On the time-dependent solution of fluid models driven by an *M*/*M*/1 queue using a probabilistic approach

S. Dharmaraja* and Shruti Kapoor

Department of Mathematics,
Indian Institute of Technology Delhi,
Hauz Khas, New Delhi 110016, India
Email: dharmar@maths.iitd.ac.in
Email: shruti85kapoor@gmail.com
*Corresponding author

Abstract: More study has been performed on the steady state analysis of fluid models. However, the transient analysis of fluid models has not been extensively carried due to complexity of the problem. This is because the system of conservation laws for which no closed form solution is available. The motivation behind this paper is to provide a new methodology to find transient distribution of buffer content for fluid models. In this paper, the time-dependent solution of a fluid model driven by an $M/M/1$ queue is derived using a simple probability approach. Finally, numerical results are illustration for the proposed approach.

Keywords: fluid models; $M/M/1$ queue; time-dependent.

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Biographical notes: S. Dharmaraja received his PhD in Mathematics from Indian Institute of Technology Madras, India in 1999. He is currently a Professor in the Department of Mathematics and joint faculty of Bharti School of Telecommunication Technology and Management at the Indian Institute of Technology Delhi, India. His current research interests include queuing models, and performance issues of wireless networks and dependability analysis of communication systems.

Shruti Kapoor is currently pursuing his PhD in Mathematics from Indian Institute of Technology Delhi, India. She is a Postgraduate in Mathematics from the Indian Institute of Technology Delhi, India. Her current research interests include queuing theory, fluid queues and Markov modelling.

1 Introduction

A fluid model is a probabilistic model used to describe the amount of fluid in a reservoir or dam, of infinite or finite capacity. The rate at which the fluid flows into the system can be analysed as being controlled by an operator which is a Markov process and is called the background process. For instance, fluid models have been widely accepted as appropriate models for recent telecommunication and manufacturing systems. This modelling approach ignores the discrete nature of the real information flow and treats it as a continuous stream. Fluid queue means that, the queue obtained from the fluid model.

Steady state analysis of the buffer content distribution has been extensively performed by various authors and different methodologies have been used to obtain the exact solution in many cases. Van Doorn and Scheinhardt (1997) gave the methodology for finding the buffer content distribution using orthogonal polynomials. Parthasarathy et al. (2002) analysed the steady state behaviour of a fluid model driven by an $M/M/1$ queue using continued fraction approach. Tardelli (2017) used recursive backward scheme for the solution. Ramaswami (1999) described the fluid models using matrix-analytic methods and provided efficient algorithm for calculating the stationary distribution. Malhotra et al. (2009) studied congestion control mechanism using fluid models and obtained the stationary distribution of the buffer occupancy, the distribution of buffer delay, and the throughput.

Steady state behaviour gives us important information of a system in long run, although to study the dynamical nature of a system, transient analysis is of critical value. Various methodologies have been studied in literature to obtain the transient buffer content distribution of fluid queues. Sericola (1998) found the transient solution of fluid queues driven by a Markov process using recurrence relations.

Further, fluid queues driven by an $M/M/1$ has already been studied in literature. Sericola et al. (2005) gave the transient solution of fluid queues driven by an $M/M/1$ is obtained via continued fractions. Fluid models driven by an $M/M/1/N$ queue was analysed by Parthasarathy and Lenin (2000), and closed form solution was obtained. Recently, Shruti et al. (2005) obtained the transient solution of fluid queue driven by a birth death process with specific rational rates and absorption. de Souza e Silva (1995) gave a methodology to find the transient distribution of cumulative reward based on probability. In this paper, the transient solution is presented based on probability concepts and obtained the solution in the form of an infinite series, where individual terms are defined using recurrence relations. This methodology of finding the transient solution is similar to finding the transient distribution of cumulative reward by de Souza e Silva (1995).

The rest of the paper is organised as follows. In the next section, the fluid model and the notation used are presented. In Section 3, the time-dependent solution of the buffer content of the fluid queue is obtained. In Section 4, numerical illustrations of these results is presented. Pointers to further research and conclusions are given in Section 5.

2 Model description

Consider a fluid model driven by an $M/M/1$ queue. In this model, the background process is a continuous time Markov chain. Let $\{X(t), t \geq 0\}$ be the underlying stochastic process of an $M/M/1$ queue, where $X(t)$ is the state of the system at time t and takes values in $S = \{0, 1, 2, ...\}$. Let λ and μ denote the mean arrival and service rates respectively. When the background process is in state $i \in S \setminus \{0\}$, a fluid commodity gets accumulated in an infinite capacity buffer with a constant rate $r > 0$. Whenever the background process is in state 0, the fluid flows out of the buffer with a constant rate $r_0 < 0$. Let $C(t)$ denote the buffer content at time t. The stochastic process $\{(X(t), C(t)), t \geq 0\}$ becomes a two-dimensional Markov process. Let

$$F_j(t, x) = P[X(t) = j, C(t) \leq x], \quad t, x \geq 0;\ j \in S$$

denote the two-dimensional distribution function of the Markov process $\{(X(t), C(t)), t \geq 0\}$. The Kolmogorov forward equations governing this process are given by Van Doorn et al. (1988):

$$\frac{\partial F_0(t,x)}{\partial t} + r_0 \frac{\partial F_0(t,x)}{\partial x} = -\lambda F_0(t,x) + \mu F_1(t,x) \tag{1}$$

$$\begin{aligned} \frac{\partial F_j(t,x)}{\partial t} + r \frac{\partial F_j(t,x)}{\partial x} &= \lambda F_{j-1}(t,x) - (\lambda + \mu) F_j(t,x) \\ &+ \mu F_{j+1}(t,x), \quad j = 1, 2, ... \end{aligned} \tag{2}$$

The initial conditions are given by

$$F_0(0, x) = 1,\ F_j(0, x) = 0 \quad \text{for } j = 1, 2, 3, \ldots$$

and boundary conditions are given by

$$F_0(t, 0) = q_0(t) \text{ and } F_j(t, 0) = 0 \quad \text{for } j = 1, 2, 3, \ldots.$$

Note that, whenever the net input rate of fluid flow into the buffer is positive, the buffer cannot stay empty.

3 Time-dependent solution using probabilistic approach

The motivation behind this paper is to provide a new methodology to find transient distribution of buffer content for fluid models. Though this model has been well studied by various researchers in the literature, the methodology proposed in this paper is unique and simple in nature.

The interest lies to obtain $P[C(t) > x]$. Let Q be the infinitesimal generator matrix of continuous time Markov chain $\{X(t), t \geq 0\}$. Let

$$Z = \{Z_n : n = 0, 1, \ldots\}$$

be a discrete time Markov chain with state space S and transition probability matrix $P = I + \frac{Q}{\lambda+\mu}$ and $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda + \mu$, that is independent of Z. Then $X(t) = Z_{N(t)}$ for $t \geq 0$. Assume that the background process has been uniformised and thus have n transitions during the period (0, t), i.e., $N(t) = n$, at times

$$0 < \tau_1 < \tau_2 < ... < \tau_n < t.$$

These events split (0, t) into n + 1 intervals with lengths

$$Y_1 = \tau_1, Y_2 = \tau_2 - \tau_1, ..., Y_{n+1} = t - \tau_n.$$

Thus, each interval is associated with a net rate whose value is based on the state of the process during the interval.

The state space S is partitioned into two subsets B_0 and B_1 such that B_0 = $\{0\}$ and B_1 = $\{1, 2, ...\}$. Define $V_n^0 = \sum_{i=0}^{i=n} 1_{Z_i \in B_0}$ and $V_n^1 = \sum_{i=0}^{i=n} 1_{Z_i \in B_1}$ where 1_A is the indicator function.

Define

$$V_n = (V_n^0, V_n^1).$$

Let k_1 and $k_2 = n + 1 - k_1$ denote the number of intervals associated with rate r_0 and r respectively. Note that, $k_1 \geq 1$ since the system starts in state 0 at time 0. Refer (k_1, k_2) as a partition of n + 1. Thus, conditioning on the number of transitions n and $V_n = (V_n^0, V_n^1) = (k_1, k_2) = \mathbf{k}$, obtain $P[C(t) > x]$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{k_1+k_2=n+1} P[N(t) = n, V_n = \mathbf{k}] P[C(t) > x \mid N(t) = n, V_n = \mathbf{k}] \\
&= \sum_{n=0}^{\infty} \sum_{k_1+k_2=n+1} P[N(t) = n] P[V_n = \mathbf{k} \mid N(t) = n] \\
&\times P[C(t) > x \mid N(t) = n, V_n = \mathbf{k}] \\
&= \sum_{n=0}^{\infty} e^{-(\lambda+\mu)t} \frac{((\lambda+\mu)t)^n}{n!} \sum_{k_2=0}^{n} G[n, (k_1, k_2)] M(t, x, n, k_2) \qquad (3)
\end{aligned}$$

where

$$M(t, x, n, k_2) = P[C(t) > x \mid N(t) = n, V_n^1 = k_2] \qquad (4)$$

and

$$G[n, (k_1, k_2)] = P[V_n = \mathbf{k} \mid N(t) = n].$$

Further,

$$G[n, (k_1, k_2)] = \sum_{i \in S} G_i[n, (k_1, k_2)]. \qquad (5)$$

Here $G_i[n, (k_1, k_2)]$ is the probability of partition **k** given that the number of transitions are n and the state visited after the last transition is i. If i and j are the states visited after the last $(n-1)^{\text{th}}$ and n^{th} transitions, then (k_1, k_2) is equal to the previous partition +1 at the entry corresponding to the net rate associated with state j. Let $p = \frac{\lambda}{\lambda+\mu}$ and $q = 1 - p$. Since, for $i \geq 1$, the only possibilities for j are $i-1$ and $i+1$, conditioning on the state visited after the $(n-1)^{\text{th}}$ transition,

$$G_0[n, (k_1, k_2)] = G_1[n-1, (k_1 - 1, k_2)] q \qquad (6)$$

and for $i \in S \setminus \{0\}$

$$G_i[n,(k_1,k_2)] = G_{i-1}[n-1,(k_1,k_2-1)]p + G_{i+1}[n-1,(k_1,k_2-1)]q. \tag{7}$$

The recursive function $G_i[n,(k_1,k_2)]$ satisfies the initial conditions

$$G_0[0,(1,0)] = \pi_0^{(0)} = 1$$
$$G_i[0,(0,1)] = \pi_i^{(0)} = 0 \text{ for } i \in S \setminus \{0\}.$$

Now, compute $M(t,x,n,k_2)$. Let $U_1, U_2, ..., U_n$ be uniformly distributed iid random variables in (0, 1) and $U_{(1)}, U_{(2)}, ..., U_{(n)}$ be their order statistics with $U_{(0)} = 0$ and $U_{(n+1)} = 1$. Then, τ_i, the time of the i^{th} transition has the same distribution as $tU_{(i)}$. Thus

$$Y_1 \equiv tU_{(1)}, Y_2 \equiv t(U_{(2)} - U_{(1)}), ..., Y_{n+1} \equiv t(1 - U_{(n)}).$$

Since Y_i's are exchangeable random variables, and also the background process is an $M/M/1$, the time spent in any state follows exponential property. Thus, rearranging the intervals, letting the first k_1 intervals to be associated with the net rate r_0 and the next k_2 intervals associated with the net rate r. Now, the event $\{C(t) > x\}$ given $\{N(t) = n, V_n^1 = k_2\}$ can be written as $\{C(t) > x \mid N(t) = n, V_n^1 = k_2\}$

$$\begin{aligned}
&= \{C(t) > x \mid N(t) = n, V_n^1 = k_2\} \\
&= \{C(t) > x \mid N(t) = n, V_n^0 = k_1\} \\
&= \{r_0(Y_1 + ... + Y_{k_1}) + r(Y_{k_1+1} + ... + Y_{n+1}) > x\} \\
&= \{r_0(tU_{(1)} + t(U_{(2)} - U_{(1)}) + ... + t(U_{(k_1)} - U_{(k_1-1)})) \\
&\quad + r(t(U_{(k_1+1)} - U_{k_1}) + ... + t(U_{(n+1)} - U_{(n)})) > x\} \\
&= \{t(r_0 - r)U_{(k_1)} + rt > x\}.
\end{aligned}$$

Thus, for $x \in [0, rt)$

$$\begin{aligned}
P[C(t) > x \mid N(t) = n, V_n^1 = k_2] &= P\left[(t(r_0 - r)U_{(k_1)} + rt) > x\right] \\
&= P\left[U_{(k_1)} < \frac{x - rt}{t(r_0 - r)}\right].
\end{aligned} \tag{8}$$

It is known that if $X_1, X_2, ..., X_n$ are $U(0,\,1)$ iid random variables, then the s^{th} order statistics $X_{(s)}$ follows $Beta(s, n+1-s)$. Therefore, the RHS of equation (8) is the cumulative distribution function of $Beta(k_1, n+1-k_1)$ evaluated at $\frac{x-rt}{t(r_0-r)}$. Thus,

$$M[t,x,n,k_2] = \sum_{j=0}^{k_1} \binom{n}{j} \left(\frac{x - rt}{t(r_0 - r)}\right)^j \left(1 - \frac{x - rt}{t(r_0 - r)}\right)^{n-j}. \tag{9}$$

Thus, the solution is obtained as, for $t \geq 0$ and $x \in [0, rt)$,

$$P[C(t) > x] = \sum_{n=0}^{\infty} e^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \sum_{k_2=0}^{n} G[n,(k_1,k_2)]M[t,x,n,k_2] \tag{10}$$

where $M(t,x,n,k_2)$ is given in equation (9) and $G[n,(k_1,k_2)]$ is given in equations (5), (6) and (7).

Further, the value of $q_0(t)$ can be obtained from equation (8) by letting $x = 0$ and state of the background process as $i = 0$ and using the recursive relation described in equation (6) as follows:

$$q_0(t) = P(C(t) > 0) = \sum_{n=0}^{\infty} \left[e^{-(\lambda+\mu)t} \frac{-[(\lambda+\mu)t]^n}{n!} \sum_{k_2=0}^{n} G[n,(k_1,k_2)] \times M[t,0,n,k_2] \right]. \quad (11)$$

Figure 1 Buffer content distribution (see online version for colours)

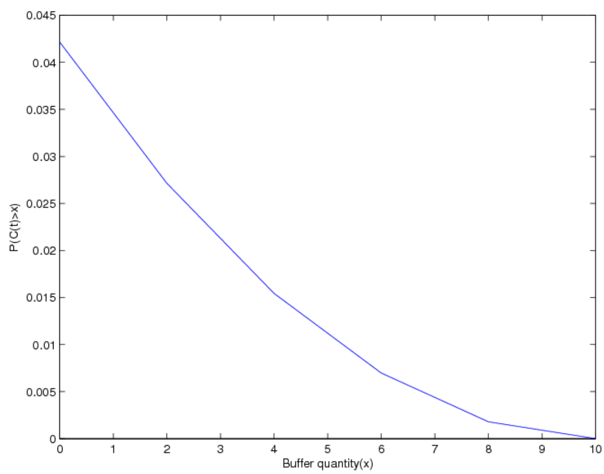
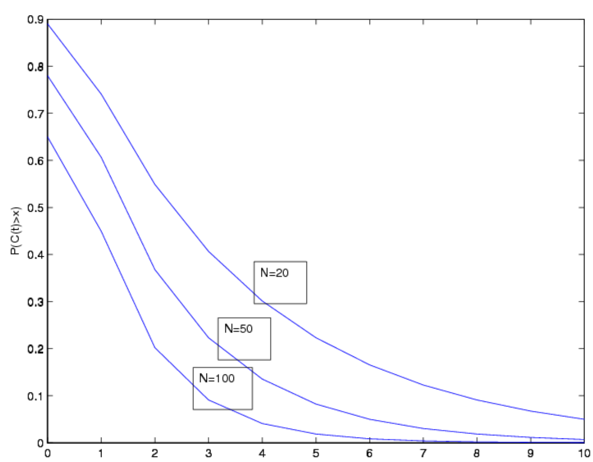


Figure 2 Buffer content distribution for various values of N (see online version for colours)



4 Numerical illustration

Analysis of fluid models driven by an $M/M/1$ queues is executed by various authors. But the result proven in this paper follows basic probabilistic approach and hence can be implemented numerically in a simple manner. In Figure 1, the distribution of buffer content for a fixed value of t and varying x is shown taking $\lambda = 1$ and $\mu = 2$.

Figure 2 shows the buffer content distribution of a fluid model driven by an $M/M/1/N$, for different values of N. The parameter values are taken to be same as for the infinite scenario. From the graph, it can be seen that as N increases the above model converges to the graph shown by fluid model driven by an $M/M/1$ queue.

5 Conclusions and future work

In this paper, a new methodology of finding the time-dependent probabilities is preseneted for a fluid model driven by an $M/M/1$ queue using basic probability. The objective of this paper is to present the solution in a simplified manner. The solution obtained is in terms of recurrence relations, and is further presented as a closed form expression. The above described methodology can be used to obtain the buffer content distribution function of other fluid models with different background other than $M/M/1$.

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