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Multi-objective optimisation for solving cooperative continuous static games using Karush-Kuhn-Tucker conditions

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Abstract: This paper introduces cooperative continuous static games (CCSG) with parameters in the cost functions of the players and in the right-hand side of the constraints. The CCSG is converted into the corresponding multi-objective nonlinear programming problem. The resulted nonlinear programming problem is converted into the single objective nonlinear programming problem through the use of the weighted sum method. A solution method for obtaining the stability set of the second kind without differentiability for the CCSG is presented using Karush-Kuhn-Tucker conditions. A numerical example is given for the illustration.

Keywords: cooperative continuous static games; CCSG; efficient solution; weighted sum method; optimal solution; Karush-Kuhn-Tucker conditions; stability.

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1 Introduction

Game theory has enormous application as in economics, engineering, biology, etc. Three major classes of games are matrix games, continuous static games and differential games. In continuous static games, the decision probabilities are involved, which need not be discrete. Moreover, continuous static games, the decisions and costs are related in a continuous rather than a discrete manner. Sakawa and Yano (1994) introduced an interactive decision making method for solving multi-objective nonlinear programming problem (MONLP) with fuzzy parameters both in objective functions and constraints. Osman (1984) gave the formulation of different formulations of continuous static games. Osman and El-Banna (1993) studied MONLP problem with fuzzy parameters in the objective functions.

Sakawa and Nishizaki (1989) considered two-person zero-sum games with fuzzy multiple payoff matrices. Stoer and Witzgall (1970) proposed the convexity and optimisation in finite dimensions. Osman et al. (1999b) presented the formulation of continuous static games, where the fuzzy numbers were considered in cost functions as well as in constraints. Vives (2005) presented the games with strategic complementarities. They also presented some new applications to the field of industrial organisation. Beckenkamp (2006) introduced a game-theoretic approach for the taxonomy of social dilemmas. Elshafei (2007) introduced an interactive approach for solving Nash cooperative continuous static games (CCSG), and also determined the stability set of the first kind corresponding to the obtained compromise solution. Potters and Suetens (2009) discussed the cooperation in games strategy involving the strategic complements as well as substitutes. Dutta and Kumar (2015) studied the multi-objective linear fractional inventory model with application of fuzzy goal programming approach.

Khalifa and ZeinEldein (2015) introduced an interactive approach for solving CCSG with fuzzy parameters in the objective functions coefficients. Very recently, Potters and Suetens (2020) proposed the optimisation incentives in various dilemma games including the strategic complementarity. The concept of equilibrium for a non-cooperative game with fuzzy goals involving fuzzy parameters has been introduced by Kacher and Larbain (2008).

Cruz and Simaan (2000) proposed theory of ordinal games where, the players are able to rank-order their decision choices against the choice by the other players instead of payoff function. Navidi et al. (2014) presented a new game theoretic-based approach for multi-response optimisation problem. Osman (1977) posed a qualitative analysis of basic notions in parametric convex programming. Daskalakis et al. (2009) investigated the complexity for computation of a Nash equilibrium. Corley et al. (2014) studied the n -person prescriptive games and presented a scalar compromise equilibrium for the same. Corley (2015) defined a mixed dual to the Nash equilibrium for n person in strategic form, where in a Nash equilibrium every players' mixed strategy maximise his/her own expected payoff for the other $n - 1$ player's strategies. Also, the comparison between the dual and the related to the mixed Nash equilibrium and both topological and algebraic conditions are given.

Kumar and Dutta (2015) developed a multi-objective linear fractional optimisation model with an application to inventory problem of multi-products in fuzzy environment. Famrooqui and Niazi (2016) introduced a comprehensive multidisciplinary state-of-the-art review and taxonomy of the game theory models of complex interactions between agents. Pakdaman and Effati (2017) investigated the bounds for the convex as well as quadratic programming problem. They also presented some applications. Rout et al. (2019) proposed the multi-objective quadratic programming problem with the fuzzy as well as probabilistic uncertainty. Khalifa (2018) introduced an interactive approach for solving multi-objective nonlinear programming problem and applied it to CCSG. Aineth and Ravindran (2018) presented a multi-objective optimisation model with an application to the selection of critical suppliers. Mrinal and Geetanjali (2018) presented a linear relaxation (LR)-optimal solution for the nonlinear optimisation model considering the variations in model parameters. Ezimadu and Nwozo (2019) studied a manufacturer-retailers dynamic cooperative advertising with retail competition. They investigated the influence of subsidy on awareness shares, retail advertising and, as a result, the payoffs matrix for all the players in the game problem.

Awaya and Krishna (2019) studied the role of communication in repeated games with private monitoring and compared the set of equilibria under two regimes. Figueroa-García et al. (2019) presented the optimal solutions for the group matrix games. They considered the interval-valued fuzzy numbers in their investigation. Liu et al. (2020) proposed an algorithm for the multi-UUV cooperative dynamic manoeuvre decision-making applying intuitionistic fuzzy game theory. Stefanini and Arana-Jiménez (2019) presented the Karush-Kuhn-Tucker conditions for the interval as well the fuzzy optimisation problems in multiple variables. They considered the total and directional generalised differentiability in their study.

Chankong and Haimes (1983) proposed a multi-objective decision making theory and methodology. Bazaraa et al. (1993) described the nonlinear programming with theory and algorithm. Osman et al. (1999a) introduced large scale continuous static games involving fuzzy parameters both in the cost functions and constraints. Marler and Arora (2010)

presented some new insights using ‘the weighted sum method for multiple objective optimisation problems. Matsumoto and Szidarovszky (2015) studied the continuous static games with some applications.

In this paper, CCSG with parameters involving in the cost functions of the players and in the right-hand side of the constraints is introduced. The stability set of the second kind corresponding to the optimal solution resulted from the use of weighted sum method, is determined.

The main contributions of the present paper are summarised below:

- Introducing the CCSG.
- Applying weighted sum method to solve the multi-objective optimisation problem.
- Solving the derived multi-objective models using Karush-Kuhn-Tucker conditions.
- Demonstrating the problem and algorithm with the help of a numerical example of two-person game problem.
- Determining the stability set of the second kind.

The outlay of the paper is organised as follows: in Section 2, the preliminaries are presented. In Section 3, CCSG are introduced as specific definition and properties. Section 4 introduces the stability set of the second kind. In Section 5, a solution method for determining the stability set of the second kind is introduced. In Section 6, a numerical example is given to clarify the solution approach. Finally, some concluding remarks are reported in Section 7.

2 Preliminaries

In this section, we discuss some definitions used in this paper.

Definition 1: Cooperative game (Khalifa, 2018).

A cooperative game is the one in which players are convinced to adopt a particular strategy through negotiations and agreements between players.

Definition 2: Continuous game (Matsumoto and Szidarovszky, 2015).

A game is called ‘continuous’ when the choice set of at least one player is the continuum to begin with, not just an extension of a discrete set to the probability distributions over that set. Let N be the number of players. If S_k is the strategy set of the player k , then its payoff function φ_k is defined on the set of all simultaneous strategies, which is represented by

$$S = S_1 \times S_2 \times \dots \times S_N, \text{ and } \varphi_k(s) \text{ for all } s \in S \text{ is a real number.}$$

Then, the game is continuous, if all sets S_k are connected and all payoff functions φ_k are piecewise continuous.

Definition 3: Static game (Khalifa, 2018).

A game is static in the sense that no time history is involved in the relationship between costs and decisions.

Definition 4: Cooperative continuous static game (Khalifa, 2018).

A static game, which is also continuous and cooperative, is referred to be cooperative continuous static game.

Definition 5: Optimal solution to a game (Figueroa-García et al., 2019).

An optimal solution to the game is said to be reached if neither player finds it beneficial to alter his strategy. In this case the game is said to be in a state of equilibrium.

Definition 6: Weighted sum method (Marler and Arora, 2010).

The weighted sum method combines all the multi-objective functions into one scalar, composite objective function using the weighted sum.

$$F(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \dots + \lambda_n f_n(x),$$

where

$$x = (x_1, x_2, \dots, x_n), \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n),$$

$$\sum_{i=1}^k \lambda_i = 1, \quad 0 \leq \lambda_i \leq 1.$$

3 Problem statement and solution concepts

Before defining the proposed problem statement and solution concepts, let us present the main notations.

3.1 Notations

The main notations are given in Table 1.

Table 1 List of notations used

k	Number of players
$P(\tau(J))$	Stability set of the second kind
\mathbb{R}	Real line
\mathbb{R}^n	n -dimensional space
$\mathbb{R}^n \times \mathbb{R}^s$	Product space of dimension $n \times s$, where $n, s \in \mathbb{R}$
$M_{a_i}^i(x, \delta)$	Convex functions on $\mathbb{R}^n \times \mathbb{R}^s$
$g(x, \delta)$	Concave functions on $\mathbb{R}^n \times \mathbb{R}^s$
$\gamma_{a_i}^i$	Arbitrary non-negative real numbers
v_t	Any real numbers, $t = 1, 2, \dots, r$
B	Solvability set
$\Psi(v)$	A regular set equal to $\{\delta \in \mathbb{R}^s : g_t(x, \delta) \geq v_t, t = 1, 2, \dots, r\}$
$\tau(J)$	Side of $\Psi(v)$

3.2 Problem statement

Consider the following CCSG with parameters in the cost functions of the players and in the right-hand side of the constraints, with k players. Assume these players have the cost as follows:

- For 1st player, cost function = $\sum_{a_1}^{l_1} \gamma_{a_1}^1 M_{a_1}^1(x, \delta)$.
- For 2nd player, cost function = $\sum_{a_2}^{l_2} \gamma_{a_2}^2 M_{a_2}^2(x, \delta)$.
- For 3rd player, cost function = $\sum_{a_3}^{l_3} \gamma_{a_3}^3 M_{a_3}^3(x, \delta)$.
- Similarly, for the k^{th} player, cost function = $\sum_{a_k}^{l_k} \gamma_{a_k}^k M_{a_k}^k(x, \delta)$.

Thus, for k player, we can write,

$$\sum_{a_1}^{l_1} \gamma_{a_1}^1 M_{a_1}^1(x, \delta), \mathbb{A} , \sum_{a_k}^{l_k} \gamma_{a_k}^k M_{a_k}^k(x, \delta), \tag{1}$$

Subject to

$$h_j(b, \delta) = 0, j = 1, 2, \mathbb{A} , n; \tag{2}$$

$$\delta \in \Psi(v) = \{ \delta \in \mathbb{R}^s : g_t(x, \delta) \geq v_t, t = 1, 2, \mathbb{A} , r \}. \tag{3}$$

Here, $M_{a_i}^i(x, \delta)$, $i = 1, 2, \dots, k$; $a_i = 1, 2, \dots, l_i$ are convex functions on $\mathbb{R}^n \times \mathbb{R}^s$, $g_t(x, \delta)$, $t = 1, 2, \dots, r$ are concave functions on $\mathbb{R}^n \times \mathbb{R}^s$, $\gamma_{a_i}^i$ are arbitrary non-negative real numbers, and v_t are any real numbers. Assume that there exists a function $b = f(\delta)$.

If the function $h_j(b, \delta)$, $j = 1, 2, \dots, n$ differentiable, then the Jacobian $\left| \frac{\partial h_j(b, \delta)}{\partial b_k} \right| \neq 0$, $j; k = 1, 2, \dots, n$, in a neighbourhood of the solution point to equation (3), which is generated by $\delta \in \Psi(v)$. It is noted that the differentiability assumptions are not needed for the functions $M_{a_i}^i(x, \delta)$ and $g_t(x, \delta)$. Furthermore, $\Psi(v)$ is a regular set.

Consequently, the corresponding multi-objective optimisation problem with parameters in both of the objective functions and in the right-hand side of the constraints, is written as

$$\text{Min} \left(\sum_{a_1=1}^{l_1} \gamma_{a_1}^1 \bar{M}_{a_1}^1(\delta), \mathbb{A} , \sum_{a_k=1}^{l_k} \gamma_{a_k}^k \bar{M}_{a_k}^k \right)^T .$$

Subject to

$$\Psi(v) = \{ \delta \in \mathbb{R}^s : \bar{g}_t(\delta) \geq v_t, t = 1, 2, \mathbb{A} , r \}, \tag{4}$$

where

- $\bar{M}_{a_i}^i(\delta), i = 1, 2, \mathbb{A}, k$ are convex functions on \mathbb{R}^s
- $\bar{g}_t(\delta), t = 1, 2, \mathbb{A}, r$ are concave functions on \mathbb{R}^s
- $\bar{M}_{a_i}^i(\delta) = M_{a_i}^i(f(\delta), \delta)$ and $\bar{g}_t(\delta) = g_t(f(\delta), \delta)$.

The problem (4) is assumed to be stable (Chankong and Haimes, 1983). In addition, the problem (4) may be treated by using the weighted sum method (Marler and Arora, 2010) as:

$$\text{Min} \sum_{i=1}^k \lambda_i \left(\sum_{a_i=1}^{l_i} \gamma_{a_i}^i \bar{M}_{a_i}^i(\delta) \right),$$

Subject to

$$\begin{cases} \Psi(v) = \{ \delta \in \mathbb{R}^s : \bar{g}_t(\delta) \geq v_t, t = 1, 2, \mathbb{A}, r \} \\ \sum_{i=1}^k \lambda_i = 1 \\ 0 \leq \lambda_i \leq 1. \end{cases} \quad (5)$$

Let $\mu_i = \lambda_i \sum_{a_i=1}^{l_i} \gamma_{a_i}^i$ and $\bar{M}_i(\delta) = \sum_{a_i=1}^{l_i} \bar{M}_{a_i}^i(\delta)$, for $i = 1, 2, \dots, k; a_i = 1, 2, \dots, l_i$.

Then, problem (5) can be rewritten as in the form

$$\text{Min} \sum_{i=1}^l \mu_i \bar{M}_i(\delta)$$

Subject to:

$$\begin{cases} \Psi(v) = \{ \delta \in \mathbb{R}^s : \bar{g}_t(\delta) \geq v_t, t = 1, 2, \mathbb{A}, r \} \\ l = l_1 + l_2 + \mathbb{A} + l_k \\ \mu_i \geq 0, i = 1, 2, \mathbb{A}, l. \end{cases} \quad (6)$$

It is obvious that the stability of problem (4) implies to the stability of problem (6) for all μ .

Suppose that problem (6) is solvable for $(\bar{\mu}, \bar{v}) \in \mathbb{R}^{s+l} - \{0\} \times \mathbb{R}^l$. Let $\tau(J)$ be the side (sides) of $\Psi(v)$, $J \subset \{1, 2, \dots, r\}$, which is defined by

$$\tau(J) = \{ \delta \in \mathbb{R}^s : \bar{g}_t(\delta) = \bar{v}_t, t \in J, \bar{g}_t(\delta) > \bar{v}_t, t \notin J \}.$$

Definition 7 (Osman, 1977): The stability set of the second kind of problem (6) corresponding to $\tau(J)$ is denoted by $P(\tau(J))$, and is defined by

$$P(\tau(J)) = \{ (\mu, v) \in \mathbb{R}_+^{s+l} \times \mathbb{R}^l : \tau(J) \text{ contains efficient solutions of problem (4)} \}.$$

Definition 8 (Osman, 1977): The solvability set of problem (6) which is denoted by B and is defined as

$$B = \{(\mu, \nu) \in \mathbb{R}_+^{*1} \times \mathbb{R}^l : \text{there exists an efficient solution of problem (4),} \\ \bar{\delta} \in E(\mu, \nu)\}.$$

4 The stability set of the second kind

Let $(\mu, \nu) \in B$ be a corresponding efficient solution $\bar{\delta}$. Since the problem (4) is stable, therefore problem (6) is also stable, and by applying the Karush-Kuhn-Tucker Saddle Point necessary optimality theorem (Bazaraa et al., 1993; Stoer and Witzgall, 1970; Stefanini and Arana-Jiménez, 2019), there exists $\bar{\mu}$ and some $\bar{u} \in \mathbb{R}^r, \bar{u} \geq 0$ such that $(\bar{\delta}, \bar{\mu}, \bar{u})$ solves the Karush-Kuhn-Tucker saddle point problem and $\bar{u}_t (g_t(\bar{\delta}) - \bar{v}_t) = 0$, i.e.,

$$\varphi(\bar{\delta}, \bar{\mu}, u, \bar{v}) \leq \varphi(\bar{\delta}, \bar{\mu}, \bar{u}, \bar{v}) \leq \varphi(\bar{\delta}, \bar{\mu}, \bar{u}, \bar{v}) \text{ for all } \delta \in \mathbb{R}^s, u \in \mathbb{R}^r, u \geq 0.$$

Here

$$\varphi(\delta, \mu, u, \nu) = \sum_{i=1}^l \mu_i \bar{M}_i(\delta) + \sum_{t=1}^r u_t (g_t(\delta) - \nu_t),$$

and

$$\mu_i \geq 0, i = 1, 2, \dots, l.$$

Let us formulate the Karush-Kuhn-Tucker saddle point conditions of problem (6) as:

$$\sum_{i=1}^l \bar{\mu}_i \bar{M}_i(\delta) + \sum_{t=1}^r u_t (g_t(\delta) - \bar{v}_t) \leq \sum_{i=1}^l \bar{\mu}_i M_i(\bar{\delta}) + \sum_{t=1}^r \bar{u}_t (g_t(\bar{\delta}) - \bar{v}_t) \\ \leq \sum_{i=1}^l \mu_i M_i(\delta) + \sum_{t=1}^r \bar{u}_t (g_t(\bar{\delta}) - \bar{v}_t) \leq \sum_{i=1}^l \bar{\mu}_i M_i(\delta) + \sum_{t=1}^r u_t (g_t(\delta) - \bar{v}_t) \\ \text{for all } \delta \in \mathbb{R}^s, u \in \mathbb{R}^r, u \geq 0.$$

Also,

$$g_t(\bar{\delta}) \leq \bar{v}_t, t = 1, 2, \dots, r \\ \bar{u}_t (g_t(\bar{\delta}) - \bar{v}_t) = 0, t = 1, 2, \dots, r \\ \bar{u}_t \geq 0, \mu_i \geq 0, i = 1, 2, \dots, l.$$

It follows that $\bar{\delta} = h(\mu)$ satisfies the Kuhn-Tucker saddle point conditions where

$$h: \mathbb{R}_+^{*1} \rightarrow \mathbb{R}^n.$$

5 Solution method

Based on the discussion mentioned above, the algorithm for obtaining the stability set of the second kind can be summarised as follows.

Inputs:

k = number of players.

Cost functions and constraints for each of the players.

Step 1 Deduce the Karush-Kuhn-Tucker saddle point conditions for problem (6).

Step 2 Construct a function $h : \mathbb{R}_+^k \rightarrow \mathbb{R}^n$ such that $\delta = h(\mu)$ and satisfies the Karush-Kuhn-Tucker saddle point conditions.

Step 3 Determine the stability set of the second kind $P(\tau(J))$ as

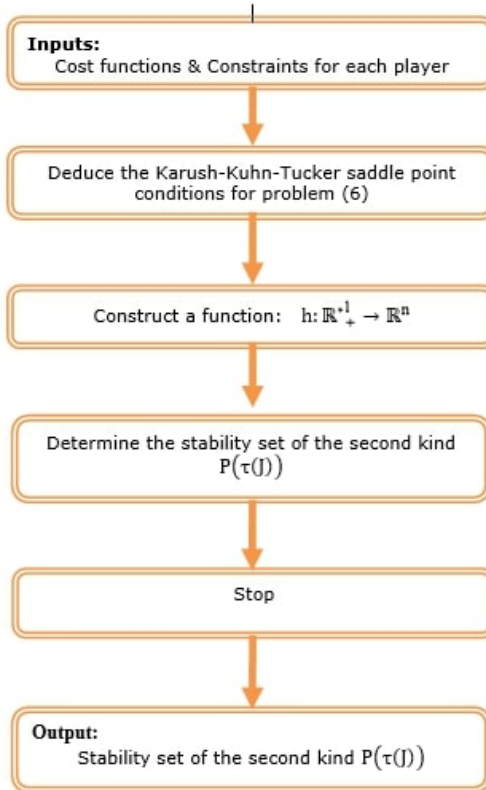
$$P(\tau(J)) = \begin{cases} (\mu, v) \in B : g_t(h(\mu)) = v_t & \text{for } t \in J \\ g_t(h(\mu)) < v_t & \text{for } t \notin J \end{cases}$$

Step 4 Stop.

Outputs: The stability set of the second kind $P(\tau(J))$ for all players.

The flowchart of the proposed algorithm is depicted in Figure 1.

Figure 1 The flowchart of the proposed algorithm (see online version for colours)



6 Numerical example

Consider a two-player game with parameters in the cost functions and in the constraints

$$\gamma^1 (\delta_1 - 1)^2 + \gamma^2 (\delta_2 - 1), \gamma^3 (\delta_1 - 2)^2 + \gamma^4 (\delta_2 - 3)^2,$$

where player I controls $\delta_1 \in \mathbb{R}$ and player II controls $\delta_2 \in \mathbb{R}$ with

$$\begin{cases} \delta_1 + 2 \delta_2 \leq v_1 \\ \delta_2 \leq v_2 \\ \delta_1 \geq 0 \\ \delta \geq 0. \end{cases} \tag{7}$$

Here, $\gamma^i, i = 1, 2, 3, 4$ are arbitrary nonnegative real numbers, $v_i, i = 1, 2$ are any real numbers.

The corresponding multi-objective nonlinear optimisation problem is

$$\text{Min} \left(\gamma^1 (\delta_1 - 1)^2 + \gamma^2 (\delta_2 - 1)^2, \gamma^3 (\delta_1 - 2)^2 + \gamma^4 (\delta_2 - 3)^2 \right)$$

Subject to

$$\Psi(v) = \{ \delta \in \mathbb{R}^2 : \delta_1 + 2 \delta_2 \leq v_1, \delta_2 \leq v_2, \delta_1, \delta_2 \geq 0 \}. \tag{8}$$

This problem is treated by using the weighted sum method (Marler and Arora, 2010) as:

$$\text{Min} \left\{ \lambda_1 \gamma^1 (\delta_1 - 1)^2 + \lambda_1 \gamma^2 (\delta_2 - 1)^2 + \lambda_2 \gamma^3 (\delta_1 - 2)^2 + \lambda_2 \gamma^4 (\delta_2 - 3)^2 \right\}$$

Subject to

$$\begin{cases} \Psi(v) = \{ \delta \in \mathbb{R}^2 : \delta_1 + 2 \delta_2 \leq v_1, \delta_2 \leq v_2, \delta_1, \delta_2 \geq 0 \} \\ \sum_{i=1}^k \lambda_i = 1 \\ 0 \leq \lambda_i \leq 1 \end{cases} \tag{9}$$

Problem (9) is a single objective nonlinear optimisation problem, which can be rewritten in the following form:

$$\text{Min} \left\{ \mu_1 (\delta_1 - 1)^2 + \mu_2 (\delta_2 - 1)^2 + \mu_3 (\delta_1 - 2)^2 + \mu_4 (\delta_2 - 3)^2 \right\}$$

Subject to

$$\Psi(v) = \{ \delta \in \mathbb{R}^2 : \delta_1 + 2 \delta_2 \leq v_1, \delta_2 \leq v_2, \delta_1, \delta_2 \geq 0 \}. \tag{10}$$

Here, the coefficients $\mu_i, i = 1, 2, 3, 4$, are written as:

$$\begin{cases} \mu_1 = \lambda_1 \gamma^1 \\ \mu_2 = \lambda_1 \gamma^2 \\ \mu_3 = \lambda_2 \gamma^3 \\ \mu_4 = \lambda_2 \gamma^4 \end{cases} \tag{11}$$

We have $J \subseteq \{1, 2, 3, 4\}$. Let $J_1 = \emptyset$, i.e., the interior side ($\delta_1 + 2\delta_2 \leq v_1$, $\delta_2 \leq v_2$, $-\delta_1 \leq 0$, $-\delta_2 \leq 0$). The first condition of the Karush-Kuhn-Tucker saddle point is as follows:

$$\begin{aligned} & \mu_1 (\delta_1 - 1)^2 + \mu_2 (\delta_2 - 1)^2 + \mu_3 (\delta_1 - 2)^2 + \mu_4 (\delta_2 - 3)^2 \\ & - \mu_1 (\bar{\delta}_1 - 1)^2 - \mu_2 (\bar{\delta}_2 - 1)^2 - \mu_3 (\bar{\delta}_1 - 2) - \mu_4 (\bar{\delta}_2 - 3)^2 \geq 0. \end{aligned} \quad (12)$$

Consider the following three cases:

6.1 Case 1: when $\mu_1, \mu_3 > 0$ and $\mu_2, \mu_4 > 0$

From the first condition of the Karush-Kuhn-Tucker Saddle Point, we obtain

$$\bar{\delta}_1 = \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} \text{ and } \bar{\delta}_2 = \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}. \quad (13)$$

Hence, $P(\tau(J_1))$ can be determined as:

$$P(\tau(J_1)) = \left\{ (\mu, v) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : v_1 > 0, v_2 > 0, \right. \\ \left. v_1 > \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} + \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, v_2 > \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, \mu_i > 0, i = 1, 2, 3, 4 \right\}. \quad (14)$$

For the side $J_2 = \{1\}$, the corresponding stability set of the second kind is

$$P(\tau(J_2)) = \left\{ (\mu, v) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : v_1 > 0, v_2 > 0, \right. \\ \left. v_1 = \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} + \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, v_2 > \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, \mu_i > 0, i = 1, 2, 3, 4 \right\}. \quad (15)$$

For the side $J_3 = \{2\}$, the corresponding stability set of the second kind is

$$P(\tau(J_3)) = \left\{ (\mu, v) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : v_1 > 0, v_2 > 0, \right. \\ \left. v_1 > \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} + \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, v_2 = \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, \mu_i > 0, i = 1, 2, 3, 4 \right\}. \quad (16)$$

For $J_4 = \{1, 2\}$. Then, in the same way, we get

$$P(\tau(J_4)) = \left\{ (\mu, v) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : v_1 > 0, v_2 > 0, \right. \\ \left. v_1 = \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} + \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, v_2 = \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}, \mu_i > 0, i = 1, 2, 3, 4 \right\}. \quad (17)$$

6.2 Case 2: when $\mu_2, \mu_4 = 0$ and $\mu_1, \mu_3 > 0$

In this case, from the first condition of the Karush-Kuhn-Tucker Saddle Point, we obtain

$$\bar{\delta}_1 = \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3}. \quad (18)$$

Then, for $J_1 = \emptyset$, we have

$$P(\tau(J_1)) = \left\{ \begin{array}{l} (\mu, \nu) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : \nu_1 > 0, \nu_2 > 0, \\ \nu_1 > \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} \mu_i > 0, i = 1, 3 \end{array} \right\}. \tag{19}$$

For $J_2 = \{1\}$, we have

$$P(\tau(J_2)) = \left\{ \begin{array}{l} (\mu, \nu) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : \nu_1 > 0, \nu_2 > 0, \\ \nu_1 = \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} \mu_i > 0, i = 1, 3 \end{array} \right\}. \tag{20}$$

For $J_3 = \{2\}$, we have

$$P(\tau(J_3)) = \left\{ \begin{array}{l} (\mu, \nu) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : \nu_1 > 0, \nu_2 > 0, \\ \nu_1 > \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} \mu_i > 0, i = 1, 3 \end{array} \right\}. \tag{21}$$

For $J_4 = \{1, 2\}$, we have

$$P(\tau(J_4)) = \left\{ \begin{array}{l} (\mu, \nu) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : \nu_1 > 0, \nu_2 > 0, \\ \nu_1 = \frac{\mu_1 + 2\mu_3}{\mu_1 + \mu_3} \mu_i > 0, i = 1, 3 \end{array} \right\}. \tag{22}$$

6.3 Case 3: when $\mu_1, \mu_3 = 0$ and $\mu_2, \mu_4 > 0$

From the first condition of the Karush-Kuhn-Tucker saddle point, we obtain

$$\bar{\delta}_2 = \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4}. \tag{23}$$

For $J_1 = \emptyset$, we have

$$P(\tau(J_1)) = \left\{ \begin{array}{l} (\mu, \nu) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : \nu_1 > 0, \nu_2 > 0, \\ \nu_1 > \frac{2\mu_2 + 6\mu_4}{\mu_2 + \mu_4}, \nu_2 > \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4} \mu_i > 0, i = 2, 4 \end{array} \right\}. \tag{24}$$

For $J_2 = \{1\}$, we have

$$P(\tau(J_2)) = \left\{ \begin{array}{l} (\mu, \nu) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : \nu_1 > 0, \nu_2 > 0, \\ \nu_1 = \frac{2\mu_2 + 6\mu_4}{\mu_2 + \mu_4}, \nu_2 > \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4} \mu_i > 0, i = 2, 4 \end{array} \right\}. \tag{25}$$

For $J_3 = \{2\}$, we have

$$P(\tau(J_3)) = \left\{ \begin{array}{l} (\mu, \nu) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : \nu_1 > 0, \nu_2 > 0, \\ \nu_1 > \frac{2\mu_2 + 6\mu_4}{\mu_2 + \mu_4}, \nu_2 = \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4} \mu_i > 0, i = 2, 4 \end{array} \right\}. \tag{26}$$

For $J_4 = \{1, 2\}$, we have

$$P(\tau(J_4)) = \left\{ \begin{array}{l} (\mu, v) \in \mathbb{R}_+^{*4} \times \mathbb{R}^2 : v_1 > 0, v_2 > 0, \\ v_1 = \frac{2\mu_2 + 6\mu_4}{\mu_2 + \mu_4}, v_2 = \frac{\mu_2 + 3\mu_4}{\mu_2 + \mu_4} \mu_i > 0, i = 2, 4 \end{array} \right\}. \quad (27)$$

7 Concluding remarks and future research

The theory of games is one of important tools for handling the multi-criteria decision making problems, which arise in the conflict of situations between intelligent players in order to choose the best strategy. The proposed research paper is based on the approach of Karush-Kuhn-Tucker conditions. In this research article, the CCSG are investigated with the parameters in cost functions of the players and in the right-hand side of the constraints. The algorithm as well as the flowchart of the proposed approach has been presented. The main objective of the proposed paper is to determine the stability set of the second kind without differentiability. In the last, a numerical example is presented to demonstrate the efficiency of the proposed method. In this numerical example, the stability set of the second kind without differentiability is determined. Overall, the proposed approach is a novel approach, which can be easily applied to determine the results.

There are some scopes for future research. One may consider the multi-objective optimisation in linguistic environment by characterising with fuzzy random numbers. In these situations, the fuzzy game theory model should be used to solve this optimisation problem.

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