accurate in the analysis of the correlation of Ghana’s stocks returns in both time and frequency domain (Eghan, 2019). The authors inferred that Haar-ARIMA approach would produce better forecasting performance compared to ARIMA model. Not only with ARIMA, but wavelet approach has also been found to work well with generalised auto regressive conditional Heteroskedasticity (GARCH), another popular statistical model for time series analysis. Wavelet-GARCH transform was found to provide more accurate results than GARCH model alone in the modelling of multivariate ENSO index (Ahasan et al., 2019).

Quite a number of research suggested that wavelet transform fits well with ARIMA when the two techniques were combined to form hybrid approaches. Dela Cruz (2019) proposed a Hybrid model of forecasting by using a time series data that was pre-transformed using Daubechies Filter with DWT. Salazar et al. (2018) claimed to obtain better forecasting results by applying the wavelet transform using the Haar cumulated wavelet function to the original time series prior to ARIMA modelling. Al-Wadi and Al-Slaihat (2019) managed to improve the forecasting accuracy by applying ARIMA modelling to time series that has been smoothened by wavelet transform.

Existing works that integrated wavelet transform and ARIMA modelling techniques generally utilised wavelet transform to de-noise the time series before modelling it with ARIMA. This process not only smoothen the time series but will also eliminate some valid information inadvertently. The ability of CWT to capture the frequency components at any one point of the time series has not been fully utilised. On top of that, there is a lack of studies on the residuals of ARIMA modelling. This paper attempts to investigate the residuals of ARIMA modelling by using CWT to analyse and to pick up useful information manifested as sinusoid frequency components from the residuals.

3 Methods and data

In this paper, two types of forecasting have been performed to test the information capturing capability of CWT from the residuals of ARIMA. First, in-sample forecasting was performed to examine the CWT captured information and the associative optimum parameters, follow by out-of-sample forecasting to further confirm the validity of the CWT captured information.

A number of financial time series from equity, commodity and currency exchange market were carefully selected to put into test. With the assumption that the process of ARIMA modelling may have left some useful information in its residuals, this paper tried to extract this information from the ARIMA residuals by applying CWT with GMW. The extracted information was expected to be able to complement ARIMA model to provide better forecasting outcomes.

The first step is to test the hybrid ARIMA+CWT concept via in-sample forecasting on the training data. First of all, ARIMA model with the best fit parameters was applied to the training data of selected financial time series. CWT was then applied to the residuals of the ARIMA process, attempting to capture additional information in terms of sinusoid waves with magnitudes, frequencies and phases. The captured information was then added back to the ARIMA fitted values obtained earlier. Both of the ARIMA fitting and ARIMA+CWT values were then compared to the actual time series value. If the ARIMA+CWT values yield smaller error, it is assumed that the CWT process has successfully captured valid information from the ARIMA residuals.

Upon successful validation of the CWT-captured sinusoid information in in-sample forecasting, the captured information will then be used in out-of-sample forecasting, with the assumption that the captured sinusoid information will extend beyond the training time series used in in-sample forecasting. First, ARIMA model is used to obtain 20 forecasting points beyond the initial training time series. The CWT-captured sinusoid(s) will then be extended from training time series and added to the 20 ARIMA out-of-sample forecasting points to obtain hybrid ARIMA+CWT out-of-sample forecasting. MAPE of both ARIMA forecasting and the hybrid ARIMA+CWT forecasting were calculated by comparing to the 20 points actual data used as test data. Lastly, comparisons were made between the accuracy of ARIMA forecast and hybrid ARIMA+CWT forecast.

The two forecasting approaches, ARIMA and hybrid ARIMA+CWT will be applied to four sets of selected financial time series. Each set comprises of two equivalent time series ended at different point of time (refer to Section 3.3, data). This setup was designed to test the capabilities of the proposed forecasting strategies under different scenarios.

3.1 ARIMA

The following iterative process have been applied in the ARIMA modelling of the selected financial time series in this paper:

1 Model identification in order to make the time series stationary. Differencing was performed on the time series and checked by unit-root test. Following the naming convention of ARIMA (p, d, q), the order of differencing was named as d. The order of p and q were estimated by analysing the autocorrelation function (ACF) and the partial autocorrelation function (PACF).

2 Model estimation using maximum likelihood. The model with lowest Akaike's information criterion (AIC) and Bayesian information criterion (BIC) was selected.

3 Diagnostic checking was performed to ensure that the selected ARIMA models fulfilled the stationary and invertibility conditions. The white noise property of the residuals in the model has been examined to ensure that the errors were normally distributed. The ACF and PACF plots of the residuals have been checked. Normality of the residuals was checked by using the normal quantile-quantile plot (Q-Q plot). Residuals were ensured to approximate white noise by checking on the Ljung-Box statistics.

Finally, the fined tuned ARIMA model that satisfied all the required conditions were used to perform in-sample as well as out-of-sample forecast.

3.2 Hybrid ARIMA+CWT

All the works in this paper has been carried out using GMWs and the scale-frequency interpretation method explained in the last paragraph of Section 2.3.

For in-sample forecasting, the hybrid ARIMA+CWT approach was formulated via the following steps:

1 ARIMA model fitting with best-fit parameters was performed on the training time series.

2 CWT with GMW was applied on the residuals of ARIMA to identify sinusoid components at each time point of the residuals. Amplitudes, frequencies, and phases information of these components were saved for subsequent processing.

3 The dominant sinusoid wave with the highest amplitude at each time point will be captured and used as a benchmark. For the remaining identified sinusoid waves at the same time point, only those with amplitudes exceeds k% [refer to Figure 5(a) for illustration] of the benchmark sinusoid's amplitude were deemed significant and captured, while those below were discarded.

4 Captured sinusoid components at every time point were then summed up point-by-point to the ARIMA fitted model in step 1) to establish in-simple forecast of hybrid ARIMA+CWT.

5 Performance of the hybrid ARIMA+CWT forecasts with different parameter (k%) were then be evaluated using MAPE. Comparisons were made between in-sample forecasting performance of hybrid ARIMA+CWT and ARIMA model (refer to Tables 2A, 2B, 2C and 2D).

The CWT captured sinusoid components at every time point of the ARIMA residuals from the training time series were saved for later use in out-of-sample forecasting. In out-of-sample forecasting, the hybrid ARIMA+CWT approach was formulated via the following steps:

1 ARIMA forecast of 20 points was obtained based on the ARIMA fitting model obtained from the training time series.

2 The CWT captured sinusoid components resulted from the earlier in-sample forecasting were extended beyond the training time series with additional 20 forecasting points. Three parameters have been used to dictate whether to consider a particular captured sinusoid component in the out-of-sample forecasting, and to determine the length to extend the chosen sinusoid.

   - The frequency rejection parameter (k%) that determine the amplitude of identified sinusoid to be considered as significant (refer to step 3 of in-sample forecasting).
   - The back tracking points (BTP) (Refer to Figure 5B for illustration) that dictate the number of data points to trace back from the last point of the training time series in order to scan for dominant sinusoid components.
   - The sinusoid probability parameter (j%) (Refer to Figures 5C-1 to 5C-2 for illustration) which is the probability threshold that determine the length to extend the chosen sinusoid. 0% means the sinusoid will definitely be extended for the 20 points forecasting, 100% means the sinusoid component will only be extended if its probability of continuation to particular forecasting point is 100%, 50% means the sinusoid component will only be extended if its probability of continuation to particular forecasting point is 50% and above. The probability was calculated based on the historical length records of the particular sinusoid in training time series.
   - The k% parameter takes values 0%, 50%, 80% and 100%. The BTP parameter takes values 20, 50, 100, 200 and 300. Whereas the j% takes values 0%, 30%, 50%, 80% and 100%.

- Combining the three parameters of k%, BTP and j%, there were 100 combinations. Hence, 100 ARIMA+CWT simulations have been performed for each of the eight time series (Refer Section 3.3 Data).

3 The selected sinusoid components that fulfilled the k%, BTP and j% conditioning parameters were then summed up point-by-point to the ARIMA forecast in step 1) to establish out-of-sample forecasting of hybrid ARIMA+CWT.

4 Performance of the hybrid ARIMA+CWT forecasts with parameters k%, BTP and j% were then be evaluated using MAPE. Comparison was made between out-of-sample forecasting performance of hybrid ARIMA+CWT and ARIMA model.

The entire methodology process flow described in section 3.2 has been summarised in Figure 6.

3.3 Data

Four financial time series were examined in this paper:

- Daily trading prices (in ringgit) of Malaysia's gold bullion coin (KEmas)
- BitCoin-USD Daily Exchange Rate (BCoin)
- Daily price of Brent Oil Futures (Brent)
- Daily index of S&P500 Index (S&P500)

Two-time frames (hereafter 2021-February and 2021-October) were derived from each of the time series stated above:

- KEmas from 18/07/2001 to 19/02/2021 (4939 data points)
- KEmas from 18/07/2001 to 01/10/2021 (5083 data points)
- BCoin from 17/09/2014 to 19/02/2021 (2330 data points)
- BCoin from 17/09/2014 to 02/10/2021 (2570 data points)
- Brent from 27/06/1988 to 19/02/2021 (8330 data points)
- Brent from 27/06/1988 to 01/10/2021 (8494 data points)
- S&P500 from 30/12/1927 to 19/02/2021 (23392 data points)
- S&P500 from 30/12/1927 to 01/10/2021 (23551 data points)

The KEmas data was obtained from Bank Negara website (Bnm.gov.my, 2021), the BCoin data was obtained from Yahoo Finance (Finance.yahoo.com, 2021), the Brent data was obtained from Investing.com (Investing.com, 2021), whereas the S&P500 data was obtained from Yahoo Finance (Finance.yahoo.com, 2021).

The last 20 data points of all the above time series were reserved and used as benchmark for out-of-sample forecasting. The rest of the earlier data points were used as training data for both in-sample and out-of-sample forecasting.

3.4 Model Performance Evaluation

Evaluation criteria of forecast accuracy used in this paper was MAPE.

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{A_i - F_i}{A_i}\right|$$

where F_i are forecasted values, A_i are actual values and n is number of points.

4 Results and discussion

The time-domain plots of all the financial time series analysed in this paper were shown from Figure 1A to Figure 4B. Each figure shows the training time series, ARIMA fitted values and its residuals.

Visual inspection into the training time series obviously shows that KEmas, BCoin, Brent and S&P500 exhibit very diverse time domain characteristics. In addition, the statistical characteristics at different time frames of each time series change significantly. Table 1A and Table 1B show the descriptive statistics of KEmas, BCoin, Brent and S&P500 of two different time frames respectively. It can be observed that all-time series are nonstationary therefore differencing is needed.

According to the outcomes from unit root tests, all the selected financial time series became stationary after first differencing. Observations to AIC and BIC and other diagnostic checking procedures led to the best fit ARIMA parameters which are listed as follow:

- ARIMA (2,1,2) for both KEmas time frames
- ARIMA (2,1,0) for both BCoin time frames
- ARIMA (0,1,0) for both Brent time frames
- ARIMA (0,1,2) for both S&P500 time frames.

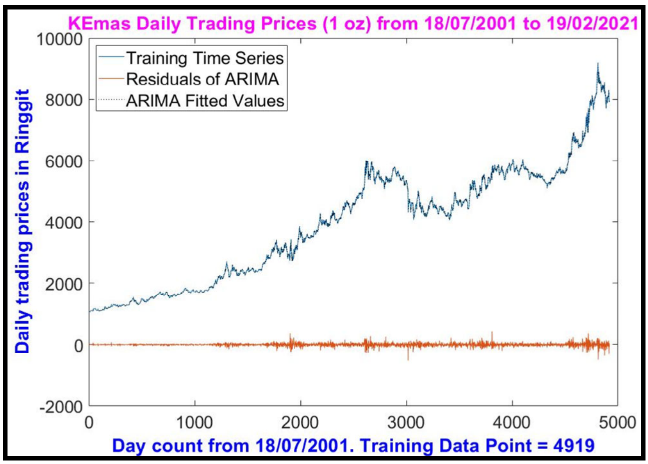
Figure 1A to Figure 4B shows the plotting of ARIMA fitted values which were labelled as ‘ARIMA Fitted Values’. Note that from Figure 1A to Figure 4B, all of the ARIMA fitted values almost overlapped with the training time series under the plotting scales used. Hence the two-plotting labelled by the legends ‘ARIMA fitted values’ and ‘training time series’ are hardly distinguishable visually in all the figures.

Comparison of in-sample forecasting performance between ARIMA and ARIMA+CWT models are shown from Table 2A to Table 2D. For all tested time series, the ARIMA+CWT model gave lower MAPE under all the frequency rejection parameter (k%). The improvement of the in-sample forecasting performance clearly indicates that the extra information extracted by CWT from the ARIMA residuals from all the training time series contain valid information.

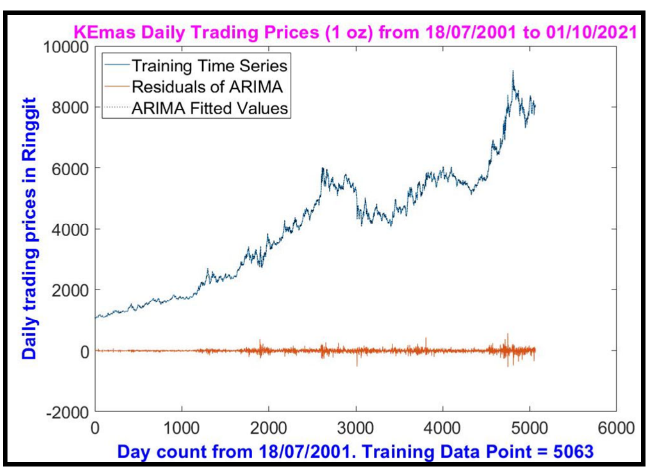
Table 3A to Table 5H shows the 5 points to 20 points out-of-sample forecasting summary results for KEmas, BCoin, Brent and S&P500 time series at various sinusoid components capturing parameters (k%, j%, and BTP) respectively. Figures in the tables indicate the percentages of all the parameters’ combinations that yield better forecasting results compared to the ARIMA model, when applied to the hybrid ARIMA+CWT models.

Table 3A, 4A, 5A and Table 3B, 4B, 5B are forecasting performance for KEmas 2021-Feb and KEmas 2021-October respectively. Table 3C, 4C, 5C and Table 3D, 4D, 5D are forecasting performance for BCoin 2021-February and BCoin 2021-October. Table 3E, 4E, 5E and Table 3F, 4F, 5F are forecasting performance for Brent 2021-February and Brent 2021-October. Table 3G, 4G, 5G and Table 3H, 4H, 5H are forecasting performance for S&P500 2021-February and S&P500 2021-October.

Figure 1 (a) KEmas 18/07/2001 to 19/02/2021 (Total 4,939 data points), (b) KEmas 18/07/2001 to 01/10/2021 (Total 5,083 data points) (see online version for colours)

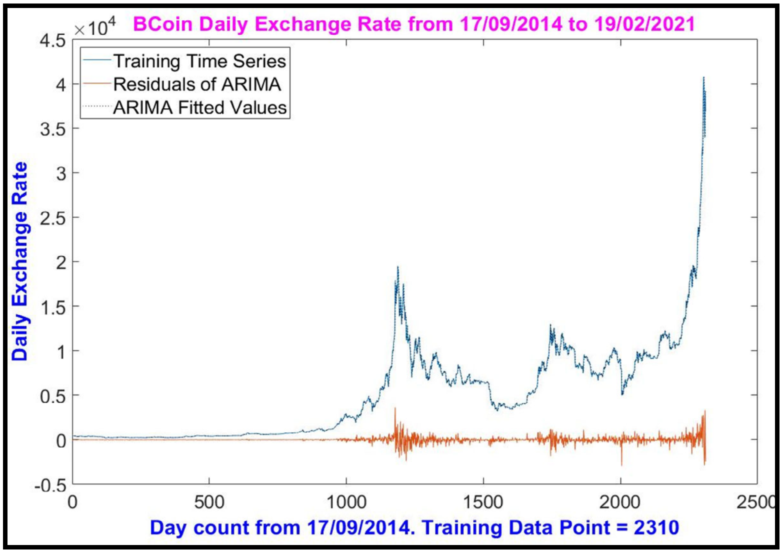


(a)

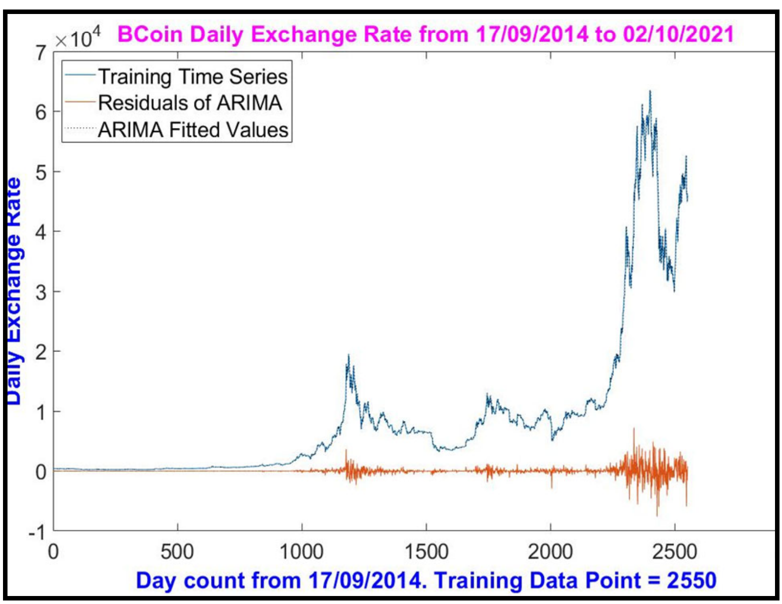


(b)

Figure 2 (a) BCoin from 17/9/2014 to 19/02/2021 (Total 2330 data points), (b) BCoin from 17/9/2014 to 02/10/2021 (2570 data points) (see online version for colours)

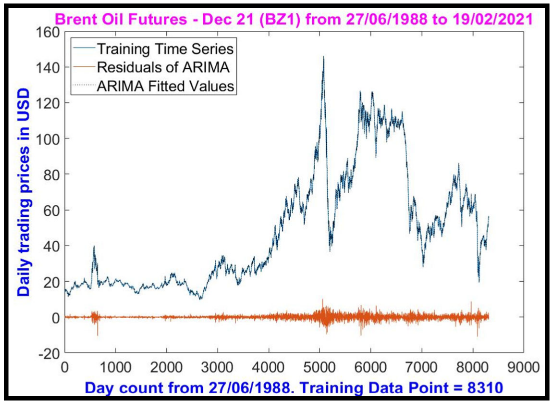


(a)

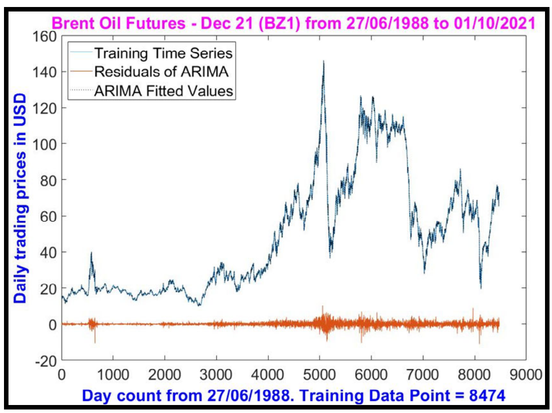


(b)

Figure 3 (a) Brent from 27/06/1988 to 19/02/2021 (Total 8330 data points), (b) Brent from 27/06/1988 to 01/10/2021 (Total 8494 data points) (see online version for colours)

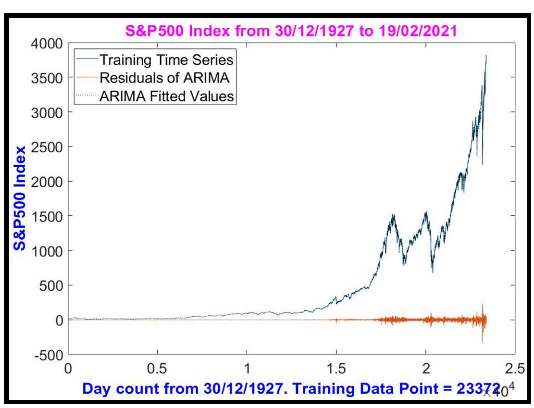


(a)



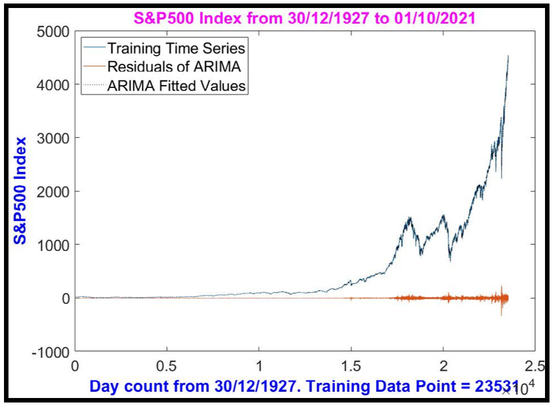
(b)

Figure 4 (a) S&P500 from 30/12/1927 to 19/02/2021 (Total 23392 data points), (b) S&P500 from 30/12/1927 to 01/10/2021 (Total 23551 data points) (see online version for colours)



(a)

Figure 4 (a) S&P500 from 30/12/1927 to 19/02/2021 (Total 23392 data points), (b) S&P500 from 30/12/1927 to 01/10/2021 (Total 23551 data points) (see online version for colours) (continued)



(b)

Table 1A Statistical characteristics of selected financial time series

Statistic	*KEmas 18/07/2001 to 19/02/2021*	*BCoin 17/09/2014 to 19/02/2021*	*Brent 27/06/1988 to 19/02/2021*	*S&P500 30/12/1927 to 19/02/2021*
Mean	3,945.39	5,375.66	48.05	496.47
Standard deviation	1,894.35	6,029.08	32.52	746.18
Kurtosis	–0.79	7.70	–0.50	3.11
Skewness	0.18	2.20	0.81	1.88
Min	1,069.00	178.10	9.64	4.40
Max	9,188.00	40,797.61	146.08	3,934.83
Count	4,939	2,330	8,330	23,392

Table 1B Statistical characteristics of selected financial time series

Statistic	*KEmas 18/07/2001 to 01/10/2021*	*BCoin 17/09/2014 to 02/10/2021*	*Brent 27/06/1988 to 01/10/2021*	*S&P500 30/12/1927 to 01/10/2021*
Mean	4,062.35	9,172.58	48.48	521.65
Standard deviation	1,979.26	13,422.70	32.36	804.09
Kurtosis	–0.81	4.45	–0.53	4.41
Skewness	0.20	2.27	0.77	2.08
Min	1,069.00	178.10	9.64	4.40
Max	9,188.00	63,503.46	146.08	4,536.95
Count	5,083	2,570	8,494	23,551

Table 2A In-sample forecasting MAPE of KEmas training time series

Modelling function	*Frequency rejection parameter (k%)*	*MAPE of KEmas time series*	
		18/07/2001 to 19/02/2021	*18/07/2001 to 01/10/2021*
ARIMA (2, 1, 2) + CWT	0%	0.4176	0.4156
ARIMA (2, 1, 2) + CWT	50%	0.4954	0.4938
ARIMA (2, 1, 2) + CWT	80%	0.5803	0.5818
ARIMA (2, 1, 2) + CWT	100%	0.6263	0.6230
ARIMA (2, 1, 2)	-	0.4176	0.7561

Table 2B In-sample forecasting MAPE of BCoin training time series

Modelling function	*Frequency rejection parameter (k%)*	*MAPE of KEmas time series*	
		17/9/2014 to 19/02/2021	*17/09/2014 to 02/10/2021*
ARIMA (2, 1, 0) + CWT	0%	1.7658	1.7592
ARIMA (2, 1, 0) + CWT	50%	2.1602	2.1976
ARIMA (2, 1, 0) + CWT	80%	2.1370	2.1810
ARIMA (2, 1, 0) + CWT	100%	2.2438	2.2917
ARIMA (2, 1, 0)	-	2.4989	2.5634

Table 2C In-sample forecasting MAPE of brent training time series

Modelling function	*Frequency rejection parameter (k%)*	*MAPE of KEmas time series*	
		27/06/1988 to 19/02/2021	*27/06/1988 to 01/10/2021*
ARIMA (0, 1, 0) + CWT	0%	0.8302	0.8284
ARIMA (0, 1, 0) + CWT	50%	1.0091	1.0068
ARIMA (0, 1, 0) + CWT	80%	1.1663	1.1648
ARIMA (0, 1, 0) + CWT	100%	1.2566	1.2547
ARIMA (0, 1, 0)	-	1.5680	1.5656

Table 2D In-sample forecasting MAPE of S&P500 training time series

Modelling function	*Frequency rejection parameter (k%)*	*MAPE of KEmas time series*	
		30/12/1927 to 19/02/2021	*30/12/1927 to 01/10/2021*
ARIMA (0, 1, 2) + CWT	0%	0.4057	0.4019
ARIMA (0, 1, 2) + CWT	50%	0.5183	0.5414
ARIMA (0, 1, 2) + CWT	80%	0.6058	0.6209
ARIMA (0, 1, 2) + CWT	100%	0.6500	0.6600
ARIMA (0, 1, 2)	-	0.7624	0.7613

Referring to Table 3, Table 4, and Table 5 series, the total at the bottom of each table sums up the performance percentages by number of forecasting points (5 points, 10 points, 15 points and 20 points). Each total is the overall forecasting performance of

ARIMA+CWT for 5 points, 10 points, 15 points and 20 points. It can be observed that these total figures of the same times series of same time frame matched each other (barring some rounding errors) when different parameters were varied for testing. For example, totals in Table 3A matches with totals in Table 4A and Table 5A, whereas totals in Table 3B matches with totals in Table 4B and Table 5B. This validates the results of the simulations.

Table 3A Out-of-sample forecasting KEmas (5 Points to 20 Points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 2) breakdown by frequency rejection (k%)

Modelling function	*Frequency rejection parameter (k%)*	*Kemas 18/07/2001 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 2) + CWT	0%	19.74%	5.26%	17.11%	14.47%	14.15%
ARIMA (2, 1, 2) + CWT	50%	6.58%	5.26%	7.89%	14.47%	8.55%
ARIMA (2, 1, 2) + CWT	80%	3.95%	0.00%	0.00%	0.00%	0.99%
ARIMA (2, 1, 2) + CWT	100%	2.63%	2.63%	3.95%	1.32%	2.63%
Total%		32.90%	13.15%	28.95%	30.26%	

Table 3B Out-of-sample forecasting KEmas (5 Points to 20 Points) percentages of ARIMA+CWT parameters to outperform ARIMA (2,1,2) breakdown by frequency rejection (k%)

Modelling function	*Frequency rejection parameter (k%)*	*Kemas 18/07/2001 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 2) + CWT	0%	15.63%	18.75%	15.63%	20.31%	17.58%
ARIMA (2, 1, 2) + CWT	50%	0.00%	3.13%	3.13%	9.38%	3.91%
ARIMA (2, 1, 2) + CWT	80%	0.00%	4.69%	4.69%	7.81%	4.30%
ARIMA (2, 1, 2) + CWT	100%	0.00%	0.00%	0.00%	1.56%	0.39%
Total%		15.63%	26.57%	23.45%	39.06%	15.63%

Table 3C Out-of-sample forecasting BCoin (5 Points to 20 Points) percentages of ARIMA+CWT parameters to outperform ARIMA (2,1,0) breakdown by frequency rejection (k%)

Modelling function	*Frequency rejection parameter (k%)*	*BCoin 17/09/2014 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 0) + CWT	0%	7.59%	8.86%	13.92%	12.66%	10.76%
ARIMA (2, 1, 0) + CWT	50%	6.33%	6.33%	13.92%	13.92%	10.13%
ARIMA (2,1, 0) + CWT	80%	21.52%	27.85%	30.38%	24.05%	25.95%
ARIMA (2, 1, 0) + CWT	100%	0.00%	8.86%	2.53%	5.06%	4.11%
Total%		35.44%	51.90%	60.75%	55.69%	35.44%

Table 3D Out-of-sample forecasting BCoin (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (2,1,0) breakdown by frequency rejection (k%)

Modelling function	*Frequency rejection parameter (k%)*	*BCoin 17/09/2014 to 02/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 0) + CWT	0%	9.64%	20.48%	18.07%	20.48%	17.17%
ARIMA (2, 1, 0) + CWT	50%	8.43%	6.02%	3.61%	4.82%	5.72%
ARIMA (2, 1, 0) + CWT	80%	19.28%	19.28%	21.69%	21.69%	20.49%
ARIMA (2, 1, 0) + CWT	100%	21.69%	27.71%	15.66%	13.25%	19.58%
Total%		59.04%	73.49%	59.03%	60.24%	

Table 3E Out-of-sample forecasting brent (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0,1,0) breakdown by frequency rejection (k%)

Modelling function	*Frequency rejection parameter (k%)*	*Brent 27/06/1988 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 0) + CWT	0%	21.43%	22.86%	25.71%	22.86%	23.22%
ARIMA (0, 1, 0) + CWT	50%	11.43%	12.86%	10.00%	15.71%	12.50%
ARIMA (0, 1, 0) + CWT	80%	2.86%	8.57%	2.86%	2.86%	4.29%
ARIMA (0, 1, 0) + CWT	100%	5.71%	5.71%	8.57%	8.57%	7.14%
Total%		41.43%	50.00%	47.14%	50.00%	

Table 3F Out-of-sample forecasting brent (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0,1,0) breakdown by frequency rejection (k%)

Modelling function	*Frequency rejection parameter (k%)*	*Brent 27/06/1988 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 0) + CWT	0%	12.50%	1.79%	3.57%	1.79%	4.91%
ARIMA (0, 1, 0) + CWT	50%	5.36%	3.57%	3.57%	5.36%	4.47%
ARIMA (0, 1, 0) + CWT	80%	0.00%	0.00%	0.00%	0.00%	0.00%
ARIMA (0, 1, 0) + CWT	100%	0.00%	0.00%	0.00%	0.00%	0.00%
Total%		17.86%	5.36%	7.14%	7.15%	

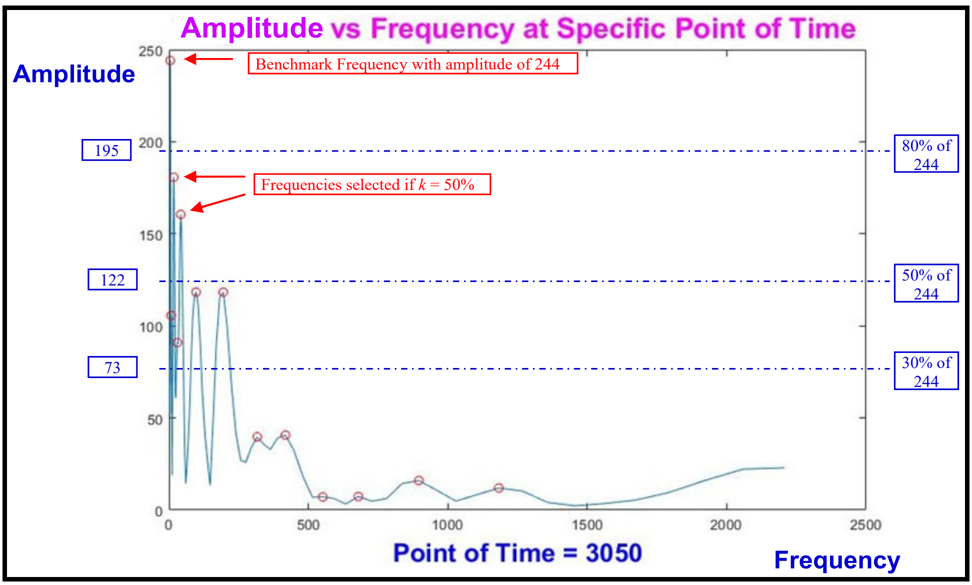
Table 3G Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0,1,2) breakdown by frequency rejection (k%)

Modelling function	*Frequency rejection parameter (k%)*	*S&P500 30/12/1927 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 2) + CWT	0%	2.99%	2.99%	5.97%	19.40%	7.84%
ARIMA (0, 1, 2) + CWT	50%	4.48%	0.00%	4.48%	14.93%	5.97%
ARIMA (0, 1, 2) + CWT	80%	5.97%	0.00%	7.46%	13.43%	6.72%
ARIMA (0, 1, 2) + CWT	100%	10.45%	2.99%	2.99%	5.97%	5.60%
Total%		23.89%	5.98%	20.90%	53.73%	

Table 3H Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0,1,2) breakdown by frequency rejection (k%)

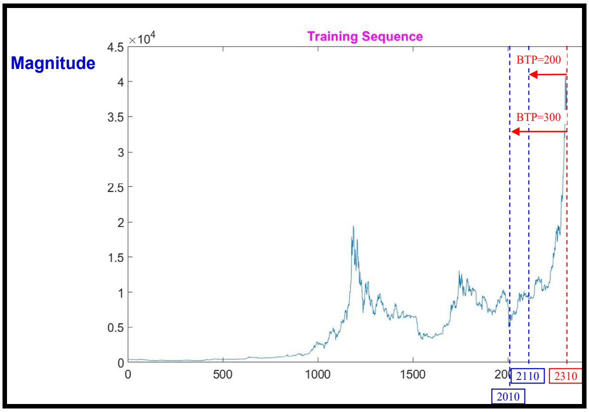
Modelling function	*Frequency rejection parameter (k%)*	*S&P500 30/12/1927 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 2) + CWT	0%	24.07%	24.07%	29.63%	33.33%	27.78%
ARIMA (0, 1, 2) + CWT	50%	3.70%	9.26%	7.41%	1.85%	5.56%
ARIMA (0, 1, 2) + CWT	80%	9.26%	9.26%	1.85%	3.70%	6.02%
ARIMA (0, 1, 2) + CWT	100%	0.00%	0.00%	0.00%	0.00%	0.00%
Total%		37.03%	42.59%	38.89%	38.88%	37.03%

Figure 5A Illustration of k% frequency rejection parameter (see online version for colours)



Notes: Illustrates the peaking sinusoid frequencies at point 3050 of a time series. There are 13 identified peaking sinusoid frequencies highlighted in red circles. When the parameter k% is set at 50%, only sinusoid frequencies with amplitude 50% and above of the benchmark frequency's amplitude will be selected. 3 sinusoid frequencies including the benchmark frequency will be selected. For k = 100%, only the benchmark frequency will be selected. For k = 80%, only the benchmark frequency will be selected. For k = 30%, 7 sinusoid frequencies including the benchmark frequency will be selected. For k = 0%, all sinusoid frequencies including the benchmark frequency will be selected.

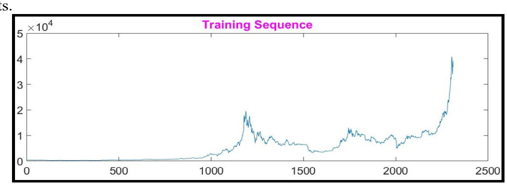
Figure 5B Illustration of BTP (see online version for colours)



Notes: Illustrates the BTP parameter of a time series with training sequence that contains 2,310 points. The last point of the training sequence is at data point 2,310. When the BTP is set at 200, only the data between point 2,110 and 2,310 will be considered to contain sinusoid frequencies that will influence the next 20 forecasting points beyond data point 2310. Significant sinusoid frequencies captured between point 2,110 and 2,310 will be examined with parameters k% and j% for their continuation in out-of-sample forecasting. When the BTP is set at 300, only the data between point 2010 and 2310 will be considered to contain sinusoid frequencies that will influence the next 20 forecasting points beyond data point 2,310. Significant sinusoid frequencies captured between point 2,010 and 2,310 will be examined with parameters k% and j% for their continuation in out-of-sample forecasting.

Figure 5C-1 Illustration of j% sinusoid frequency probability (see online version for colours)

Assuming that a time series as shown below was examined to reveal its significant sinusoid frequencies at every data points.



Assuming that a number of sinusoid frequencies (including sinusoid X) were discovered at different data points in the training sequence.

The observation of sinusoid X is shown in the following table for every detection at different data point with respective length.

Data points where sinusoid X was detected	205	350	564	785	1,035	1,254	1,638	1,884	2,087	2,111	2,239	2,310
Detected sinusoid X length	6	2	3	2	9	2	3	9	7	5	4	1

- minimum length = 2
- maximum length = 9

Based on observation of sinusoid X in the training sequence, probabilities for sinusoid *X* to have length ranges from 2 to 9 in the out-of-sample forecasting was calculated and assigned as shown in the following table.

Sinusoid x length	2	3	4	5	6	7	8	9
Assigned probability	100.00%	87.50%	75.00%	62.50%	50.00%	37.50%	25.00%	12.50%

Figure 5C-2 Illustration of j% sinusoid frequency probability

In the case where sinusoid X was detected at the last point of the training sequence, its length position was captured and the probabilities for it to extend into the out-of-sample forecasting region were calculated.

Assuming the length position of sinusoid X at last point of the training sequence is 1, table shows the probabilities for sinusoid X to extend into the 20 out-of-sample forecasting points with respective j% parameter setting.

Sinusoid X length position from last data point	*Forecasting point count*	*Assigned probability*	*j% parameter setting*				
			0%	*30%*	*50%*	*80%*	*100%*
1	-						
2	1	100.00%	Y	Y	Y	Y	Y
3	2	87.50%	Y	Y	Y	Y	-
4	3	75.00%	Y	Y	Y	-	-
5	4	62.50%	Y	Y	Y	-	-
6	5	50.00%	Y	Y	Y	-	-
7	6	37.50%	Y	Y	-	-	-
8	7	25.00%	Y	-	-	-	-
9	8	12.50%	Y	-	-	-	-
10	9	0.00%	Y	-	-	-	-
11	10	0.00%	Y	-	-	-	-
12	11	0.00%	Y	-	-	-	-
13	12	0.00%	Y	-	-	-	-
14	13	0.00%	Y	-	-	-	-
15	14	0.00%	Y	-	-	-	-
16	15	0.00%	Y	-	-	-	-
17	16	0.00%	Y	-	-	-	-
18	17	0.00%	Y	-	-	-	-
19	18	0.00%	Y	-	-	-	-
20	19	0.00%	Y	-	-	-	-
21	20	0.00%	Y	-	-	-	-

Notes: ‘Y” in the table indicates that sinusoid X was extended into the forecasting point. calculation were based on sinusoid x’s amplitude, frequency and phase.

Figure 5C-3 Illustration of j% sinusoid frequency probability (see online version for colours)

Assuming the length position of sinusoid X at last point of the training sequence is 3, table shows the probabilities for sinusoid X to extend into the 20 out-of-sample forecasting points with respective *j*% Parameter Setting.

Sinusoid X Length Position from Last Data Point	Forecastin g Point Count	Assigned Probability	*j*% Parameter Setting				
			0%	30%	50%	80%	100%
3	-						
4	1	75.00%	Y	Y	Y	-	-
5	2	62.50%	Y	Y	Y	-	-
6	3	50.00%	Y	Y	Y	-	-
7	4	37.50%	Y	Y	-	-	-
8	5	25.00%	Y	-	-	-	-
9	6	12.50%	Y	-	-	-	-
10	7	0.00%	Y	-	-	-	-
11	8	0.00%	Y	-	-	-	-
12	9	0.00%	Y	-	-	-	-
13	10	0.00%	Y	-	-	-	-
14	11	0.00%	Y	-	-	-	-
15	12	0.00%	Y	-	-	-	-
16	13	0.00%	Y	-	-	-	-
17	14	0.00%	Y	-	-	-	-
18	15	0.00%	Y	-	-	-	-
19	16	0.00%	Y	-	-	-	-
20	17	0.00%	Y	-	-	-	-
21	18	0.00%	Y	-	-	-	-
22	19	0.00%	Y	-	-	-	-
23	20	0.00%	Y	-	-	-	-

Note: ‘Y” in the table indicates that sinusoid X was extended into the forecasting point. Calculation was based on sinusoid X’s amplitude, frequency and phase.

Figure 6 Methodology process flow (see online version for colours)

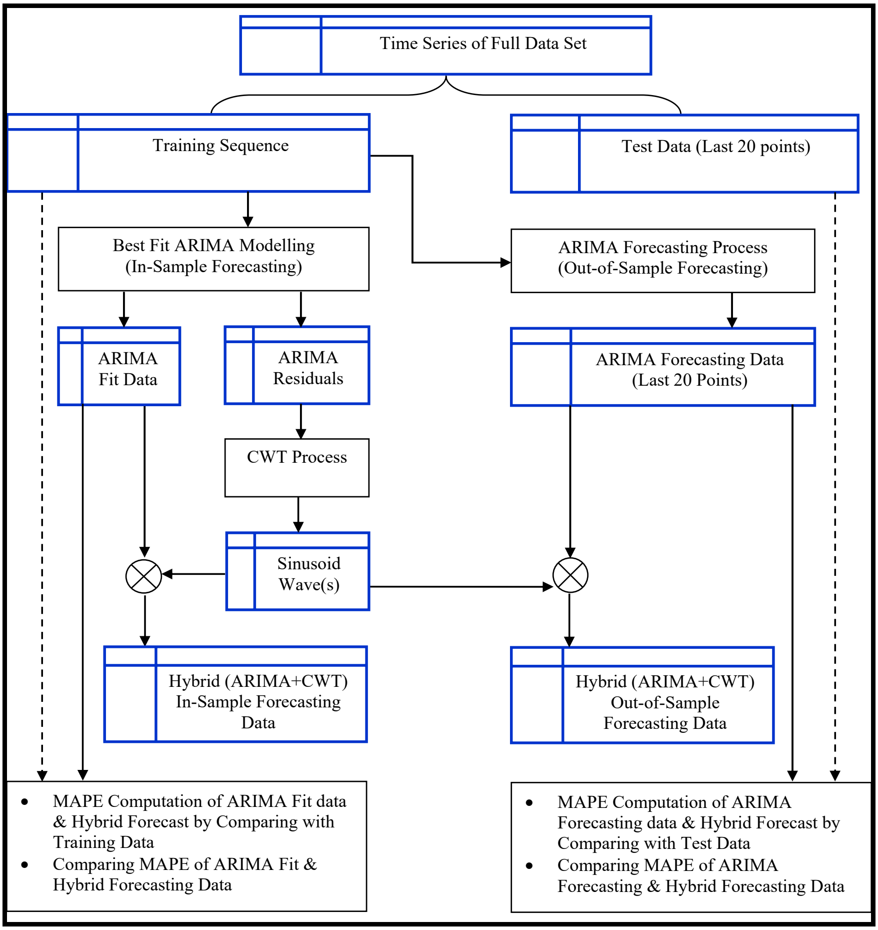


Table 4A Out-of-sample forecasting KEmas (5 Points to 20 Points), percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 2), breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*Kemas 18/07/2001 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 2) + CWT	20	5.26%	0.00%	2.63%	2.63%	2.63%
ARIMA (2, 1, 2) + CWT	50	6.58%	1.32%	3.95%	3.95%	3.95%
ARIMA (2, 1, 2) + CWT	100	9.21%	3.95%	9.21%	6.58%	7.24%
ARIMA (2, 1, 2) + CWT	200	6.58%	3.95%	6.58%	7.89%	6.25%
ARIMA (2, 1, 2) + CWT	300	5.26%	3.95%	6.58%	9.21%	6.25%
Total%		32.89%	13.17%	28.95%	30.26%	

Table 4B Out-of-sample forecasting KEmas (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 2) breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*Kemas 18/07/2001 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 2) + CWT	20	3.13%	7.81%	7.81%	12.50%	7.81%
ARIMA (2, 1, 2) + CWT	50	1.56%	3.13%	3.13%	4.69%	3.13%
ARIMA (2, 1, 2) + CWT	100	4.69%	7.81%	6.25%	9.38%	7.03%
ARIMA (2, 1, 2) + CWT	200	1.56%	4.69%	3.13%	7.81%	4.30%
ARIMA (2, 1, 2) + CWT	300	4.69%	3.13%	3.13%	4.69%	3.91%
Total%		15.63%	26.57%	23.45%	39.07%	

Table 4C Out-of-sample forecasting BCoin (5 Points to 20 Points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 0) breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*BCoin 17/09/2014 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 0) + CWT	20	6.33%	7.59%	12.66%	8.86%	8.86%
ARIMA (2, 1, 0) + CWT	50	5.06%	7.59%	11.39%	7.59%	7.91%
ARIMA (2, 1, 0) + CWT	100	6.33%	10.13%	16.46%	12.66%	11.40%
ARIMA (2, 1, 0) + CWT	200	10.13%	12.66%	11.39%	13.92%	12.03%
ARIMA (2, 1, 0) + CWT	300	7.59%	13.92%	8.86%	12.66%	10.76%
Total%		35.44%	51.89%	60.76%	55.69%	

Table 4D Out-of-sample forecasting BCoin (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 0) breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*BCoin 17/09/2014 to 02/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 0) + CWT	20	12.05%	12.05%	8.43%	9.64%	10.54%
ARIMA (2, 1, 0) + CWT	50	8.43%	19.28%	14.46%	14.46%	14.16%
ARIMA (2, 1, 0) + CWT	100	10.84%	15.66%	10.84%	13.25%	12.65%
ARIMA (2, 1, 0) + CWT	200	13.25%	10.84%	13.25%	12.05%	12.35%
ARIMA (2, 1, 0) + CWT	300	14.46%	15.66%	12.05%	10.84%	13.25%
Total%		59.03%	73.49%	59.03%	60.24%	

Table 4E Out-of-sample forecasting brent (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 0) breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*Brent 27/06/1988 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 0) + CWT	20	7.14%	5.71%	7.14%	7.14%	6.78%
ARIMA (0, 1, 0) + CWT	50	14.29%	14.29%	12.86%	14.29%	13.93%

Table 4E Out-of-sample forecasting brent (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 0) breakdown by back tracking points (BTP) (continued)

Modelling function	*Back tracking points (BTP)*	*Brent 27/06/1988 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 0) + CWT	100	8.57%	7.14%	10.00%	8.57%	8.57%
ARIMA (0, 1, 0) + CWT	200	5.71%	11.43%	8.57%	10.00%	8.93%
ARIMA (0, 1, 0) + CWT	300	5.71%	11.43%	8.57%	10.00%	8.93%
Total%		41.42%	50.00%	47.14%	50.00%	

Table 4F Out-of-sample forecasting brent (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 0) breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*Brent 27/06/1988 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 0) + CWT	20	5.36%	0.00%	0.00%	1.79%	1.79%
ARIMA (0, 1, 0) + CWT	50	5.36%	3.57%	3.57%	3.57%	4.02%
ARIMA (0, 1, 0) + CWT	100	1.79%	0.00%	0.00%	0.00%	0.45%
ARIMA (0, 1, 0) + CWT	200	5.36%	1.79%	1.79%	1.79%	2.68%
ARIMA (0, 1, 0) + CWT	300	0.00%	0.00%	1.79%	0.00%	0.45%
Total%		17.87%	5.36%	7.15%	7.15%	

Table 4G Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 2) breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*S&P500 30/12/1927 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 2) + CWT	20	7.46%	1.49%	0.00%	0.00%	2.24%
ARIMA (0, 1, 2) + CWT	50	8.96%	2.99%	5.97%	8.96%	6.72%
ARIMA (0, 1, 2) + CWT	100	4.48%	1.49%	5.97%	8.96%	5.23%
ARIMA (0, 1, 2) + CWT	200	1.49%	0.00%	2.99%	16.42%	5.23%
ARIMA (0, 1, 2) + CWT	300	1.49%	0.00%	5.97%	19.40%	6.72%
Total%		23.88%	5.97%	20.90%	53.74%	

Table 4H Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 2) breakdown by back tracking points (BTP)

Modelling function	*Back tracking points (BTP)*	*S&P500 30/12/1927 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 2) + CWT	20	5.56%	7.41%	7.41%	9.26%	7.41%
ARIMA (0, 1, 2) + CWT	50	5.56%	9.26%	12.96%	9.26%	9.26%
ARIMA (0, 1, 2) + CWT	100	9.26%	11.11%	12.96%	11.11%	11.11%

Table 4H Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 2) breakdown by back tracking points (BTP) (continued)

Modelling function	*Back tracking points (BTP)*	S&P500 30/12/1927 to 01/10/2021				
		5	10	15	20	Average%
ARIMA (0, 1, 2) + CWT	200	9.26%	7.41%	3.70%	9.26%	7.41%
ARIMA (0, 1, 2) + CWT	300	7.41%	7.41%	1.85%	0.00%	4.17%
Total%		37.05%	42.60%	38.88%	38.89%	

Table 5A to Table 5H show the forecasting outcome grouped by sinusoid probability parameter (*j*%). Table 5A and 5B are for KEmas time series at two different time frames, Table 5C and 5D are for BCoin, Table 5E and 5F are for Brent, and Table 5G and 5H are for S&P500. Both Brent 2021-February and 2021-October time series obtained best forecasting results with 50% sinusoid probability parameter. Both S&P500 2021-Februaruy and 2021-October time series obtained best forecasting results with 30% sinusoid probability parameter. KEmas and BCoin time series did not show any clear pattern. Except for the KEmas 2021-February, better forecasting could be achieved with *j*% lower than 50%.

Table 4A to Table 4H shows the forecasting outcome grouped by the parameter BTP. Table 4A and 4B are for KEmas time series at two different time frames, Table 4C and 4D are for BCoin, Table 4E and 4F are for Brent, and Table 4G and 4H are for S&P500. Data in the tables show that better forecasting performance could be achieved with BTP of 100 points or less. Generally, more BTP did not improve forecasting performance in all of time series tested. This observation suggested that future data movements of the tested time series depend more to the recent historical data points. In addition, different BTP values were required to achieve optimum forecasting performance for time frames 2021-February and 2021-October of the same time series. This is due to the differences in volatility of different time frames.

Looking through the data from Table 3A to Table 5H, out-of-sample forecasting comparison can be made among the tested parameters of k%, *j*% and BTP and across the time series of KEmas, BCoin, Brent and S&P500. For comparison among the parameters, it can be observed that the k% parameter gave the most consistent results at different time frames of the same time series. The best performing k% values are always the same for 2021-Februaruy and 2021-October. This indicates that the known best performing *k*% parameter in an earlier time frame should be reused in the forecasting of later time frames. The same conclusion cannot be made for BTP and j%, as the best performing values are not consistent at different time frames. For comparison across time series, hybrid ARIMA+CWT has higher chance to outperform ARIMA in the tested BCoin time series. The results are consistent for both 2021-Februaruy and 2021-October time frames. However, more different time frames should be put into tests to confirm the results obtained.

From the result compilations of hundreds of simulations, hybrid ARIMA+CWT approach was able to outperform the ARIMA approach when the appropriate parameters were used. The drawback is that an additional CWT process has to be performed on the residuals after the ARIMA modelling, which incurs additional penalty in time series analysis in terms of processing time. DWT, a discrete version of wavelet transform that uses a finite set of wavelets where the resulting wavelet coefficients are only defined at a particular set of scales and locations, may be able to improve the efficiency. However, unlike CWT, where the finer discretisation of scales and shifting in its implementation yields higher-fidelity analysis of scales and shifting, DWT is sparse because the signal is down sampled at each successive DWT scale. The high-fidelity analysis result is essential to financial time series analysis as it enables better localisation of transients and better characterisation of oscillatory movement in the time series. Taking into consideration of the pros and cons of CWT and DWT, CWT is still the preferred choice if detailed sinusoid information is to be extracted from every point of the time series.

Table 5A Out-of-sample forecasting KEmas (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 2) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*Kemas 18/07/2001 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 2) + CWT	0%	2.63%	1.32%	2.63%	3.95%	2.63%
ARIMA (2, 1, 2) + CWT	30%	6.58%	2.63%	3.95%	3.95%	4.28%
ARIMA (2, 1, 2) + CWT	50%	6.58%	5.26%	5.26%	5.26%	5.59%
ARIMA (2, 1, 2) + CWT	80%	10.53%	3.95%	10.53%	10.53%	8.89%
ARIMA (2, 1, 2) + CWT	100%	6.58%	0.00%	6.58%	6.58%	4.94%
Total%		32.90%	13.16%	28.95%	30.27%	

Table 5B Out-of-sample forecasting KEmas (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 2) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*Kemas 18/07/2001 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 2) + CWT	0%	1.56%	1.56%	3.13%	14.06%	5.08%
ARIMA (2, 1, 2) + CWT	30%	4.69%	10.94%	9.38%	14.06%	9.77%
ARIMA (2, 1, 2) + CWT	50%	1.56%	6.25%	3.13%	3.13%	3.52%
ARIMA (2, 1, 2) + CWT	80%	7.81%	7.81%	7.81%	7.81%	7.81%
ARIMA (2, 1, 2) + CWT	100%	0.00%	0.00%	0.00%	0.00%	0.00%
Total%		15.62%	26.56%	23.45%	39.06%	

Table 5C Out-of-sample forecasting BCoin (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 0) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*BCoin 17/09/2014 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 0) + CWT	0%	6.33%	11.39%	13.92%	10.13%	10.44%
ARIMA (2, 1, 0) + CWT	30%	8.86%	12.66%	16.46%	15.19%	13.29%
ARIMA (2, 1, 0) + CWT	50%	3.80%	11.39%	15.19%	12.66%	10.76%
ARIMA (2, 1, 0) + CWT	80%	10.13%	10.13%	7.59%	10.13%	9.50%
ARIMA (2, 1, 0) + CWT	100%	6.33%	6.33%	7.59%	7.59%	6.96%
Total%		35.45%	51.90%	60.75%	55.70%	

Table 5D Out-of-sample forecasting BCoin (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (2, 1, 0) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*BCoin 17/09/2014 to 02/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (2, 1, 0) + CWT	0%	13.25%	18.07%	15.66%	16.87%	15.96%
ARIMA (2, 1, 0) + CWT	30%	14.46%	13.25%	13.25%	13.25%	13.55%
ARIMA (2, 1, 0) + CWT	50%	13.25%	14.46%	14.46%	14.46%	14.16%
ARIMA (2, 1, 0) + CWT	80%	10.84%	15.66%	9.64%	9.64%	11.45%
ARIMA (2, 1, 0) + CWT	100%	7.23%	12.05%	6.02%	6.02%	7.83%
Total%		59.03%	73.49%	59.03%	60.24%	

Table 5E Out-Of-sample forecasting brent (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 0) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*Brent 27/06/1988 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 0) + CWT	0%	5.71%	2.86%	1.43%	4.29%	3.57%
ARIMA (0, 1, 0) + CWT	30%	8.57%	15.71%	17.14%	17.14%	14.64%
ARIMA (2, 1, 0) + CWT	50%	17.14%	21.43%	18.57%	18.57%	18.93%
ARIMA (0, 1, 0) + CWT	80%	2.86%	2.86%	2.86%	2.86%	2.86%
ARIMA (0, 1, 0) + CWT	100%	7.14%	7.14%	7.14%	7.14%	7.14%
Total%		41.42%	50.00%	47.14%	50.00%	

Table 5F Out-of-sample forecasting brent (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 0) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*Brent 27/06/1988 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 0) + CWT	0%	0.00%	0.00%	1.79%	0.00%	0.45%
ARIMA (0, 1, 0) + CWT	30%	3.57%	1.79%	1.79%	1.79%	2.24%
ARIMA (2, 1, 0) + CWT	50%	7.14%	1.79%	1.79%	3.57%	3.57%
ARIMA (0, 1, 0) + CWT	80%	7.14%	0.00%	0.00%	0.00%	1.79%
ARIMA (0, 1, 0) + CWT	100%	0.00%	1.79%	1.79%	1.79%	1.34%
Total%		17.85%	5.37%	7.16%	7.15%	

Table 5G Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 2) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*S&P500 30/12/1927 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 2) + CWT	0%	13.43%	0.00%	4.48%	13.43%	7.84%
ARIMA (0, 1, 2) + CWT	30%	8.96%	4.48%	10.45%	16.42%	10.08%
ARIMA (2, 1, 2) + CWT	50%	1.49%	1.49%	1.49%	8.96%	3.36%

Table 5G Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 2) breakdown by sinusoid probability (j%) (continued)

Modelling function	*Sinusoid probability (j%)*	*S&P500 30/12/1927 to 19/02/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 2) + CWT	80%	0.00%	0.00%	1.49%	11.94%	3.36%
ARIMA (0, 1, 2) + CWT	100%	0.00%	0.00%	2.99%	2.99%	1.50%
Total%		23.88%	5.97%	20.90%	53.74%	

Table 5H Out-of-sample forecasting S&P500 (5 points to 20 points) percentages of ARIMA+CWT parameters to outperform ARIMA (0, 1, 2) breakdown by sinusoid probability (j%)

Modelling function	*Sinusoid probability (j%)*	*S&P500 30/12/1927 to 01/10/2021*				
		5	*10*	*15*	*20*	*Average%*
ARIMA (0, 1, 2) + CWT	0%	12.96%	12.96%	7.41%	9.26%	10.65%
ARIMA (0, 1, 2) + CWT	30%	12.96%	16.67%	12.96%	9.26%	12.96%
ARIMA (0, 1, 2) + CWT	50%	9.26%	11.11%	9.26%	9.26%	9.72%
ARIMA (0, 1, 2) + CWT	80%	1.85%	1.85%	3.70%	5.56%	3.24%
ARIMA (0, 1, 2) + CWT	100%	0.00%	0.00%	5.56%	5.56%	2.78%
Total%		37.03%	42.59%	38.89%	38.90%	

Table 6 Out-of-sample forecasting (5 points to 20 points) percentages for ARIMA+CWT parameters that outperform ARIMA for 5 points forecasting to continue to 10 points, 15 points and 20 points forecasting

Time series – time frames	*Continuity percentages*			
	5	*10*	*15*	*20*
KEmas – 2021-February	-	36.00%	72.00%	56.00%
KEmas – 2021-October	-	90.00%	90.00%	90.00%
BCoin – 2021-February	-	100.00%	89.29%	89.29%
BCoin – 2021-October	-	83.67%	61.22%	65.31%
Brent – 2021-February	-	93.10%	86.21%	89.66%
Brent – 2021- October	-	20.00%	20.00%	30.00%
S&P500 – 2021-February	-	18.75%	31.25%	50.00%
S&P500 – 2021-October	-	90.00%	65.00%	70.00%

5 Conclusions

In-sample forecasting was performed on KEmas, BCoin, Brent and S&P500 time series. From the results obtained, CWT managed to extract additional useful information from the ARIMA residuals of fitted time series of KEmas, BCoin, Brent and S&P500 respectively.

For out-of-sample forecasting, the outcome from the hybrid ARIMA+CWT approach is encouraging. It is obvious that the hybrid ARIMA + CWT approach managed to outperform the ARIMA model when the appropriate combinations of parameters were used. For certain time series, the superior shorter time frame forecasting performance may even be projected to longer time frame forecasting under the same combination of parameters.

Conclusively, the findings in this paper pave the way for new research extensions. Firstly, the hybrid ARIMA + CWT approach shows great potential in performing out-of-sample forecasting. Secondly, more detail investigation is needed to identify the parameters that produce optimum and more consistent forecasting results. Lastly, this paper makes full use of the concept of CWT scale-frequency mapping, the magnitude of success is affected by the accuracy of the chosen mapping method, which is highly dependent on the choice of wavelet function and its frequency characteristics. Further investigation could be carried out using other wavelet functions.

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