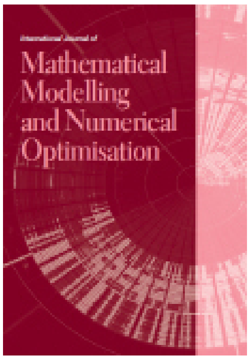


International Journal of Mathematical Modelling and Numerical Optimisation

ISSN online: 2040-3615 - ISSN print: 2040-3607
https://www.inderscience.com/ijmmno

Numerical solution of Lane-Emden pantograph delay differential equation: stability and convergence analysis

Nikhil Sriwastav, Amit K. Barnwal

DOI: 10.1504/IJMMNO.2023.10049441

Article History:

Received:	07 April 2022
Accepted:	13 June 2022
Published online:	19 December 2022

Numerical solution of Lane-Emden pantograph delay differential equation: stability and convergence analysis

Nikhil Sriwastav and Amit K. Barnwal*

Department of Mathematics and Scientific Computing,
Madan Mohan Malaviya University of Technology,
Gorakhpur-273010, India
Email: nikhilsrivastava416908@gmail.com
Email: akbmsc@mmmut.ac.in
*Corresponding author

Abstract: In this work, a collocation approach-based Bernstein operational matrix of differentiation method is used for obtaining the numerical solution of a class of modified Lane-Emden equation with delay in pantograph sense. The proposed numerical algorithm provides numerical solution by discretising the Lane-Emden pantograph delay differential equation into a system of algebraic equations which can be solved directly using any mathematical software. The consistency of the proposed numerical technique is verified with the convergence analysis of the proposed algorithm. The stability analysis of the model is also given using the Lyapunov function. Test examples and graphical representations of their solutions are included to illustrate the applicability and superiority of the proposed method over existing methods.

Keywords: Lane-Emden equation; pantograph delay differential equation; PDDE; Bernstein polynomials; collocation method; convergence analysis.

Reference to this paper should be made as follows: Sriwastav, N. and Barnwal, A.K. (2023) 'Numerical solution of Lane-Emden pantograph delay differential equation: stability and convergence analysis', *Int. J. Mathematical Modelling and Numerical Optimisation*, Vol. 13, No. 1, pp.64–83.

Biographical notes: Nikhil Sriwastav is a PhD Scholar under the supervision of Dr. Amit K. Barnwal in the Department of Mathematics and Scientific Computing, Madan Mohan Malaviya University of Technology Gorakhpur, India. His area of research is 'numerical solution of singular boundary value problems'. He has completed his post-graduation from the Deen Dayal Upadhyaya Gorakhpur University Gorakhpur, India in 2016.

Amit K. Barnwal has received his MSc from the Banaras Hindu University Varanasi, India and PhD in Mathematics from the IIT Kharagpur, India. Currently, he is an Assistant Professor at the Madan Mohan Malaviya University of Technology Gorakhpur, India. His area of research is 'singular boundary-value problems' and 'cryptography'.

1 Introduction

As a special form of a differential equation, delay differential equations (DDEs) have become a great criterion in the modelling of many mathematical models such as population dynamics, epidemiology, and neural networks, and inventory control with supply delay (see Sulem and Tapiero, 1996; Kuang, 1993; Aziz and Amin, 2016, and the references therein). In DDEs, solution at certain time instant depends on the past time. Delay involves automatically in the mathematical model of any dynamical system if someone wants the correct evaluation of parameter involves in it. Many areas of science and technology involves DDEs, for example chemical processes (Epstein, 1992), economic growth (Keller, 2010), network motif modelling (Glass et al., 2021), and mathematical models of infectious disease (Tipsri and Chinviriyasit, 2015; Chinnathambi et al., 2021; Nelson and Perelson, 2002), etc.

Pantograph delay differential equations (PDDEs) are the special types of DDEs. The PDDEs are differential equations with proportional delay, arises in many mathematical models such as control system, probability, quantum mechanics, population studies, electrodynamics (see Kuang, 1993; Bahşi and Çevik, 2015; Ghomanjani and Shateyi, 2020; Anakira et al., 2022, and references therein). The pantograph was a tool of electric locomotive used by British Railways to collect electric current from overloaded lines. Ockendon and Tayler (1971) introduced a first order PDDE

$$y'(t) = ay(t) + by(\alpha t), \ t > 0. \tag{1.1}$$

This differential equations is used to model the motion of pantograph head on electric locomotive, where a and b are real constants and $0 < \alpha < 1$. In view of the frequent occurrence of the DDEs in mathematical modelling of various physical phenomena, it is necessary to find an analytical or a numerical approach to deal with such problems. It is quite a tough job to solve PDDEs analytically. Therefore most researchers adopt a numerical approach to solve the PDDEs. Often spectral method (Adam et al., 2016; Liu et al., 2019), pseudo-spectral method (Breda et al., 2005; Mahmoudi et al., 2020), finite element methods (Deng et al., 2007; Qin et al., 2019), tau methods (Raslan et al., 2019), and polynomial approximation methods (Sedaghat et al., 2012; Ernst and Soleymani, 2019; Gülsu et al., 2011; Yuzbasi and Savasaneril, 2020) are some numerical methods being used to solve PDDEs.

Another area of the differential equation is a singular differential equation, which often arises while developing models of several phenomena of mathematical physics, astrophysics, and biochemistry (see Sahu and Mohapatra, 2021; Lane, 1870; Srivastava, 1962; McCrea, 1939; Hao et al., 2018, and references therein). Lane-Emden equation, Emden-Fowler equation and Emden-Chandrashekhar equation are some well known singular differential equations (Roul, 2019; Wong, 1975; Shi et al., 2016; Chandrasekhar, 1972). Adel and Sabir (2020) have developed a new mathematical model by merging the two prominent areas PDDEs and Lane-Emden equation of differential equation, known as the Lane-Emden PDDEs. They have proposed a numerical technique based on the Bernoulli polynomials and collocation approach for the numerical solution of Lane-Emden PDDEs. A class of Lane-Emden PDDE is given by

$$x^{-\rho}\frac{d^n}{dx^n}\left(x^{\rho}\frac{d^m}{dx^m}y(\alpha x)\right)+g(y)=f(x). \tag{1.2}$$

Here $\rho \geq 1$ is a real constants, represents the shape factor. Differential equation (1.2) for $m = n = 1$ is given with a set of initial conditions as

$$\alpha\frac{d^2}{dx^2}y(\alpha x)+\frac{\rho}{x}\frac{d}{dx}y(\alpha x)+g(y)=f(x) \tag{1.3}$$

subject to

$$y(0)=\beta, \quad y'(0)=0. \tag{1.4}$$

In literature, very few studies (Adel and Sabir, 2020; Izadi and Srivastava, 2021) have been done on numerical solution of singular PDDEs (1.3) with initial conditions (1.4). Several numerical techniques are available to find an approximate solution to singular initial value problems (SIVPs), namely the Adomian decomposition method (Pourgholi and Saeedi, 2015; Kumar and Umesh, 2020), variation iteration method (Verma et al., 2021), homotopy perturbation method (Roul and Warbhe, 2017), Greens function and decomposition method (Singh, 2020; Singh et al., 2015), modified decomposition method (Singh and Wazwaz, 2016), homotopy analysis method (Bataineh et al., 2009), and approximation with polynomials (Zheng and Yang, 2009; Zhou and Xu, 2016; Sahu and Ray, 2017; Hosseini et al., 2017), etc. A method based on approximation with polynomials is easy to code on any mathematical software.

Certain mathematical models represent the practical application of Lane-Emden PDDE (see Ciaraldi-Schoolmann, 2012; Xu et al., 2016, and the references therein). A tumour growth model (Xu et al., 2016) given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma}{\partial r}\right)=\Gamma\sigma, \quad 0<r<R(t), t>0,$$

where $\sigma = \sigma(r, t - \tau(t))$, consists a functional delay $\tau(t)$. A pantograph delay is a particular form of functional delay defined by $\tau(t) = (1 - \alpha)t$. Another model proposed by Ciaraldi-Schoolmann (2012) also shows the practical significance of Lane-Emden PDDE.

Our aim in this paper is to present a numerical algorithm based on the Bernstein basis polynomials and its operational matrix of differentiation (Yousefi and Behroozifar, 2010; Shahni and Singh, 2020a, 2020b, 2021) to solve SIVPs (1.3)–(1.4). We discretise the singular differential equation with appropriate collocation points to get rid of singularity. Thus SIVPs (1.3)–(1.4) are transformed to an equivalent system of algebraic equations, which can be solved easily using mathematical software. The accuracy of the proposed methodology is examined through absolute error norm, and presented in comparison with the exact solution and with some well known existing techniques. The viability of the proposed numerical technique is examined through convergence analysis of the numerical scheme. Lyapunov function is constructed to analyse the stability of the solution of SIVPs (1.3)–(1.4).

The work of this article is organised as follows: Section 2 introduces the basics of the Bernstein basis polynomial and relatable properties. In Section 3, the methodology and the convergence analysis are sentenced. Section 4 contains the stability analysis using Lyapunov function. The accuracy and reliability of the methodology is tested through five numerical examples in Section 5. The conclusion and final remarks are given in Section 6.

2 Bernstein polynomials

Bernstein polynomials were introduced by Sergei Natanovich Bernstein (1880–1968) in order to prove the Weierstrass approximation theorem in a constructive manner. Some basics and relevant studies on the Bernstein polynomials are as follows:

2.1 Basics

The Bernstein polynomials of degree n on the interval $[0, 1]$, are defined as

$$\begin{cases} B_n^i(x) = \binom{n}{i} x^i (1-x)^{n-i}, & 0 \le i \le n \\ 0, & i < 0, i > n, \end{cases}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}, n \in \mathbb{N}, i = 0, 1, ..., n.$ Here, $B_n^0(x), B_n^1(x), \ldots, B_n^n(x)$ are called Bernstein basis polynomials, collectively forms a complete basis for the vector space of all polynomials of degree not more than n and with real coefficients. Some pertinent observations on the Bernstein polynomials are

1 the Bernstein polynomials are non-negative function i.e. $B_n^i(x) \ge 0, \forall x \in [0, 1]$, and $i = 0, 1, \ldots, n$

2 $B_n^0(0) = B_n^n(1) = 1$

3 $B_n^i(0) = B_n^i(1) = 0$, for $1 \le i \le n-1$

4 the sum of all the Bernstein basis polynomials for any n is 1, i.e., $\sum_{i=0}^{n} B_n^i(x) = 1$.

2.2 Functional approximation

A function $y(x) \in L_2[0, 1]$ can be estimated with Bernstein basis polynomials in a linear combination as

$$y(x) \approx y_N(x) = \sum_{i=0}^{N} a_i B_N^i(x) = A^T B(x), \tag{2.1}$$

where

$$A^T = [a_0, a_1, ..., a_N], \quad \text{and} \quad B(x) = [B_N^0(x), B_N^1(x), ..., B_N^N(x)]^T. \tag{2.2}$$

The Bernstein polynomial $B_N^i(x)$ can be expressed in the series of integer power of x, as

$$B_N^i(x) = \binom{N}{i} x^i (1-x)^{N-i} = \sum_{j=0}^{N-i} (-1)^j \binom{N}{i} \binom{N-i}{j} x^{i+j}. \tag{2.3}$$

Equation (2.1) can be expressed in the matrix form, by add of equation (2.3) as

$$y_n(x) = A^T D X(x), \tag{2.4}$$

where

$$D = \begin{bmatrix} (-1)^0 \binom{N}{0} & (-1)^1 \binom{N}{0}\binom{N-0}{1} & \dots & (-1)^{N-0}\binom{N}{0}\binom{N-0}{N-0} \\ 0 & (-1)^0 \binom{N-1}{1} & \dots & (-1)^{N-1}\binom{N}{1}\binom{N-1}{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (-1)^0 \binom{N}{N} \end{bmatrix}, \tag{2.5}$$

and

$$X(x) = [1, x, ..., x^N]^T. \tag{2.6}$$

2.3 The operational matrix of differentiation

The operational matrix of derivative of the set $X(x)$ is defined as follows

$$X'(x) = [0, 1, 2x, ..., Nx^{N-1}]^T \tag{2.7}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & N & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{bmatrix}. \tag{2.8}$$

Thus, the derivatives of the function $y_n(x)$ in terms of Bernstein basis is given by

$$y_N^{(k)}(x) = A^T D C^k X(x), \quad k = 1, 2, ... \tag{2.9}$$

where

$$C = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & N & 0 \end{bmatrix}. \tag{2.10}$$

The matrix C is of order $(N+1)$, called operational matrix of differentiation.

2.4 Approximation of a function and its derivate with delay

A function $y(x) \in L_2[0,1]$ with the delay α, i.e., $y(\alpha x)$ can be approximated using the Bernstern basis polynomials as

$$y(\alpha x) \approx y_N(\alpha x) = \sum_{i=0}^{N} a_i B_N^i(\alpha x) = A^T B(\alpha x), \tag{2.11}$$

which can also be expressed as

$$y_N(\alpha x) = A^T D X(\alpha x). \tag{2.12}$$

Furthermore, the approximation of derivatives of $y(\alpha x)$ using the Bernstein basis polynomials are given as follows

$$y_N^k(\alpha x) = A^T D C^k X(\alpha x). \tag{2.13}$$

3 Methodology and convergence analysis

This section comprises the numerical scheme to find the approximate solution of the SIVPs (1.2)–(1.3) in terms of the Bernstein polynomials. The working of the scheme involves the Bernstein polynomials and a set of appropriate collocation points. The Bernstein polynomials and its derivatives transform differential equation (1.2) into a matrix equation, given by

$$\begin{aligned} &\alpha X(\alpha x)\left(C^T\right)^2\left(D^T\right)^{-1} A + \frac{\rho}{x} X(\alpha x) C^T \left(D^T\right)^{-1} A \\ &+ g\left(X(x)\left(D^T\right)^{-1} A\right) = f(x). \end{aligned} \tag{3.1}$$

The collocation points x_{i-1}, have introduced here to solve equation (3.1) as

$$x_{i-1} = \frac{i}{N+1}, \quad i = 1, 2, ..., N+1, \text{ (for any positive integer } N). \tag{3.2}$$

Collocation points (3.2) transform matrix form (3.1) into a system of nonlinear algebraic equation

$$\alpha \bar{\bar{X}} (C^T)^2 \left(D^T\right)^{-1} A + H \bar{\bar{X}} C^T \left(D^T\right)^{-1} A + g\left(\bar{X}\left(D^T\right)^{-1} A\right) = F. \tag{3.3}$$

Here,

$$\bar{\bar{X}} = \begin{bmatrix} 1 & \alpha x_0 & (\alpha x_0)^2 & \cdots & (\alpha x_0)^N \\ 1 & \alpha x_1 & (\alpha x_1)^2 & \cdots & (\alpha x_1)^N \\ 1 & \alpha x_2 & (\alpha x_2)^2 & \cdots & (\alpha x_2)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha x_N & (\alpha x_N)^2 & \cdots & (\alpha x_N)^N \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix},$$

$$H = \begin{bmatrix} \frac{\rho}{x_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{x_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\rho}{x_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\rho}{x_N} \end{bmatrix}, \quad F = [f(x_0), f(x_1), \cdots, f(x_N)]^T.$$

The application of the Bernstein polynomials and its derivatives, transform initial conditions (1.3), into a pair of algebraic equations given by

$$X(0)\left(D^T\right)^{-1} A = \beta, \quad X(0)C^T\left(D^T\right)^{-1} A = 0. \tag{3.4}$$

In order to get the solution of SIVPs (1.2)–(1.3), recombine a system of $(N+1)$ nonlinear algebraic equation by replacing any two equation of system (3.1) having $(N+1)$ equations, with the pair of algebraic equations (3.4). The solution of the recombined system using Newton-Raphson iteration method for unknown coefficients a_i's provides the solution of SIVPs (1.2)–(1.3) by replacing the values of a_i's in equation (2.1).

To show the convergence of the methodology, the Bernstein polynomials (Levasseur, 1984) have used for Weierstrass approximation theorem.

Theorem 3.1: If $y(x)$ be a continuous function on $[0,1]$, and let $B_n(y,x) = \sum_{i=0}^{n} B_n^i(x) y\left(\frac{i}{n}\right)$ be the Bernstein polynomial of degree n in terms of Bernstein basis, then $B_n(y,x)$ converges to $y(x)$, uniformly.

Proof: Some results on the Bernstein polynomials are as follows:

$$\sum_{i=0}^{n} \binom{n}{i} B_n^i(x) = 1, \tag{3.5}$$

$$\sum_{i=0}^{n} \binom{n}{i} \left(\frac{i}{n}\right) B_n^i(x) = x, \tag{3.6}$$

$$\sum_{i=0}^{n} \binom{n}{i} \left(\frac{i}{n}\right)^2 B_n^i(x) = \frac{n-1}{n}x^2 + \frac{x}{n}. \tag{3.7}$$

The difference of $B_n(y,x)$ and $y(x)$ is given by

$$B_n(y,x) - y(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_n^i(x) - y(x).1.$$

Using relation (3.5), in the following equation, we have

$$B_n(y,x) - y(x) = \sum_{i=0}^{n} \left\{ y\left(\frac{i}{n}\right) - y(x) \right\} B_n^i(x), \tag{3.8}$$

$$\implies |B_n(y,x) - y(x)| \leq \sum_{i=0}^{n} \left| y\left(\frac{i}{n}\right) - y(x) \right| B_n^i(x). \tag{3.9}$$

Since the function $y(x)$ is uniformly continuous on $[0, 1]$, thus there exist a positive real number δ for a given real number $\epsilon > 0$, so that

$$|x_1 - x_2| < \delta, \quad \implies \quad |y(x_1) - y(x_2)| < \epsilon. \tag{3.10}$$

Corresponding to the real number $\delta > 0$ and $x \in [0, 1]$, we can divide the set of nodes $\frac{i}{n}$ into two sets $A = \{\frac{i}{n} : |\frac{i}{n} - x| < \delta\}$ and $B = \{\frac{i}{n} : |\frac{i}{n} - x| \geq \delta\}$. Thus the series on the right hand side of inequality (3.9), can be divided into two series $\sum'$ and $\sum''$, as follows:

$$\begin{aligned} |B_n(y,x) - y(x)| \leq &\sum_{i=0}^{n}{}' \left| y\left(\frac{i}{n}\right) - y(x) \right|_{\left(\frac{i}{n} \in A\right)} B_n^i(x) \\ &+ \sum_{i=0}^{n}{}'' \left| y\left(\frac{i}{n}\right) - y(x) \right|_{\left(\frac{i}{n} \in B\right)} B_n^i(x). \end{aligned} \tag{3.11}$$

Let ϵ is given corresponding to the real number δ, such that

$$\left| y\left(\frac{i}{n}\right) - y(x) \right| < \frac{\epsilon}{2}, \text{ for } \left| \frac{i}{n} - x \right| < \delta. \tag{3.12}$$

Now, for $\left| \frac{i}{n} - x \right| \geq \delta$, we have

$$1 \leq \frac{\left(\frac{i}{n} - x\right)^2}{\delta^2}. \tag{3.13}$$

Let $|f(x)| \leq M$, then by using relation (3.13), we have

$$\begin{aligned} &\sum_{i=0}^{n}{}'' \left| y\left(\frac{i}{n}\right) - y(x) \right|_{\left(\frac{i}{n} \in B\right)} B_n^i(x) \\ &\leq \frac{1}{\delta^2} \sum_{i=0}^{n}{}'' \left(\frac{i}{n} - x\right)^2 \left| y\left(\frac{i}{n}\right) - y(x) \right|_{\left(\frac{i}{n} \in B\right)} B_n^i(x), \end{aligned} \tag{3.14}$$

$$< \frac{2M}{\delta^2} \sum_{i=0}^{n}{}'' \left(\frac{i}{n} - x\right)^2 B_n^i(x). \tag{3.15}$$

Using the results (3.5), (3.6) and (3.7) in the above inequality, we have

$$\sum_{i=0}^{n}{}'' \left| y\left(\frac{i}{n}\right) - y(x) \right|_{\left(\frac{i}{n} \in B\right)} B_n^i(x) < \frac{2M}{\delta^2} \left(\frac{x(1-x)}{n} \right). \tag{3.16}$$

$$< \frac{M}{2\delta^2 n}. \tag{3.17}$$

For the positive real number $\epsilon > 0$, there exist natural number N, such that for all $n \geq N$, $\frac{M}{2\delta^2 n} < \frac{\epsilon}{2}$. Therefore for all $x \in [0,1]$, we have

$$|B_n(y,x) - y(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.18}$$

Thus the Bernstein polynomial $B_n(y,x)$ converges to $y(x)$, uniformly. □

4 Stability analysis

4.1 The Lane-Emden PDDE as an autonomous system

The autonomous system, equivalent to a particular form of Lane-Emden pantograph differential equation has been established to analyse the stability of its equilibrium point at $\alpha = \frac{1}{2}$.

Theorem 4.1: The Lane-Emden pantograph differential equation

$$\frac{1}{2}\frac{d^2 y\left(\frac{1}{2}x\right)}{dx^2} + \frac{2}{x}\frac{dy\left(\frac{1}{2}x\right)}{dx} + y(x)^n = 0 \tag{4.1}$$

is equivalent to the following autonomous system of equation

$$\frac{du}{dt} = p, \tag{4.2}$$

$$\frac{dp}{dt} = -2F^1(u,p), \tag{4.3}$$

with the function $F^1(u,p)$ given by

$$F^1(u,p) = \frac{1}{2}\left[-\frac{7n-15}{n-1}p - \frac{8(3n-5)}{(n-1)^2}u + 2^{\frac{5-3n}{1-n}} B^{n-1}u^n\right]. \tag{4.4}$$

Proof: To transform the Lane-Emden pantograph differential equation (4.1), to an autonomous system, a set of variables (u,t) is given by

$$y(x) = Bx^{\frac{2}{1-n}}u(x), x = e^{-t}, \tag{4.5}$$

where $B > 0$, is a constant. The new set of variables transforms differential equation (4.1) to the following second order differential equation

$$\begin{aligned}&\frac{d^2}{dt^2}u\left(\frac{1}{2}e^{-t}\right) - \frac{7n-15}{n-1}\frac{d}{dt}u\left(\frac{1}{2}e^{-t}\right)\\ &- \frac{8(3n-5)}{(n-1)^2}u\left(\frac{1}{2}e^{-t}\right) + 2^{\frac{5-3n}{1-n}} B^{n-1}u^n(e^{-t}) = 0.\end{aligned} \tag{4.6}$$

In order to analyse the stability condition of equilibrium point of the autonomous system (4.2)–(4.3), the index n will be restrict to the range $(0,\infty)\setminus\{1\}$. The second order differential equation (4.6), is equivalent to the autonomous system (4.2)–(4.3) with the introduction of new variable $p = du/dt$. Also, we have $F^1(0,0) = 0$.

Since, $x \in (0,1]$. So, the new variable $t = \ln\left(\frac{1}{x}\right)$ lies in the set $[0,\infty)$. We observe that, the variables $u(e^{-t})$ and $u(\frac{1}{2}e^{-t})$ are similar for large value of t.

Remark: All the critical points of the autonomous system (4.2)–(4.3) lies on $p = 0$ and, they are solution of the equation

$$F^1(u,0) = \frac{1}{2}\left[-\frac{8(3n-5)}{(n-1)^2}u + 2^{\frac{5-3n}{1-n}} B^{n-1}u^n\right] = 0. \tag{4.7}$$

Thus, we have two critical points given by

$$C_0 = (0,0), \tag{4.8}$$

$$C_n = \left(\frac{1}{B}\left(\frac{8(3n-5)}{2^{\frac{5-3n}{1-n}}(n-1)^2}\right)^{1/n-1}, 0\right) \quad \text{for } n > \frac{5}{3}, \tag{4.9}$$

and

$$C_n = \left((-1)^{\frac{1}{n-1}}\frac{1}{B}\left(\frac{8(5-3n)}{2^{\frac{5-3n}{1-n}}(n-1)^2}\right)^{1/n-1}, 0\right) \tag{4.10}$$
$$\text{for } n \in \left(0, \frac{5}{3}\right)\setminus\{1\},$$

□

4.2 Stability analysis using Lyapunov function (Boehmer and Harko, 2010)

In order to analyse the stability of autonomous system (4.2)–(4.3), some relatable definitions are given as follows:

Definition 4.2 (Lyapunov function): A Russian mathematician Aleksandr Mikhailovich Lyapunov (1857-1918) had presented a continuous function $V : R^n \to R$, in his PhD dissertation for stability analysis of autonomous system $\dot{x} = f(x), x \in R^n$. The function V known as Lyapunov function satisfies following axioms in a neighbourhood D of equilibrium point x_0 of system $\dot{x} = f(x)$

1 V should be differentiable in $D\setminus\{x_0\}$

2 $V(x) > V(x_0)$

3 $\dot{V}(x) \leq 0 \forall x \in D$.

If one can establish a Lyapunov function in a neighbourhood D of equilibrium point x_0. This guaranties the asymptotic stability of the equilibrium point x_0. The best quality of this function is that, it enables us to analyse the stability of autonomous system without its explicit solution.

Theorem 4.3: The system of autonomous differential equation (4.2)–(4.3), is asymptotically stable for $1 < n < \frac{5}{3}$ at C_0. The critical point C_n is asymptotically stable for $1 < n < \dfrac{15}{7}$.

Proof: In order to construct a Lyapunov function $V(u, p)$ using variable gradient method, we set

$$\Delta V(u,p) = \begin{bmatrix} -\frac{8(3n-5)}{(n-1)^2}u + 2^{\frac{5-3n}{1-n}} B^{n-1}u^n \\ p \end{bmatrix} \tag{4.11}$$

such that the critical points C_0 and C_n satisfy the homogeneous equation $\Delta V = 0$. Thus, we have

$$V(u,p) = \frac{1}{2}p^2 - \frac{4(3n-5)}{(n-1)^2}u^2 + 2^{\frac{5-3n}{1-n}} B^{n-1} \frac{u^{n+1}}{n+1}. \tag{4.12}$$

If function $V(u, p)$ in equation (4.12) is Lyapunov function, then there exist local minima at critical points C_0 and C_n. We can verify it by using Hessian $H(V)$ of equation (4.12), given by

$$H(V) = \begin{bmatrix} -\dfrac{8(3n-5)}{(n-1)^2} + n2^{\frac{5-3n}{1-n}} B^{n-1}u^{n-1} & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.13}$$

The eigenvalues of this Hessian matrix at the critical point C_0 are given as

$$\lambda_1 = -\frac{8(3n-5)}{(n-1)^2}, \quad \lambda_2 = 1. \tag{4.14}$$

Since $\lambda_1, \lambda_2 > 0$ for $n \in \left(0, \dfrac{5}{3}\right) \setminus \{1\}$. Thus, Lyapunov function V has local minima near C_0 for $n \in \left(0, \dfrac{5}{3}\right) \setminus \{1\}$. Also, the eigenvalues of the Hessian matrix $H(V)$ at C_n are given by

$$\lambda_1 = -\frac{8n^2 - 29n + 40}{(n-1)^2}, \quad \lambda_2 = 1. \tag{4.15}$$

The analysis for existence of local minima at C_n states that $\lambda_1, \lambda_2 > 0$ for $n \in (0, \infty) \setminus \{1\}$. Thus, Lyapunov function V has local minima near C_n for $n \in (0, \infty) \setminus \{1\}$.

The function $V(u,p)$ also satisfies,

$$\frac{dV}{dx} = \frac{\partial V}{\partial u}\frac{du}{dx} + \frac{\partial V}{\partial p}\frac{dp}{dx} = \frac{(7n-15)}{(n-1)}p^2. \tag{4.16}$$

Thus, we have

$$\dot{V} < 0, \quad \text{for } 1 < n < \frac{15}{7}. \tag{4.17}$$

Hence, the equilibrium point C_0 is asymptotic stable for $1 < n < \frac{5}{3}$ and the equilibrium point C_n is asymptotically stable for $1 < n < \frac{15}{7}$. The Lyapunov function constructed above cannot enable us to know the global stability analysis. □

5 Numerical testing and discussion

In this section, five numerical problems have been tested using the new methodology developed for second order nonlinear Lane-Emden PDDE in order to show the efficiency and accuracy of the method. The absolute error norm $e = |Exact - Approximate|$ is used to compare the approximate solution with exact solution along with existing numerical techniques (Adel and Sabir, 2020; Izadi and Srivastava, 2021).

Example 5.1:

$$\frac{1}{2}\frac{d^2}{dx^2}y\left(\frac{1}{2}x\right) + \frac{2}{x}\frac{d}{dx}y\left(\frac{1}{2}x\right) + y^3 = 4 + 3x^2 + 3x^4 + x^6, \tag{5.1}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{5.2}$$

The exact solution of SIVPs (5.1)–(5.2) is $1 + x^2$. Implementation of the proposed numerical technique developed in Section 3 on Problem 5.1 provides the approximate solution for $N = 2$ which is equal to the exact solution. This shows that the new approach has high adaptability to solve such problems. The values of Bernstein coefficients are provided in Table 1.

Table 1 Chebyshev coefficients at different values of N for Example 5.1

	a_0	a_1	a_2
$N = 2$	1.0	1.0	2.0

Example 5.2:

$$\frac{1}{2}\frac{d^2}{dx^2}y\left(\frac{1}{2}x\right) + \frac{3}{x}\frac{d}{dx}y\left(\frac{1}{2}x\right) + e^y = e^{1+x^3} + \frac{15}{4}x, \tag{5.3}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{5.4}$$

The exact solution of IVP (5.3)–(5.4) is $1 + x^3$.

Figure 1 Comparison of results for Example 5.2, (a) numerical vs. exact (b) absolute error (e) (see online version for colours)

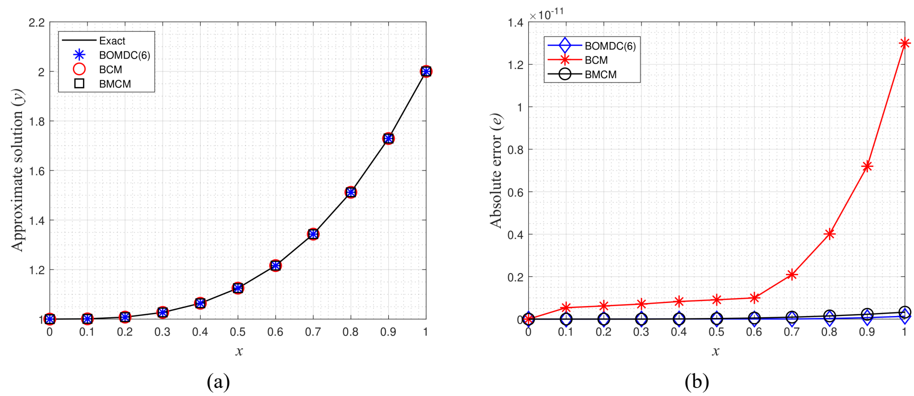


Table 2 Comparison of BOMDC(N) with exact solution at $N = 3$ and $N = 4$ of Example 5.2

x	*Exact*	$N = 3$	e	$N = 4$	e	*BCM*	e	*BMCM*	e
0.0	1.0000	1.0000	0.00000	1.0000	0.00000	1.0000	0.00000	1.0000	0.00000
0.2	1.0080	1.0080	5.7E-11	1.0080	1.5E-15	1.0080	6.2E-13	1.0080	1.6E-15
0.4	1.0640	1.0640	1.4E-10	0.0640	6.6E-15	1.0640	8.3E-13	1.0640	7.8E-15
0.6	1.2160	1.2160	1.1E-10	1.2160	5.5E-15	1.2160	1.0E-12	1.2160	5.0E-14
0.8	0.5120	1.5120	1.4E-10	1.5120	2.7E-14	1.5120	4.0E-12	1.5120	1.5E-13
1.0	2.0000	2.0000	7.8E-10	2.0000	1.3E-13	2.0000	1.3E-11	2.0000	3.2E-13

The approximate solution using the proposed numerical technique, Bernstein operational matrix of differentiation and collocation approach (BOMDC(N)) has presented in Table 2 in comparison with exact solution and some existing methods including Bernoulli collocation method (BCM) (Adel and Sabir, 2020) and Bessel matrix with collocation method (BMCM) (Izadi and Srivastava, 2021). The numerical results of BCM is presented at $N = 6$ and BMCM is presented at $M = 3$. The graphical representation of the approximate solutions using different numerical methods against the exact solution has given in Figure 1(a). One cannot differentiate the solution graphs for different methods without legends. So, we have also drawn the graph for absolute errors in Figure 1(b), which shows the excellency of BOMDC(N) over BCM and BMCM. Table 2 concludes that the maximum absolute error decreases significantly from the order 10^{-10} to 10^{-13} as N increases from 3 to 4. To verify the numerical results of BOMDC(N), the Bernstein coefficients are given in Table 3.

Table 3 Bernstein coefficients at different values of N for Example 5.2

	a_0	a_1	a_2	a_3	a_4
$N = 3$	1.0	1.0	0.9999999993306705	2.0000000007865645	
$N = 4$	1.0	1.0	1.0000000000000000	1.2499999999999349	2.0000000000001332

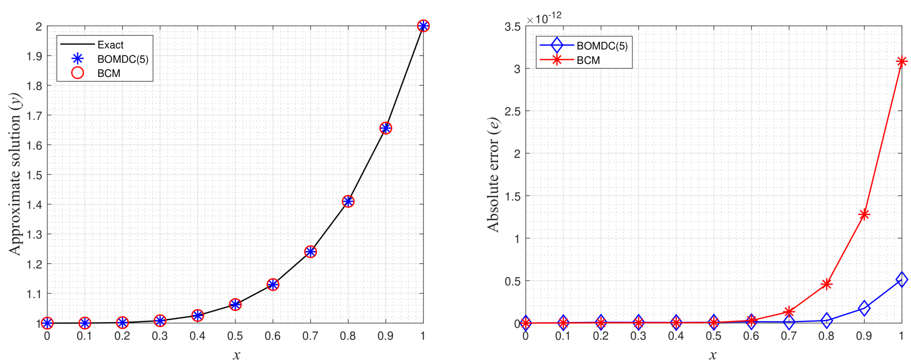
Example 5.3:

$$\frac{1}{2}\frac{d^2}{dx^2}y\left(\frac{1}{2}x\right) + \frac{3}{x}\frac{d}{dx}y\left(\frac{1}{2}x\right) + y^2 = x^8 + 2x^4 + 3x^2 + 1, \tag{5.5}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{5.6}$$

Figure 2 Comparison of results for Example 5.3, (a) numerical vs. exact (b) absolute error (e) (see online version for colours)



(a) (b)

Table 4 Comparison of BOMDC(N) with exact solution at $N = 4$ and $N = 5$ of Example 5.3

x	*Exact*	$N = 4$	e	$N = 5$	e	BCM	e
0.0	1.0000	1.0000	0.00000	1.0000	0.00000	1.0000	0.00000
0.2	1.0016	1.0016	1.8E-13	1.0016	8.8E-15	1.0016	3.7E-15
0.4	1.0256	1.0256	2.4E-13	1.0256	6.8E-15	1.0256	4.4E-15
0.6	1.1296	1.1296	1.8E-12	1.1296	1.6E-14	0.1296	3.0E-14
0.8	1.4096	1.4096	9.6E-12	1.4096	2.9E-14	1.4096	4.5E-13
1.0	2.0000	2.0000	3.1E-11	2.0000	5.1E-13	2.0000	3.0E-12

Table 5 Bernstein coefficients at different values of N for Example 5.3

	a_0	a_1	a_2	a_3	a_4	a_5
$N = 4$	1.0	1.0	1.0000000000023	0.9999999999909	2.0000000000318	
$N = 5$	1.0	1.0	1.0000000000000	0.9999999999997	1.2000000000004	1.9999999999994

The exact solution of SIVPs (5.5)–(5.6) is $1 + x^4$. The proposed methodology converts the SIVPs into a system of nonlinear algebraic equations. The solution of these algebraic equations for unknown Bernstein coefficients provides the numerical solution of the IVP (5.5)–(5.6). The comparative discussion of the proposed technique BOMDC with BCM (Adel and Sabir, 2020) is given in Table 4 and Figure 2. The values of approximate solution using present method are being presented in Table 4 for $N = 4$ and $N = 5$, at different values of x. The numerical results of BCM are given at $N = 6$. The absolute error given in Table 4, and plotted in Figure 2(b) makes us sure, that the proposed methodology performs better than the existing technique BCM. To check the viability of the proposed technique, the values of the Bernstein coefficients are given in Table 5.

Example 5.4:

$$\begin{aligned}&\frac{1}{2}y''\left(\frac{1}{2}x\right) + \frac{2}{x}y'\left(\frac{1}{2}x\right) + y(x) + \frac{1}{3}y^3(x) + \frac{2}{5}y^5(x)\\&= 3 + (1 + x^2) + \frac{1}{3}(1 + x^2)^3 + \frac{2}{5}(1 + x^2)^5,\end{aligned} \tag{5.7}$$

subject to

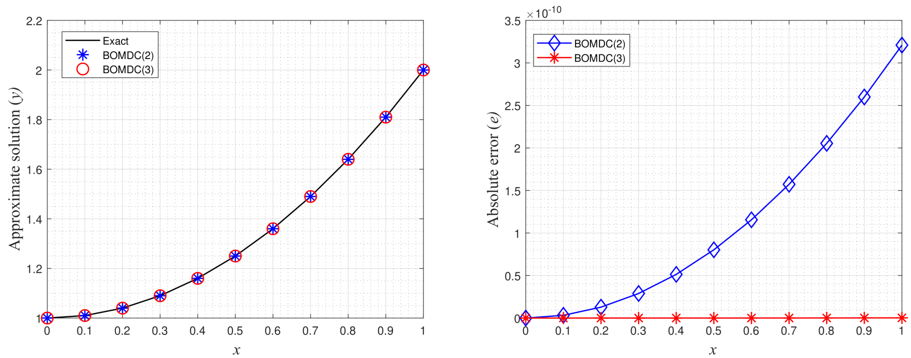
$$y(0) = 1, \quad y'(0) = 0. \tag{5.8}$$

The exact solution of Problem 5.4 is $1 + x^2$. The qualitative and quantitative representation of approximate solution in comparison with exact solution has presented in Table 6, and Figure 3. The approximate solution using proposed numerical technique has given in Table 6 and plotted in Figure 3(a), for $N = 2$ and $N = 3$. One cannot differentiate between the solution graphs for $N = 2$, $N = 3$ and also the exact solution in the Figure 3(a) without legend. To overcome this situation, we have also given the absolute error in Table 6 and plotted in Figure 3(b). Table 6 concludes that the absolute error decreases significantly as N increases from 2 to 3.

Table 6 Comparison of BOMDC(N) with exact solution at $N = 2$ and $N = 3$ of Example 5.4

x	*Exact*	*N = 2*	e	*N = 3*	e
0.0	1.0000	1.0000	0.00000	1.0000	0.00000
0.2	1.0400	1.0400	1.2E-11	1.0400	4.2E-15
0.4	1.1600	1.1600	5.1E-11	1.1600	2.6E-15
0.6	1.3600	1.3600	1.1E-10	1.3600	2.6E-14
0.8	1.6400	1.6400	2.0E-10	1.6400	1.0E-13
1.0	2.0000	2.0000	3.2E-10	2.0000	2.4E-13

Figure 3 Comparison of results for Example 5.4, (a) numerical vs. exact (b) absolute error (e) (see online version for colours)



(a) (b)

Table 7 Bernstein coefficients at different values of N for Example 5.4

	a_0	a_1	a_2	a_3
$N = 2$	1.0	1.0	2.0000000003209352	
$N = 3$	1.0	1.0	1.3333333333332697	2.0000000000002473

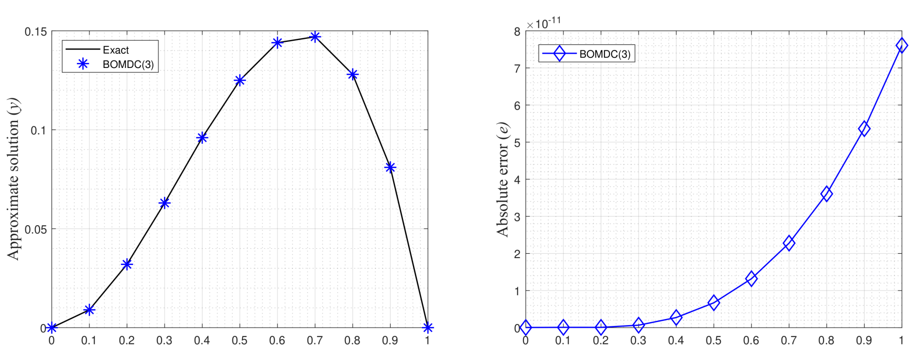
Example 5.5:

$$\frac{1}{2}y''\left(\frac{1}{2}x\right) + \frac{2}{x}y'\left(\frac{1}{2}x\right) + e^{\frac{y}{2}} = 3 - 3x + e^{\frac{1}{2}(x^2 - x^3)}, \tag{5.9}$$

subject to

$$y(0) = 0, \quad y'(0) = 0. \tag{5.10}$$

Figure 4 Comparison of results for Example 5.5, (a) numerical vs. exact (b) absolute error (e) (see online version for colours)



(a) (b)

Table 8 Bernstein coefficients at different values of N for Example 5.5

	a_0	a_1	a_2	a_3
$N = 2$	0.0	0.0	0.333333333400134	
$N = 3$	0.0000000000000431	0.0000000000000447	0.3333333333258025	0.0000000000760721

Table 9 Comparison of BOMDC(N) with exact solution at $N = 2$ and $N = 3$ of Example 5.5

x	*Exact*	$N = 2$	e	$N = 3$	e
0.0	0.0000	0.0000	0.00000	0.0000	$4.3E - 14$
0.2	0.0320	0.0133	1.8E-02	0.0320	7.5E-14
0.4	0.0960	0.0533	4.2E-02	0.0960	2.7E-12
0.6	0.1440	0.1200	2.4E-02	0.1440	1.3E-11
0.8	0.1280	0.2133	8.5E-02	0.1280	3.6E-11
1.0	0.0000	0.3333	3.3E-01	0.0000	7.6E-11

The exact solution of SIVPs (5.9)–(5.10) is $x^2 - x^3$. The qualitative and quantitative representation of approximate solution in comparison with exact solution has presented in Table 9 and Figure 4. The approximate solution using proposed numerical technique has given in Table 9, for $N = 2$ and $N = 3$, also plotted in Figure 3(a), for $N = 3$. To check the accuracy of the method, the absolute errors are computed and given in Table 9. Table 9 concludes that the absolute error decreases significantly as N increases from 2 to 3.

6 Conclusions

A robust numerical method based on the Bernstein operational matrix of differentiation and collocation approach has been introduced to find the numerical solution of a class of PDDE. The leading supremacy of the proposed technique is its algorithm and computer programming. The programming of the methodology is easy to implement on any mathematical software. It can be easily implemented on different test examples with slight modification in code. The other advantage of this technique is its high precision results in terms of absolute error norms. The proposed technique deal the nonlinear problems with highly nonlinear terms $(e^y, e^{\frac{y}{2}})$ with excellent accuracy of order 10^{-11} to 10^{-13} for very small values of $N \leq 4$. The convergence analysis of the proposed numerical technique and Lyapunov stability analysis of Lane-Emden PDDE is also given to show the efficiency and applicability of the numerical algorithm.

Acknowledgements

This work is supported by SERB, New Delhi (Grant No. ECR/2017/000560).

References

Adam, A., Bashier, E., Hashim, M. and Patidar, K. (2016) 'Fitted galerkin spectral method to solve delay partial differential equations', *Mathematical Methods in the Applied Sciences*, Vol. 39, No. 11, pp.3102–3115.

Adel, W. and Sabir, Z. (2020) 'Solving a new design of nonlinear second-order Lane-Emden pantograph delay differential model via Bernoulli collocation method', *The European Physical Journal Plus*, Vol. 135, No. 5, p.427.

Anakira, N., Jameel, A., Hijazi, M., Alomari, A-K. and Man, N. (2022) 'A new approach for solving multi-pantograph type delay differential equations', *International Journal of Electrical & Computer Engineering*, Vol. 12, No. 2, pp.1859–1868, ISSN: 2088-8708.

Aziz, I. and Amin, R. (2016) 'Numerical solution of a class of delay differential and delay partial differential equations via Haar wavelet', *Applied Mathematical Modelling*, Vol. 40, Nos. 23–24, pp.10286–10299.

Bahşi, M.M. and Çevik, M. (2015) 'Numerical solution of pantograph-type delay differential equations using perturbation-iteration algorithms', *Journal of Applied Mathematics*, Vol. 2015, pp.1–10.

Bataineh, A.S., Noorani, M.S.M. and Hashim, I. (2009) 'Homotopy analysis method for singular ivps of Emden-Fowler type', *Communications in Nonlinear Science and Numerical Simulation*, Vol. 14, No. 4, pp.1121–1131.

Boehmer, C.G. and Harko, T. (2010) 'Nonlinear stability analysis of the Emden-Fowler equation', *Journal of Nonlinear Mathematical Physics*, Vol. 17, No. 4, pp.503–516.

Breda, D., Maset, S. and Vermiglio, R. (2005) 'Pseudospectral differencing methods for characteristic roots of delay differential equations', *SIAM Journal on Scientific Computing*, Vol. 27, No. 2, pp.482–495.

Chandrasekhar, S. (1972) 'A limiting case of relativistic equilibrium', in O'Raifertaigh, L. in honor of Synge, J.L. (Eds.): *General Relativity*, pp.185–199, Clarendon Press, Oxford.

Chinnathambi, R., Rihan, F.A. and Alsakaji, H.J. (2021) 'A fractional-order model with time delay for tuberculosis with endogenous reactivation and exogenous reinfections', *Mathematical Methods in the Applied Sciences*, Vol. 44, No. 10, pp.8011–8025.

Ciaraldi-Schoolmann, F. (2012) *Modeling Delayed Detonations of Chandrasekhar-Mass White Dwarfs*, PhD thesis, Technische Universität München.

Deng, K., Xiong, Z. and Huang, Y. (2007) 'The Galerkin continuous finite element method for delay-differential equation with a variable term', *Applied Mathematics and Computation*, Vol. 186, No. 2, pp.1488–1496.

Epstein, I.R. (1992) 'Delay effects and differential delay equations in chemical kinetics', *International Reviews in Physical Chemistry*, Vol. 11, No. 1, pp.135–160.

Ernst, P.A. and Soleymani, F. (2019) 'A legendre-based computational method for solving a class of Itô stochastic delay differential equations', *Numerical Algorithms*, Vol. 80, No. 4, pp.1267–1282.

Ghomanjani, F. and Shateyi, S. (2020) 'Solving a quadratic riccati differential equation, multi-pantograph delay differential equations, and optimal control systems with pantograph delays', *Axioms*, Vol. 9, No. 3, p.82.

Glass, D.S., Jin, X. and Riedel-Kruse, I.H. (2021) 'Nonlinear delay differential equations and their application to modeling biological network motifs', *Nature communications*, Vol. 12, No. 1, pp.1–19.

Gülsu, M., Gürbüz, B., Öztürk, Y. and Sezer, M. (2011) 'Laguerre polynomial approach for solving linear delay difference equations', *Applied Mathematics and Computation*, Vol. 217, No. 15, pp.6765–6776.

Hao, T-C., Cong, F-Z. and Shang, Y-F. (2018) 'An efficient method for solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions and error estimate', *Journal of Mathematical Chemistry*, Vol. 56, No. 9, pp.2691–2706.

Hosseini, E., Barid Loghmani, G., Heydari, M. and Wazwaz, A-M. (2017) 'A numerical study of electrohydrodynamic flow analysis in a circular cylindrical conduit using orthonormal bernstein polynomials', *Computational Methods for Differential Equations*, Vol. 5, No. 4, pp.280–300.

Izadi, M. and Srivastava, H. (2021) 'An efficient approximation technique applied to a non-linear Lane-Emden pantograph delay differential model', *Applied Mathematics and Computation*, Vol. 401, p.126123.

Keller, A.A. (2010) 'Generalized delay differential equations to economic dynamics and control', *American-Math*, Vol. 10, pp.278–286.

Kuang, Y. (1993) *Delay Differential Equations: With Applications in Population Dynamics*, Academic Press, New York.

Kumar, M. and Umesh (2020) 'Numerical solution of Lane-Emden type equations using Adomian decomposition method with unequal step-size partitions', *Engineering Computations*, Vol. 38, No. 1, pp.1–11.

Lane, H.J. (1870) 'On the theoretical temperature of the sun, under the hypothesis of a gaseous mass maintaining its volume by its internal heat, and depending on the laws of gases as known to terrestrial experiment', *American Journal of Science*, Vol. 2, No. 148, pp.57–74.

Levasseur, K.M. (1984) 'A probabilistic proof of the Weierstrass approximation theorem', *The American Mathematical Monthly*, Vol. 91, No. 4, pp.249–250.

Liu, H., Lü, S. and Chen, H. (2019) 'Spectral approximations for nonlinear fractional delay diffusion equations with smooth and nonsmooth solutions', *Taiwanese Journal of Mathematics*, Vol. 23, No. 4, pp.981–1000.

Mahmoudi, M., Ghovatmand, M. and Skandari, M.N. (2020) 'A new convergent pseudospectral method for delay differential equations', *Iranian Journal of Science and Technology, Transactions A: Science*, Vol. 44, No. 1, pp.203–211.

McCrea, W. (1939) 'An introduction to the study of stellar structure', *Nature*, Vol. 144, No. 3638, pp.130–131.

Nelson, P.W. and Perelson, A.S. (2002) 'Mathematical analysis of delay differential equation models of HIV-1 infection', *Mathematical Biosciences*, Vol. 179, No. 1, pp.73–94.

Ockendon, J.R. and Tayler, A.B. (1971) 'The dynamics of a current collection system for an electric locomotive', *Proceedings of the Royal Society of London A. Mathematical and Physical Sciences*, Vol. 322, No. 1551, pp.447–468.

Pourgholi, R. and Saeedi, A. (2015) 'A numerical method based on the Adomian decomposition method for identifying an unknown source in non-local initial-boundary value problems', *International Journal of Mathematical Modelling and Numerical Optimisation*, Vol. 6, No. 3, pp.185–197.

Qin, H., Zhang, Q. and Wan, S. (2019) 'The continuous galerkin finite element methods for linear neutral delay differential equations', *Applied Mathematics and Computation*, Vol. 346, pp.76–85.

Raslan, K.R., Ali, K.K., Mohamed, E.M. et al. (2019) 'Spectral tau method for solving general fractional order differential equations with linear functional argument', *Journal of the Egyptian Mathematical Society*, Vol. 27, No. 1, pp.1–16.

Roul, P. (2019) 'A new mixed madm-collocation approach for solving a class of Lane-Emden singular boundary value problems', *Journal of Mathematical Chemistry*, Vol. 57, No. 3, pp.945–969.

Roul, P. and Warbhe, U. (2017) 'A new homotopy perturbation scheme for solving singular boundary value problems arising in various physical models', *Zeitschrift für Naturforschung A*, Vol. 72, No. 8, pp.733–743.

Sahu, P.K. and Ray, S.S. (2017) 'Chebyshev wavelet method for numerical solutions of integro-differential form of Lane-Emden type differential equations', *International Journal of Wavelets, Multiresolution and Information Processing*, Vol. 15, No. 02, pp.1750015.

Sahu, S.R. and Mohapatra, J. (2021) 'Numerical investigation for solutions and derivatives of singularly perturbed initial value problems', *International Journal of Mathematical Modelling and Numerical Optimisation*, Vol. 11, No. 2, pp.123–142.

Sedaghat, S., Ordokhani, Y. and Dehghan, M. (2012) 'Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials', *Communications in Nonlinear Science and Numerical Simulation*, Vol. 17, No. 12, pp.4815–4830.

Shahni, J. and Singh, R. (2020a) 'An efficient numerical technique for Lane-Emden-Fowler boundary value problems: Bernstein collocation method', *The European Physical Journal Plus*, Vol. 135, No. 6, pp.1–21.

Shahni, J. and Singh, R. (2020b) 'Numerical results of Emden-Fowler boundary value problems with derivative dependence using the Bernstein collocation method', *Engineering with Computers*, Article No. 38, No. 1, pp.1–10.

Shahni, J. and Singh, R. (2021) 'Numerical solution of system of Emden-Fowler type equations by Bernstein collocation method', *Journal of Mathematical Chemistry*, Vol. 59, No. 4, pp.1117–1138.

Shi, Y., Han, Z. and Sun, Y. (2016) 'Oscillation criteria for a generalized Emden-Fowler dynamic equation on time scales', *Advances in Difference Equations*, Vol. 2016, No. 1, pp.1–12.

Singh, R. (2020) 'Solving coupled Lane-Emden equations by Green's function and decomposition technique', *International Journal of Applied and Computational Mathematics*, Vol. 6, No. 3, pp.1–14.

Singh, R., Nelakanti, G. and Kumar, J. (2015) 'Approximate solution of two-point boundary value problems using Adomian decomposition method with Green's function', *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, Vol. 85, No. 1, pp.51–61.

Singh, R. and Wazwaz, A-M. (2016) 'Numerical solution of the time dependent Emden-Fowler equations with boundary conditions using modified decomposition method', *Appl. Math. Inf. Sci.*, Vol. 10, No. 2, pp.403–408.

Srivastava, S. (1962) 'A new solution of the Lane-Emden equation of index N = 5', *The Astrophysical Journal*, Vol. 136, pp.680–681.

Sulem, A. and Tapiero, C.S. (1996) 'Inventory control with supply delays, on going orders and emergency supplies', *IFAC Proceedings Volumes*, Vol. 29, No. 8, pp.109–115.

Tipsri, S. and Chinviriyasit, W. (2015) 'The effect of time delay on the dynamics of an SEIR model with nonlinear incidence', *Chaos, Solitons & Fractals*, Vol. 75, pp.153–172.

Verma, A.K., Kumar, N., Singh, M. and Agarwal, R.P. (2021) 'A note on variation iteration method with an application on Lane-Emden equations', *Engineering Computations*, Vol. 38, No. 10, pp.3932–3943.

Wong, J.S. (1975) 'On the generalized Emden-Fowler equation', *Siam Review*, Vol. 17, No. 2, pp.339–360.

Xu, S., Wei, X. and Zhang, F. (2016) 'A time-delayed mathematical model for tumor growth with the effect of a periodic therapy', *Computational and Mathematical Methods in Medicine*, Vol. 2016, pp.1–8.

Yousefi, S. and Behroozifar, M. (2010) 'Operational matrices of Bernstein polynomials and their applications', *International Journal of Systems Science*, Vol. 41, No. 6, pp.709–716.

Yuzbasi, S. and Savasaneril, N.B. (2020) 'Hermite polynomial approach for solving singular perturbated delay differential equations', *Journal of Science and Arts*, Vol. 20, No. 4, pp.845–854.

Zheng, X. and Yang, X. (2009) 'Techniques for solving integral and differential equations by Legendre wavelets', *International Journal of Systems Science*, Vol. 40, No. 11, pp.1127–1137.

Zhou, F. and Xu, X. (2016) 'Numerical solutions for the linear and nonlinear singular boundary value problems using Laguerre wavelets', *Advances in Difference Equations*, Vol. 2016, No. 1, pp.1–15.