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Hesitant fuzzy sets with non-uniform linguistic terms: an application in multi-attribute decision making

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Abstract: The paper introduces a novel methodology for solving multi-attribute decision-making problems under hesitant fuzzy linguistic environment. It includes non-uniform, non-regular, or arbitrarily defined linguistic terms in hesitant fuzzy linguistic term set. The proposed methodology takes both normal and non-normal fuzzy numbers to represent linguistic terms in HFLTS. The combined approach of the concept of existence in ranking of fuzzy sets, α -cuts of fuzzy numbers, and ordering relations for hesitant fuzzy sets is used to value each alternative numerically. Binary integer programming is used to verify the consistency level of pairwise comparison matrix conforming to specified linguistic preferences as per the decision maker's expressions. The pairwise comparison matrices are aggregated over attributes to obtain the aggregated pairwise comparison matrix. The derived aggregated matrix calculates dominance/non-dominance levels of alternatives and selects best alternative. The proposed method is demonstrated with a numerical example, compared with similar methods and the advantages are highlighted.

Keywords: non-uniform linguistic term set; hesitant fuzzy linguistic term set; fuzzy number comparison; binary integer programming; aggregated pairwise comparison matrix; dominance degree.

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credit, he has publications in the journals such as *Decision Support Systems*, *Expert Systems with Applications*, *Computers & Operations Research*, *International Journal of Electronic Commerce*, *Fuzzy Sets and Systems*, *International Journal of Production Research*, etc.

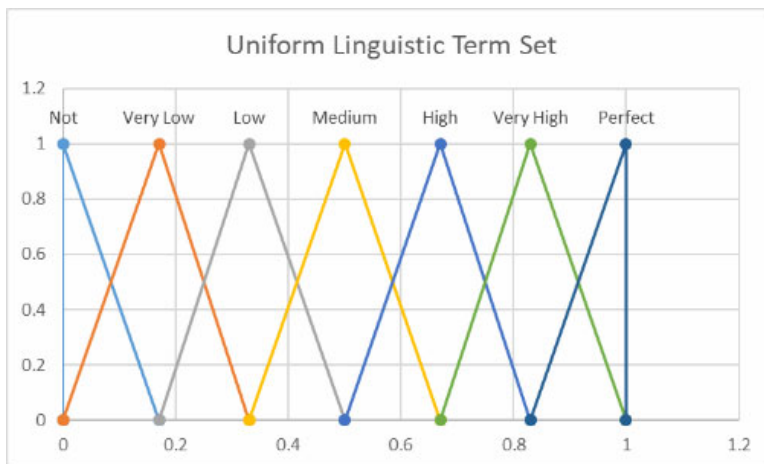
1 Introduction

The decision-making problems, in general, revolve around the identification and determination of the best decision alternative among several candidate alternatives. In real-world decisions, the selection of an alternative is difficult, mainly due to the impreciseness or vagueness inherited in the valuation of the alternatives. The fuzzy set has been successfully applied to handle such types of vague, imprecise, or uncertain information (Zimmermann, 1978; Kaufman, 1975; Kosko, 1986; Rao et al., 1988). However, the theory of fuzzy sets as a methodology to handle the impreciseness or vagueness is unsuccessful when the imprecise data arises from two or more information sources simultaneously. This difficulty is multiplied when an alternative need to be assessed over the multiple attributes and the attribute values are judged from multiple information sources. In real-world decision problems, often the data from these information sources are unclear, imprecise, and expressed in linguistic terms. Because of lack of necessary information, needed expertise, or adequate knowledge, a decision maker (DM) confuses and inclines to rely on multiple information in alternative valuations. The theory of hesitant fuzzy set (HFS) is an appropriate tool to undertake multiple information and express the uncertainty involved through its many membership values in the decision-making process. This is because by definition an element of HFS has several possible membership values and one can emulate it to multiple information bases. HFS has attracted the increasing attention of researchers and industry practitioners as the state of hesitation is very common in real-world decision-making situations (Torra, 2010; Lan et al., 2017; Li et al., 2015).

The attributes, in general, in real-world multi-attribute decision making (MADM) are more conveniently expressed in day-to-day linguistic terms rather than the exact numeric ones. For example, a buyer easily expresses his/her desire for the mileage of a car linguistically as ‘good’, ‘average’, or ‘above average’, etc. To model this situation, the concept of hesitant fuzzy linguistic term sets (HFLTS) is used in many papers in the existing literature (Torra, 2010; Wei et al., 2014; Chen and Hong, 2014). For MADM problems with linguistic information in its attribute values, the vital issue is to combine the attribute values that are defined in different dimensions. Several aggregation operators are available to aggregate the linguistically defined attribute values in MADM as long as the attribute valuations are in a single linguistic term (Yager, 1995, 2003, 2004; Wei et al., 2014; Rodriguez et al., 2013; Liao et al., 2020; Rodriguez et al., 2012; Samanta and Basu, 2020). The aggregation process becomes difficult when the DM hesitates and provides attribute valuations in multiple linguistic terms. The problem lies with the effective aggregation of HFLTS containing non-uniform linguistic terms across the attributes in MADM.

In the literature, in almost all MADM problems with attribute values as HFLTS, the linguistic terms are regular or uniform from a pre-defined linguistic term sets and the semantics of the linguistic terms are also pre-specified. Existing procedures of fuzzy linguistic approach require to subjectively choose the linguistic descriptors from the pre-defined linguistic term set and their semantics (Herrera and Martinez, 2000) to represent DM's opinion. In the case of HFLTS, it is a prerequisite that the attribute values as hesitant fuzzy element (HFE) are an ordered finite set of consecutive linguistic terms from the pre-defined linguistic term set. When a DM obtains the information from multiple sources, it is unreasonable to expect that the information will follow the above restrictions. Moreover, with the rapid development of the economy and the digitisation of society, uncertainty and fuzziness are paramount in decision-making situations, and the techniques based on hesitant fuzzy linguistic terms with uniform pre-defined linguistic terms is not competent enough for handling the varieties of uncertainties and confusing characteristics of the DM that persist in real-life problems. Further, restriction of linguistic terms to a pre-defined set with specified semantics may not concur to the true opinion of the DM in alternative evaluation. For instance, a buyer may desire to purchase a car with a price *somewhat low*. The linguistic expression *somewhat low* may not match any of the linguistic terms given in S_1 . Approximating a linguistic term from S_1 to a specific linguistic expression may not reflect the true opinion of the buyer and may lead to loss of information. Figure 1 shows the uniform pre-defined linguistic term set with pre-specified semantics.

Figure 1 Linguistic term set (see online version for colours)



The linguistic term set in Figure 1 corresponds to the terms:

$$S_1 = \{S_0 (Not), S_1 (Very low), S_2 (Low), S_3 (Medium), S_4 (High), S_5 (Very high), S_6 (Perfect)\}$$

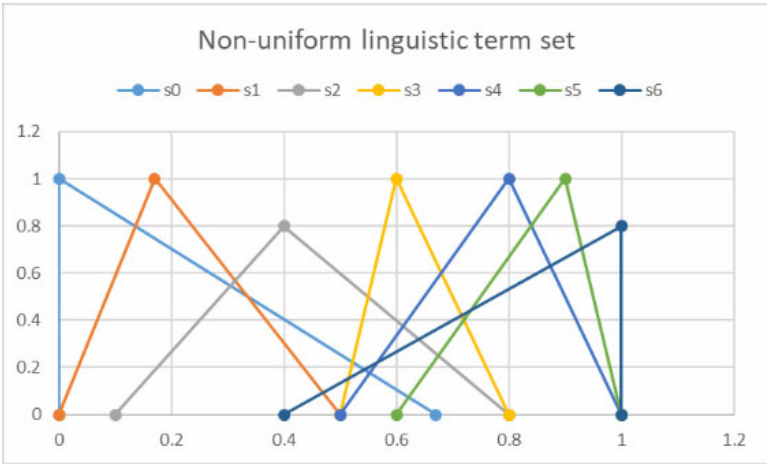
The semantics of the linguistic terms in S_1 are as shown below:

S_0 (not)	(0, 0, 0.17)
S_1 (very low)	(0, 0.17, 0.33)
S_2 (low)	(0.17, 0.33, 0.5)
S_3 (medium)	(0.33, 0.5, 0.67)
S_4 (high)	(0.5, 0.67, 0.83)
S_5 (very high)	(0.67, 0.83, 1)
S_6 (perfect)	(0.83, 1, 1)

In the above, the linguistic terms are uniformly defined and represented as fuzzy numbers. For example, the fuzzy number representation of semantics of the linguistic term ‘medium’ is (0.33, 0.5, 0.67). From Figure 1, the interpretation of the linguistic term ‘medium’ as: full satisfaction level at point 0.5 and the satisfaction level gradually decreases when it deviates from 0.5 and becomes zero at points 0.33 and 0.67. The membership value of the fuzzy numbers is same as the satisfaction level.

Our work addresses the issue mentioned above by realising the expert’s feelings, for instance, *somewhat low* and constructs a fitting linguistic term with matching semantics, say $s_1 \in S_2$, as shown in Figure 2. The construction of linguistic terms using an expert’s thoughts may result in a non-uniform linguistic term set. In the non-uniform set, the linguistic terms are generated from experts’ statements and are not restricted to choose from a pre-defined linguistic term set as prevalent in the existing literature. Even, non-normal fuzzy numbers can be counted as a linguistic term in the non-uniform linguistic term set. The linguistic terms s_2 and s_6 as non-normal fuzzy numbers with heights of 0.8 are shown in Figure 2. The main advantage of using HFLTS with non-uniform linguistic terms is to build an appropriate linguistic term set representing the alternative valuations in real terms.

Figure 2 Non-uniform linguistic term set (see online version for colours)



The non-uniform linguistic term set S_2 is shown in Figure 2.

$$S_2 = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$$

The semantics of the linguistic terms in S_2 are as follows:

s_0	(0.0, 0.0, 0.67)
s_1	(0.0, 0.17, 0.5)
s_2	(0.8, (0.1, 0.4, 0.8))
s_3	(0.5, 0.6, 0.8)
s_4	(0.5, 0.8, 1.0)
s_5	(0.6, 0.9, 1.0)
s_6	(0.8, (0.4, 1.0, 1.0))

One of the prime motives of the non-uniform linguistic term set is to get rid of unjustifiably large or unduly small deviations from the genuine assessment of the alternatives. This desirable characteristic of HFLTS with non-uniform linguistic terms generalises the decision-making process, making it fit to be used in many real fields of decision-making environments.

To rank the alternatives in MADM, our paper compares the alternatives pairwise, as the decision-making methods based on this principle are a popular tool in MADM problems (Chen and Hong, 2014; Farhadinia, 2016; Lan et al., 2017; Li et al., 2015; Sellak et al., 2018; Rodriguez et al., 2013). As the alternatives are assessed in HFLTS, the comparison requires every distinct linguistic term of an alternative to being individually compared with the linguistic terms of the other alternatives in each attribute. As the linguistic terms are in the form of fuzzy numbers, we have used the concept of existence (Chang and Lee, 1994) to evaluate the linguistic terms numerically. The numerical evaluation of fuzzy number $(s(x), \mu_s(x))$, is derived in two parts; the left part ' s_L ' and the right part ' s_R '. For $\alpha \in [0, 1]$, the minimum value of s_α is made equivalent to ' s_L ' and the maximum of s_α to ' s_R ', where s_α is the α -cut of the fuzzy number $(s(x), \mu_s(x))$. Thus we have,

$$s_L = \text{Min}(x | \mu_s(x) \geq \alpha) \text{ and } s_R = \text{Max}(x | \mu_x(x) \geq \alpha) \quad \alpha \in [0, 1]$$

Following the procedure (Lee and Li, 1993), we can derive the preference degree of the linguistic term s_i over s_j by taking the difference of evaluation of $d_{ij} = \text{Max}(0, (\mu_{s_{iR}}^{-1}(w) - \mu_{s_{jL}}^{-1}(w)))$ ($s_i \in A_1, s_j \in A_2, \mu_{s_{iR}}$ and $\mu_{s_{jL}}$) are the right part membership values of s_i and left part of s_j). Motivated by the work given in Lan et al. (2017), the ordering relation amongst HFS is used to obtain the complete pairwise comparison matrix of the alternatives in numeric terms in each attribute.

To validate the preference levels amongst the alternatives in the pairwise comparison matrix, it is essential to verify its level of consistency. In most cases, it is too difficult to obtain a comparison matrix without any inconsistency. As the pairwise comparisons are in the form of additive preferences, our focus is on additive consistency measurement. As the original preference values are in linguistic terms, it is essential to match the derived numerical preferences of the pairwise comparison matrix to their linguistic counterparts.

The matching procedure of numerical comparison values to their respective linguistic equivalents is defined as follows:

Let the alternative A_i is preferred over A_j with degree of preference $P(A_i \gg A_j) = \alpha_{ij} \in [0, 1]$. If α_{ij} falls in the domain of the linguistic terms, say s_p (s_{pL}, s_{pM}, s_{pR}), s_q (s_{qL}, s_{qM}, s_{qR}), and s_k (s_{kL}, s_{kM}, s_{kR}), i.e., $\alpha_{ij} \in [s_{pL}, s_{pR}]$, $\alpha_{ij} \in [s_{qL}, s_{qR}]$, and $\alpha_{ij} \in [s_{kL}, s_{kR}]$, then it is necessary to select a single linguistic term and a specific preference value (equivalent to α_{ij}) from its domain depending upon its impact to the maximum consistency. Additionally, if for some other alternative pairs A_s and A_t ($s, t \neq i, j$), the pairwise preference level is the same as that of A_i and A_j , i.e., $P(A_s \gg A_t) = \alpha_{ij}$, this may not necessarily match to the same linguistic term and the domain interval therein as in the case of $P(A_i \gg A_j)$. This is because the preference level α_{ij} lies in a range of values in intervals corresponding to the domains of the linguistic terms of s_p , s_q , and s_k . The linguistic term and the specific value in the domain interval of the said linguistic term that contributes to maximum consistency is chosen as the equivalent value of α_{ij} .

To our knowledge, we did not find any methodology which verifies the level of consistency of the pairwise preference of the alternatives assessed through HFLTS with non-uniform linguistic terms. In the light of the work given in Li et al. (2019), our paper introduces a binary integer programming (BIP) model that not only measures the level of consistency of the pairwise comparison matrix but also conforms each comparison indices to the linguistic terms and the preference values in the domain interval that are consistent to the behaviour of the DM.

The MCDM problems, in general, determine the ranking of the alternatives and select the best amongst the candidate alternatives. The priority vector representing the preference of alternatives and its determination are available in Xia and Xu (2014) and Wang and Parkan (2005). The fuzzy priority weights of pairwise comparison matrices based on logarithmic calculus are given in Hecke (2021). The majority of these methods generate the weights in crisp numerical terms. Because of uncertainty in real-world decision problems and the involvement of human judgements, incorporating the exact numerical weights may lead to unreliable and inconsistent results in preference relations. Following the procedure in Xia and Xu (2014), our work generates interval weights as priority vectors of the alternatives from the given pairwise comparison matrix to make it more parallel to the real-world decision-making problems.

To select the best alternative or rank them in MADM, it is necessary to combine the priority vectors of alternatives over the attributes. We put forward a distance-based aggregation approach to combine the priority vectors of alternatives that are in interval weights in the line of the procedure given in Xu et al. (2014). This aggregation results in determining the attribute weights. The weights of the attributes are further used in the pairwise comparison matrices to determine the aggregated comparison matrix. The aggregated pairwise comparison matrix aids in identifying the non-dominance degree of each alternative in MADM. The alternative with the least dominance is selected as the best alternative. The other alternatives are ranked accordingly.

There are several methods available in the literature that deals with MADM under a hesitant fuzzy linguistic environment (Yavuz et al., 2015; Tang et al., 2019; Zhang et al., 2018; Rodriguez et al., 2012). In Farhadinia (2016), the concept of entropy is used in HFLTS to determine the weights of the attributes. Here, the necessary operations on HFLTS are based on an index of the linguistic terms, and the semantics of the terms are not taken into consideration. In Yavuz et al. (2015), hierarchical hesitant fuzzy linguistic values are used for vehicle selection problem. This paper uses a pre-defined set of

linguistic terms for alternative vehicle evaluation. Fuzzy linguistic modelling based on discrete fuzzy numbers is used to manage HFLTS in Riera et al. (2015). Here, the semantics of the fuzzy numbers representing the linguistic terms are taken subjectively. In the work given in Liao et al. (2020), the hesitant degree-based correlation measures for HFLTS are used for solving MCDM. The concept of HFS and HFLTS are both used for the solution of MCDM in Chen and Hong (2014). An outranking approach of MCDM based on HFLTS is given in Wang et al. (2015). The envelope of HFS to define the distance measure amongst the HFLTS and to outrank the alternatives subsequently is investigated in Wang et al. (2014). Multi-criteria relational clustering along with HFLTS to outrank the alternatives in MCDM is given in Sellak et al. (2018). In Rodriguez et al. (2013), a group decision making problem is handled considering HFLTS as attribute values. The model uses comparative linguistic expressions (CLE) to solve group decision making problem. In Durand and Trucl (2018), the linguistic expressions are assigned with weights and the set of linguistic terms are from a pre-defined linguistic term set. In Liu et al. (2019), a very interesting and useful methodology to deal with the uncertainties and fuzziness in hesitant fuzzy linguistic terms using type-2 fuzzy sets is introduced. The limitation of the work is the use of pre-specified linguistic terms. An MCDM based on the hesitant fuzzy linguistic ORESTE method is given in Liao et al. (2018) for the supplier selection problem. This method has taken the factors as a preference, indifference, and incomparability to solve the MCDM problem. The concept of TODIM, PROMETHE, and HFN are also used in Liao et al. (2018). The work given in Halouani (2021) deals with multi criteria group decision-making methodology based on hesitant fuzzy linguistic terms. This methodology uses mentality parameter for interval-valued hesitant fuzzy linguistic term sets before prescribing the preference level of the alternatives.

In almost all the works mentioned above, the linguistic terms are either pre-specified or their indices are used for comparison of linguistic terms. The restrictions imposed on HFLTS with the pre-specified linguistic term set may not indicate the true opinion of the DM in alternative assessments especially when the DM receives information from multiple sources and hesitates to concentrate on a single-pointed decision. Our paper addresses these issues by gathering non-uniform set of linguistic terms from different information sources and ranking them by using the concepts of existence methodology of ranking fuzzy numbers (Chang and Lee, 1994).

1.1 Research challenges and gaps

It has been reasonably cited in many research papers that the assessment of alternatives or objects in linguistic terms is quite acceptable and more preferred in real-world decisions. Linguistic way of expressions is more parallel to human thinking and closely related to the human thought processes and reasoning. Several research works are available that deals with fuzzy linguistic terms (Chen and Hong, 2014; Durand and Trucl, 2018; Farhadinia, 2016; Liao et al., 2020). However, some issues are still unanswered in this context.

- a In almost all existing research works, the linguistic terms are uniform, pre-defined with pre-specified semantics in each linguistic term. Further, the HFLTS are restricted to an ordered finite subset of consecutive linguistic terms. In majority of the procedures, it is required to choose such pre-defined linguistic terms subjectively

to represent the decision-maker's preferences. Limiting the linguistic terms in HFLTS in this manner may not represent the decision maker's judgements in a true sense. The restrictions on pre-defined linguistic term set with pre-specified semantics does not support the opinions when the opined information is from multiple sources. Therefore, the real challenge lies in not only identifying the linguistic terms but also evaluating them numerically to represent the genuine cognitive thinking of the human judgements.

- b The majority of the existing literature takes the index values of linguistic terms for the pairwise comparison of alternatives. The semantics of the linguistic terms is not taken into consideration. Ignoring the semantics especially when the HFLTS contains non-uniform and unbalanced linguistic terms may cause information loss and may not represent the accurate facts. For example, considering the linguistic terms 'S3 (medium)' and 'S5 (very high)' in Figure 1, majority of the procedures do take the linguistic term indices '3' and '5' instead of the semantics corresponding to 'S3' and 'S5'. Consideration of semantics especially when the linguistic terms are non-uniform is a challenging task.
- c Generally, the validity of the pairwise comparison matrices of the alternatives is measured through its consistency level. Existing methodologies measure the consistency level when the alternatives are assessed in uniformly defined linguistic terms. The measurement of consistency level when the alternatives are assessed in non-uniform linguistic terms taking into consideration the personalised individual semantics, if any, is still a research gap.
- d The alternatives are compared pairwise in each attribute representing the local comparison. However, the need is to obtain the alternative comparisons globally representing all the attributes in MADM using an appropriate aggregation operator is still a research gap.

1.2 Motivation and contributions of this study

The proposed methodology attempts to eliminate the above-mentioned shortcomings. The contributions of this work are summarised below:

- 1 The proposed work includes the non-uniform, non-regular, or arbitrarily defined linguistic terms in HFLTS contrary to prevalent uniform, pre-defined linguistic terms in the literature. This will widen the scope of the HFLTS-MADM applications. Additionally, the use of irregular or non-uniform linguistic terms is more towards the DM's cognitive thinking, thus overcoming the flaws mentioned in (a).
- 2 The semantics of the linguistic terms are identified and used to derive the degree of preference of an alternative over the other using a combined approach of Concept of Existence in the ranking of fuzzy sets, α -cuts of fuzzy numbers, and specific ordering relations on HFS. This results in minimum information loss.
- 3 We propose binary integer programming (BIP) model for the consistency measurement to overcome the flaw mentioned in (c). Our work not only measures the level of consistency of the pairwise comparison matrix within the threshold but also conform each comparison indices to the linguistic terms specified by the DM.

- 4 A weighted aggregation approach is proposed to aggregate the pairwise comparison matrices across the attributes to arrive at an aggregated pairwise comparison matrix. The aggregated matrix determines the non-dominance degree of each alternative and accordingly ranks them for the MADM problem.

1.3 Structure of the paper

The preliminaries needed for our work is given in Section 2. Section 3 is devoted to the methodology for deriving the pairwise comparison of the alternatives. In Section 4, we have verified the consistency measure of the pairwise comparison matrices of the alternatives. In Section 5, we have formulated the proposed problem. This includes the derivation of priority vectors of the alternatives as interval numbers in each attribute. The calculation of the degree of non-dominance of the alternatives is also done in this section. In Section 6, we have given a numerical example to highlight the proposed procedure. Finally, in Section 7, we have compared our methodology with similar works. In Section 8, we concluded the result.

2 Preliminaries

2.1 Hesitant fuzzy set

A set $E \subseteq X$ is said to be a HFS in X when E is defined as:

$$E = \left\{ \langle x, h_E(x_i) \rangle \mid x_i \in X, i = 1, 2, \dots, k \right\}$$

where $h_E(x_i)$ denotes the set of possible membership values of element $x_i \in X$ in E . Let $\text{card}(h_E(x_i)) = l_i$, l_i being the number of membership values of the element x_i .

2.2 Hesitant fuzzy linguistic term set

Let X be a set and $x_i \in X$ ($i = 1, 2, \dots, n$). Let S be a set of linguistic terms with odd cardinality,

$$S = \{s_\alpha \mid \alpha = 1, 2, \dots, 2k + 1\}.$$

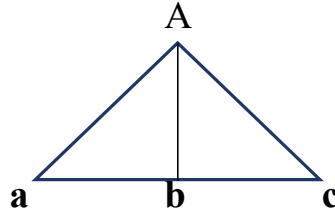
The HFLTS is defined as follows:

$$H_S = \left\{ \langle x_i, h_s(x_i) \rangle \mid x_i \in X \right\}$$

where $h_s(x_i)$ is an ordered finite subset of the consecutive linguistic terms from S .

2.3 Fuzzy number evaluation

Let us take a fuzzy number (a, b, c) , represented graphically below:



The numerical measure of the fuzzy number A is (Chang and Lee, 1994):

$$OM(A) = \int_0^1 \omega(w) [\chi_1(w)\mu_{A_L}^{-1}(w) + \chi_2(w)\mu_{A_R}^{-1}(w)] dw$$

$$\chi_1(w) + \chi_2(w) = 1$$

The evaluation of fuzzy number A contains two parts: left part A_L and right part A_R , where

$$A_L = \int_0^1 \omega(w)\chi_1(w)\mu_{A_L}^{-1} dw \text{ and } A_R = \int_0^1 \omega(w)\chi_2(w)\mu_{A_R}^{-1}(w)dw.$$

Note that, the membership function $\mu_A(x)$ can be normal or non-normal.

3 Pairwise comparison of alternatives

In this section, we have explained the comparison of alternatives pairwise when the alternatives are assessed as HFLTS. The linguistic terms in HFLTS are arbitrarily considered as opined by the DM. They are neither from a pre-defined linguistic term set nor in consecutive linguistic terms as prevailing in the literature. As the alternatives are assessed as HFLTS in each attribute, it is necessary to compare their attribute values to get the complete pairwise comparison amongst the alternatives. This is explained as follows:

Let there are m number of alternatives and n number of attributes and their valuations over the attributes are in HFLTS as shown in the matrix H below.

$$H = \begin{matrix} & C_1 & \cdots & C_n \\ \begin{matrix} A_1 \\ \vdots \\ A_m \end{matrix} & \begin{bmatrix} h_{s_{11}} & \cdots & h_{s_{1n}} \\ \vdots & \ddots & \vdots \\ h_{s_{m1}} & \cdots & h_{s_{mn}} \end{bmatrix} \end{matrix} \tag{3.1}$$

The HFLTS $h_{s_{ij}}$ ($i, j = 1, 2, \dots, m$) are subsets consisting of linguistic terms in the form of fuzzy numbers where $h_{s_{ij}} = (s_{i1}, s_{i2}, \dots, s_{il_{ij}})$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) represents the value of i^{th} alternative in j^{th} attribute in HFLTS containing l_{ij} number of linguistic terms.

Take two alternatives A_i and A_k ($i, k = 1, 2, \dots, m$). In the j^{th} attribute, they are $h_{s_{ij}} = (s_{i1}, s_{i2}, \dots, s_{il_{ij}})$ and $h_{s_{kj}} = (s_{k1}, s_{k2}, \dots, s_{lk_j})$. To compare A_i and A_k in the j^{th} attribute, it needs to compare the HFLTS $h_{s_{ij}}$ and $h_{s_{kj}}$. In other words, it is necessary to compare every distinct pair of linguistic terms $s_{ij} \in h_{s_{ij}}$ with $s_{kj} \in h_{s_{kj}}$ ($s_i \neq s_k$).

Before the comparison of linguistic terms that are in the form of fuzzy numbers, initially, we need to evaluate the linguistic terms $s_{ij} \in h_{s_{ij}}$ and $s_{kj} \in h_{s_{kj}}$ numerically. As per the procedure (Chang and Lee, 1994), we have the evaluation of s_{ij} as:

$$OM(s_{ij}) = \int_0^1 \omega(w) [\chi_1(w)\mu_{s_{ijL}}^{-1}(w) + \chi_2(w)\mu_{s_{ijR}}^{-1}(w)] dw \quad (3.2)$$

where $OM(s_{ij})$ in equation (3.2) represents the evaluation of the linguistic term (fuzzy number) s_{ij} depending on its semantics.

The left part $OM(s_{ijL}) = \int_0^1 \omega(w) [\chi_1(w)\mu_{s_{ijL}}^{-1}(w)] dw$ and the right part $OM(s_{ijR}) = \int_0^1 \omega(w) [\chi_2(w)\mu_{s_{ijR}}^{-1}(w)] dw$ respectively represent the left and right evaluation of linguistic term s_{ij} (fuzzy number).

$\chi_1(w)$ and $\chi_2(w)$ are the weights of the left and right part of evaluation respectively with $\chi_1(w) + \chi_2(w) = 1$.

$$\omega(w) = \frac{w}{0.5(w_{hgt}^*)^2}, \quad w_{hgt}^* \text{ is the height of the fuzzy number.}$$

Similarly, we can have an evaluation of the linguistic term s_{kj} .

The degree of superiority of s_{ij} over s_{kj} to the maximum extent possible is derived in equation (3.3) by comparing the right part of s_{ij} with left part of s_{kj} . The comparison of fuzzy numbers using the left and right part is found in detail in Lee and Li (1993). Thus $\forall s_i \neq s_k$ we have:

$$P(s_{ij} \gg s_{kj}) = \begin{cases} (OM(s_{ijR}) - OM(s_{kjL})), & \text{if } OM(s_{ijR}) > OM(s_{kjL}) \\ 0, & \text{otherwise} \end{cases} \quad (3.3)$$

Taking all the linguistic terms, in $h_{s_{ij}}$ and $h_{s_{kj}}$, and using the ordering relation procedure in HFS (Lan et al., 2017), we have the preference relation of A_i over A_k in j^{th} attribute as:

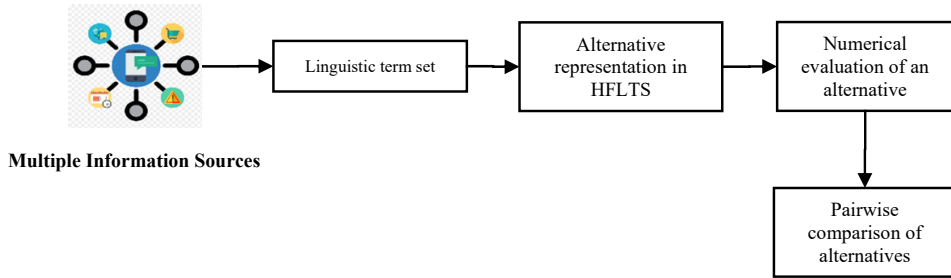
$$P(A_i > A_k) = a_{ik}^j = \frac{1}{mn} \sum_{s_{ij} \in h_{s_{ij}}, s_{kj} \in h_{s_{kj}}, s_i \neq s_k} P(s_{ij} \gg s_{kj}) \quad \forall i, k \quad (3.4)$$

Using equation (3.4), we have the additive preference of the pairwise comparison of alternatives in j^{th} attribute as:

$$PC_j = \begin{matrix} & A_1 & \cdots & A_m \\ \begin{matrix} A_1 \\ \vdots \\ A_m \end{matrix} & \begin{bmatrix} a_{11}^j & \cdots & a_{1m}^j \\ \vdots & \ddots & \vdots \\ a_{m1}^j & \cdots & a_{mm}^j \end{bmatrix} \end{matrix} \quad (3.5)$$

The procedure for obtaining pairwise comparison matrix of alternatives is shown in Figure 3.

Figure 3 Procedure for computing pairwise comparison matrices of alternatives (see online version for colours)



4 Consistency level measurement

The pairwise comparison of alternatives consisting of hesitant linguistic elements is very common and widely used in MADM problems. The pairwise comparison matrix becomes consistent when the transitivity and reciprocity in the additive/multiplicative sense are not violated. In the majority of the real-world decision situations, it is too difficult to present a comparison matrix with zero inconsistency. Therefore, an important aspect of pairwise comparison is to fix a threshold value for measuring the level of consistency. In some sense, the level of consistency above 90% guarantees the consistency of the preferences (comparisons) amongst the alternatives. Further, the need for a pre-fixed consistency threshold is to better manage any unnecessary and unavoidable human judgements that are very common in the comparison process.

The pairwise comparison indices derived in Section 3 are in crisp numerical numbers. However, as the original preferences of the DM are given in linguistic terms, the DM may not necessarily accept the derived crisp numerical preferences as his/her genuine preferences. The actual preference may lie in the interval domain of the linguistic term corresponding to the preference indices in the pairwise preference matrix. Therefore, it is essential and ideal to associate the pairwise preference indices to right linguistic terms and the associated preference value in its domain. This is explained through the following example:

Let the preference level of i^{th} alternative over k^{th} alternative in j^{th} attribute is $a_{ik}^j = 0.56$. The single numeric term 0.56 may not necessarily lead to full consistency according to the choice of the DM. However, it may so happen that another preference index other than 0.56 in the domain of the linguistic term that contains the number 0.56 may provide the desired consistency level in the pairwise preference matrix. This justifies and necessitates an identification of an appropriate linguistic term from the linguistic term set S_2 associating the number 0.56. This process may alter the preference index value 0.56 to $s_{ik}^* = s_{0.56}^*$ as the real preference interpretation of the DM.

From the above explanation, our work searches the linguistic term $s(0.56) \in S_2$ and a specific value in the domain of $s(0.56)$ as the right match for 0.56. However, if there are two or more such linguistic terms containing the number 0.56 in their domains, this becomes difficult to select a suitable linguistic term and the domain value to match the number 0.56. Our work takes this aspect and selects a linguistic term and the

corresponding domain value depending on its role in providing maximum consistency in the pairwise preference matrix. For instance, in Figure 2, the domains of the linguistic terms s_2, s_3, s_4 and s_6 contain the number 0.56, i.e., $0.56 \in [s_{2L}, s_{2R}], 0.56 \in [s_{3L}, s_{3R}], 0.56 \in [s_{4L}, s_{4R}], 0.56 \in [s_{6L}, s_{6R}]$. Out of the above four linguistic domains, only one is suitable to represent the pairwise preference $a_{ik}^j = 0.56$. Our work introduces a new methodology based on binary integer programming (BIP) to select a fitting linguistic term and the corresponding domain specific value that gives maximum consistency in the preference matrix.

Several methods are available to minimise the inconsistency in the pairwise comparison of alternatives (Zhang et al., 2016; Ishizaka and Lusti, 2004). Though the existing approaches suitably maintain the consistency level within the threshold, these methods are unsuitable to examine the level of consistency when the pairwise comparisons are in HFLTS with non-uniform linguistic terms. We have introduced a methodology to derive the consistency level of the pairwise preference matrix using BIP as shown below:

The pairwise comparison matrix shown in equation (3.4) is reproduced below:

$$PC_j = \begin{matrix} & A_1 & \dots & A_m \\ A_1 & \begin{bmatrix} a_{11}^j & \dots & a_{1m}^j \\ \vdots & \ddots & \vdots \\ A_m & \begin{bmatrix} a_{m1}^j & \dots & a_{mm}^j \end{bmatrix} \end{bmatrix} \end{matrix} \quad (4.1)$$

In equation (4.1), the matrix entries are in numerical terms. We need to confirm each term a_{ik}^j to an originally defined linguistic term $s(a_{ik}^j)$ from the non-uniform linguistic term set $S_2 = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ shown in Figure 2.

Let a_{ik}^j falls in the domains of $\{s_{i_1}, s_{i_2}, \dots, s_{i_k}\} \in S_2$, i.e., $a_{ik}^j \in [s_{i_pL}, s_{i_pR}]$ ($p = 1, 2, \dots, k$). Assume $s_{a_{ik}^j}$ as the variable corresponding to the preference index a_{ik}^j . Thus, we have $s_{i_pL} \leq s_{a_{ik}^j} \leq s_{i_pR}$ ($p = 1, 2, \dots, k$). We need to select a single domain interval from the k number of domain intervals $[s_{i_pL}, s_{i_pR}]$ for consistency verification. The proposed BIP model not only verifies the level of consistency within the threshold but also identifies the correct linguistic term and a specific value in its domain for the right interpretation of a_{ik}^j . The BIP model in our work is an extended version of the work given in Li et al. (2019). Thus we have,

$$\text{Max } CI(PC_j)$$

where

$$CI(PC_j) = 1 - \frac{4 \sum_{i,k,z=1; i < k < z}^m |s_{a_{ik}^j} + s_{a_{kz}^j} - s_{a_{iz}^j} - 0.5|}{m(m-1)(m-2)} \quad (4.2)$$

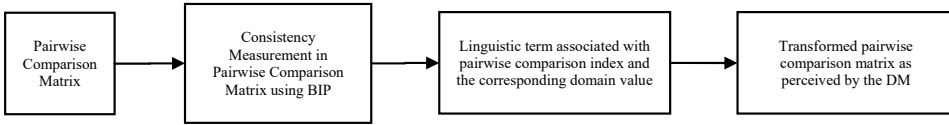
Subject to

$$s_{a_{ik}^j} \leq s_{i_pR} y_1 + M(1 - y_1)$$

$$\begin{aligned}
 s_{a_{ik}^j} &\geq s_{i_1^j L} y_1 \\
 s_{a_{ik}^j} &\leq s_{i_2^j R} y_2 + M(1 - y_2) \\
 s_{a_{ik}^j} &\geq s_{i_2^j L} y_2 \\
 &\vdots \\
 s_{a_{ik}^j} &\geq s_{i_k^j R} y_{ik} + M(1 - y_{ik}) \\
 s_{a_{ik}^j} &\geq s_{i_k^j L} y_{ik} \\
 y_1 + y_2 + \dots + y_{ik} &= 1, \quad y_r \in \{0, 1\} \quad (r = 1, 2, \dots, i_k) \\
 s_{a_{ik}^j} &= 0.5 \quad \text{for } i = k \\
 (i, k, z = 1, 2, \dots, m) &\quad \forall j.
 \end{aligned}$$

The solution $s_{a_{ik}^j}^*$ not only shows the transformed preference index of the i^{th} alternative over k^{th} alternative in j^{th} attribute in the domain of a linguistic term as interpreted by the DM but also verifies the desired consistency of the preference index in Figure 4.

Figure 4 Transformed pairwise comparison matrix after consistency verification



5 Problem formulation

In this section, we have formulated a MADM model under HFLTS containing non-uniform linguistic terms. The problem formulation in essence consists of the following four steps:

- 1 derivation of priority vectors of the alternatives in intervals in each attribute
- 2 aggregation of priority vectors over the attributes and identification of attribute weights
- 3 weighted aggregation of pairwise comparison matrices over the attributes to obtain the aggregated pairwise comparison matrix for the MADM problem
- 4 to find the non-dominance degree of each alternative and finally to rank the alternatives.

5.1 Priority vector of the alternatives

Fuzzy set is a tool to determine the preference relations pairwise amongst the alternatives in each attribute when a decision-maker is unsure about the preference of one alternative over another. The pairwise comparison matrix, thus formed, contains fuzzy elements that capture the uncertainty due to subjectivity and incompleteness in human thinking. The pairwise preference relations (additive or multiplicative) amongst the alternatives, considering the above facts, provide the preference ranking of alternatives as priority vectors. Several methods are available to derive the priority vectors of the alternatives from pairwise comparison matrices (Lan et al., 2017; Xia and Xu, 2014). Following the procedure given in Xia and Xu (2014), we have derived the priority vectors of the alternatives as interval weights.

The reason to derive the interval weights of the alternatives is to manage the uncertainties in real-world decision-making problems that are often expressed as human judgements. The exact numerical numbers representing the preferences of the alternatives may not be consistent with the cognitive judgement of human beings. The procedure for deriving the priority vectors is shown below:

The pairwise comparison matrix in equation (3.4) is reproduced below:

$$PC_j = \begin{matrix} & A_1 & \cdots & A_m \\ A_1 & \begin{bmatrix} a_{11}^j & \cdots & a_{1m}^j \\ \vdots & \ddots & \vdots \\ a_{m1}^j & \cdots & a_{mm}^j \end{bmatrix} & & \end{matrix} \quad (\forall j = 1, 2, \dots, n) \quad (5.1.1)$$

According to the work in Xia and Xu (2014), if the priority vectors of the alternatives are $w = ([w_1^-, w_1^+], [w_2^-, w_2^+], \dots, [w_m^-, w_m^+])$, based on additive consistency, we have

$$a_{ik}^j \in [0.5(w_i^- - w_k^+ + 1), 0.5(w_k^+ - w_i^- + 1)]$$

The solution to following mathematical programming model provides the priority vectors of the alternatives in j^{th} attribute ($j = 1, 2, \dots, n$):

$$\text{Min} \sum_{i \neq k} \left((a_{ik}^j - 0.5(w_i^- - w_k^+ + 1))^p + (0.5(w_i^+ - w_k^- + 1) - a_{ik}^j)^p \right)$$

Such that

$$\begin{aligned} a_{ik}^j - 0.5(w_i^- - w_k^+ + 1) &\geq 0, & i, k = 1, 2, \dots, m \\ 0.5(w_i^+ - w_k^- + 1) - a_{ik}^j &\geq 0, & i, k = 1, 2, \dots, m \\ \sum_{i \neq k} w_i^+ + w_k^- &\geq 1, & i, k = 1, 2, \dots, m \\ \sum_{i \neq k} w_i^- + w_k^+ &\leq 1 & i, k = 1, 2, \dots, m \\ w_i^+ &\geq w_i^- & i = 1, 2, \dots, m \end{aligned} \quad (5.1.2)$$

5.2 Aggregation of the alternative priority vectors across the attributes

From equation (5.1.2), the priority vectors of the alternatives in j^{th} attribute are obtained as:

$$w_{ij} = [w_{ij}^-, w_{ij}^+] \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n$$

To select the best alternative, it is required to aggregate $[w_{ij}^-, w_{ij}^+]$ over the attributes. This necessitates identifying an aggregation operator ‘Agg’ such that

$$\text{Agg}_j [w_{ij}^-, w_{ij}^+] \tag{5.2.1}$$

The aggregation value in (5.2.1) represents the valuation of A_i for the whole MADM problem.

There are many aggregation tools (Wei et al., 2014; Rodriguez et al., 2013; Liao et al., 2020; Rodriguez et al., 2012; Yager, 1995, 2003, 2004) available in the literature that aggregate the alternative assessments over the attributes and are mostly defined as crisp numbers. In our case, these values are in the form of closed intervals. In general, the priority vectors of the alternatives differ from one attribute to another, i.e., $[w_{ij}^-, w_{ij}^+] \neq [w_{il}^-, w_{il}^+] (\forall j, l)$. However, an alternative’s final ranking cannot be different for different attributes and should congregate to one value. This requires some sort of compromise or give and take amongst the attributes to reach out at a single valuation of the alternatives in the entire MADM problem. In other words, it is necessary to arrive at some sort of consensus amongst the attributes to rank the alternatives in MADM.

The work is given in Xu et al. (2014) aggregates the opinions of the experts when the experts opine their preferences of the alternatives in interval values. Motivated by the work given in Xu et al. (2014), we have minimised the distance between the priority vectors of the alternatives to arrive at a consensus amongst the attributes. A particular set of weights whose attachment to the attributes will help to minimise the distance amongst the priority vectors corresponding to each attribute. Quadratic programming is used in our work to minimise the distance and to identify the relevant attribute weights. The following steps explain the procedure:

Step 1 Take (A_1, A_2, \dots, A_m) and (C_1, C_2, \dots, C_n) be the set of m alternatives and n attributes, respectively. Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ as the weights of the attributes.

Step 2 Let $[w_{ij}^-, w_{ij}^+] (i = 1, 2, \dots, m)$ be the interval-valued preferences of the i^{th} alternative (priority vector) in j^{th} attribute. For an alternative A_i , the preference values in j^{th} and l^{th} attributes are respectively $[w_{ij}^-, w_{ij}^+]$ and $[w_{il}^-, w_{il}^+]$. The aggregated value of A_i in j^{th} and l^{th} attributes is the minimum of the squared weighted distance between the valuations of A_i in j^{th} and l^{th} attributes as shown below:

$$\text{Min} \left\{ d \left(\alpha_j [w_{ij}^-, w_{ij}^+] - \alpha_l [w_{il}^-, w_{il}^+] \right) \right\}^2 \tag{5.2.2}$$

Step 3 Generalising equation (5.2.2) over all the alternatives and the attributes, we can obtain the minimum value of the squared weighted distance of the priority vectors across the attributes by solving the following quadratic programming problem

$$\text{Min } \sum_{j=1}^n \sum_{l=1, j \neq l}^n \sum_{i=1}^m \left[(a_j w_{ij}^- - a_l w_{il}^-)^2 + (\alpha_j w_{ij}^+ - \alpha_l w_{il}^+)^2 \right]$$

Subject to

$$\sum_{l=1}^n \infty_l = 1 \tag{5.2.3}$$

Step 4 The solution of the quadratic programming problem in equation (5.2.3) derives the weights of the attributes ∞_l ($l = 1, 2, \dots, n$). The weights thus derived in equation (5.2.3) minimise the distance amongst the attributes and acts as an aid to assess the alternatives at a congregated value across the attributes.

5.3 Degree of non-dominance of the alternatives

In this subsection, we have derived the non-dominance degrees of the alternatives. The alternative with a minimum non-dominance degree is selected as the best alternative. In continuation of the steps in the previous section, the procedure is as follows:

Step 5 Take the pairwise comparison matrices of the alternatives corresponding to each attribute. The weighted value of the comparison matrices across the attributes are derived as follows:

For j^{th} attribute, we have $\infty_j A_j = (\beta_{ik}^j) = (\alpha_j a_{ik}), i, k = 1, 2, \dots, n$

Thus we have

$$\infty_j A_j = \begin{matrix} & A_1 & \cdots & A_n \\ A_1 & \left[\begin{matrix} \beta_{11}^j & \cdots & \beta_{1n}^j \\ \vdots & \ddots & \vdots \\ A_n & \beta_{n1}^j & \cdots & \beta_{nn}^j \end{matrix} \right] \end{matrix} \tag{5.3.1}$$

Where the entry β_{ik}^j represents the weighted preference value of the i^{th} alternative over the k^{th} alternative in j^{th} attribute.

The preference level of the i^{th} alternative over the k^{th} alternative over all the attributes in MADM ($i, k = 1, 2, \dots, n$) is as shown below.

$$(\beta_{ik})_{n \times n} = \sum_{j=1}^m \infty_j A_j = \begin{matrix} & A_1 & \cdots & A_n \\ A_1 & \left[\begin{matrix} \sum_{j=1}^m \beta_{11}^j & \cdots & \sum_{j=1}^m \beta_{1n}^j \\ \vdots & \ddots & \vdots \\ A_n & \sum_{j=1}^m \beta_{n1}^j & \cdots & \sum_{j=1}^m \beta_{nn}^j \end{matrix} \right] \end{matrix} \tag{5.3.2}$$

The non-dominance degree of A_i is obtained as:

$$\begin{aligned} &\text{Non-Dom}(A_i) \\ &= 1 - \text{Max} \left(\sum_{j=1}^m \beta_{ji}^j, \sum_{j=1}^m \beta_{2i}^j, \dots, \sum_{j=1}^m \beta_{ni}^j \right) \quad (i = 1, 2, \dots, n) \end{aligned} \tag{5.3.3}$$

Figure 5 Process to compute the non-dominance degree of alternatives

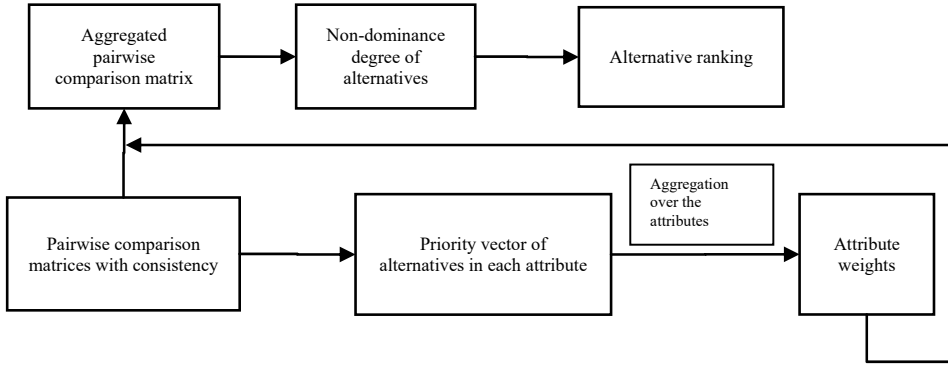
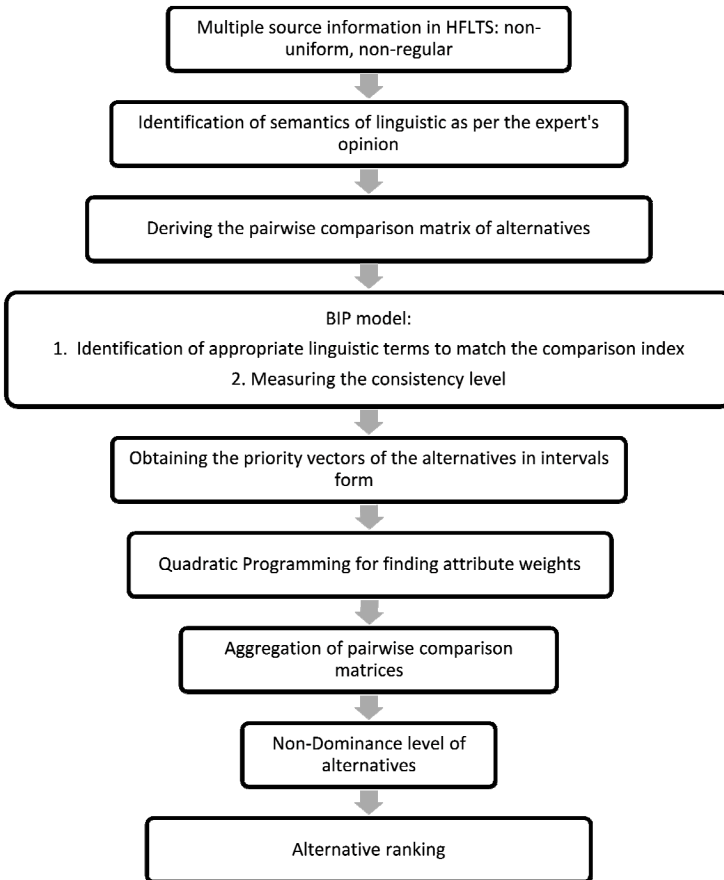


Figure 6 Stepwise procedure of the proposed work



In the matrix given in equation (5.3.2), more the entries corresponding to an alternative in a column are towards zero, less it is dominated by the other alternatives. The value $\text{Max} (\sum_{j=1}^m \beta_{1j}^j, \sum_{j=1}^m \beta_{2j}^j, \dots, S = \sum_{j=1}^m \beta_{ni}^j)$ in equation (5.3.3) derives the maximum value at which the alternative A_i is being dominated by another alternative. Thus, we have the non-dominance degree of A_i as shown in equation (5.3.3). In other words, we can say that an alternative is less dominated if the matrix entries corresponding to it have smaller values with zeroes indicating complete non-dominance.

The alternative having the highest value of the non-dominance degree is selected as the best alternative and so on.

The process for obtaining the non-dominance degrees of the alternatives after verifying the consistency level is shown in Figure 5.

The comprehensive procedure of our work is shown in Figure 6.

6 Numerical example

Let us take a car purchasing example in which a buyer prefers the attributes:

- 1 price
- 2 maintenance cost
- 3 mileage
- 4 comfort.

Consider four alternative models of cars that are available in the market. Based on the available information from multiple sources, let the buyer assesses the cars as hesitant fuzzy linguistic terms in each attribute as shown in Table 1. We need to select the best car or to rank the available cars according to the buyer’s preferences.

The linguistic term set formed after assessing the buyer’s choices is:

$$S = \{None, Very\ low, Low, Medium, High, Very\ high, Definitely\ high\}$$

Table 1 Buyer’s linguistic rating of the attributes in HFS

Car models	C_1	C_2	C_3	C_4
A_1	{ s_1, s_4, s_5 }	{ s_2, s_4 }	{ s_1, s_2 }	{ s_4 }
A_2	{ s_2, s_3 }	{ s_0, s_1 }	{ s_4 }	{ s_5, s_6 }
A_3	{ s_6 }	{ s_3, s_5 }	{ s_2, s_4, s_5 }	{ s_1, s_2, s_3 }
A_4	{ s_5, s_6 }	{ s_4, s_5, s_6 }	{ s_3, s_6 }	{ s_1, s_2, s_4 }

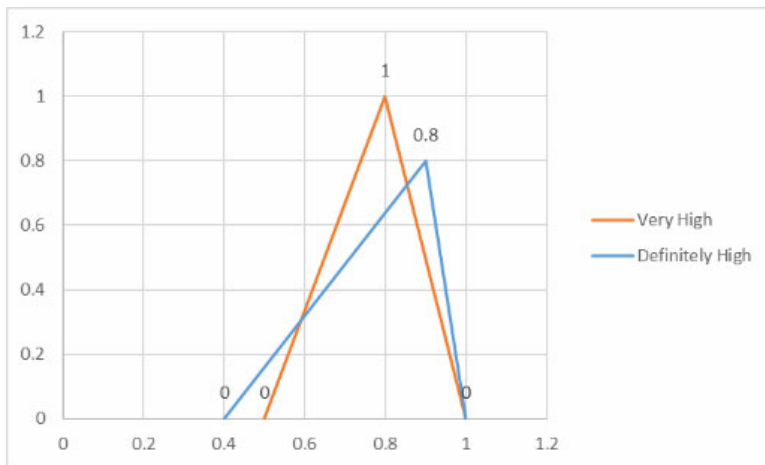
As per the buyer’s opinion, take the semantics of the linguistic terms as fuzzy numbers in column 2 of Table 2.

Table 2 Semantics of linguistic terms

Linguistic terms	Semantics	Left part	Right part	Valuation of left part	Valuation of right part
s_0 (none)	(0.0, 0.05, 0.1)	$y = 20x$	$y = -20x + 2$	0.037	0.066
s_1 (very low)	(0.0, 0.2, 0.4)	$y = 5x$	$y = -5x + 2$	0.134	0.266
s_2 (low)	(0.1, 0.4, 0.7)	$y = 3.33x - 0.33$	$y = -3.33x + 2.33$	0.3	0.5
s_3 (medium)	(0.8,(0.2, 0.6, 0.8))	$y = 2x - 0.4$	$y = -4x + 3.2$	0.465	0.665
s_4 (high)	(0.3, 0.7, 0.9)	$y = 2.5x - 7.5$	$y = -5x + 4.5$	0.566	0.766
s_5 (very high)	(0.5, 0.8, 1.0)	$y = 3.33x - 1.67$	$y = -5x + 5$	0.702	0.866
s_6 (definitely high)	(0.8,(0.4, 0.9, 1.0))	$y = 1.6x - 0.64$	$y = -8x + 8$	0.731	0.931

Graphically, the linguistic terms s_5 (very high) and s_6 (definitely high) are shown in Figure 7.

Figure 7 Normal (very high) and non-normal (definitely high) linguistic terms (see online version for colours)



The straight-line equation joining the points (0.5, 0) and (0.8, 1) is ‘ $y = 3.33x - 1.67$ ’ and is called as the left part of the fuzzy number *very high*. Similarly, the equation of the straight-line ‘ $y = -5x + 5$ ’ joining (0.8, 1) and (1, 0) is the right part of the fuzzy number *very high*. The left and right parts of other linguistic terms are given in columns 3 and 4 of Table 2.

Using equation (3.3), we have compared the alternatives pairwise. For example, the comparison of the alternatives A_3 and A_4 in attribute C_1 is done in the following steps:

Step 1 The degree to which the alternative A_3 is greater than A_4 is calculated by subtracting the sum of the left part of each linguistic term in A_4 $\{s_5, s_6\}$ from that of the right part of the linguistic terms in A_3 $\{s_6\}$. This is explained below:

The equation of the right part of s_6 is $y = -8x + 8$. Thus, we have the valuation of the right part $o(s_{6R})$ as:

$$o(s_{6R}) = \frac{1}{\left(\frac{0.8^2}{2}\right)} \int_0^{0.8} \frac{-y}{8} (y-8) dy = 0.931$$

The equation of the left part of s_5 is $y = 3.33x - 1.67$. Thus, we have the valuation of the left part as

$$o(s_{5L}) = \frac{1}{\left(\frac{1^2}{2}\right)} \int_0^1 \frac{y}{3.33} (y+1.67) dy = 0.702$$

The right and left valuations of other linguistic terms are shown in columns 5 and 6 in Table 2.

Using equation (3.4), we have the pairwise comparisons of the alternatives. For example, the comparison index between the alternatives A_3 and A_4 is shown below:

$$P(A_3 > A_4) = \frac{(0.931 - 0.702)}{2} = 0.114$$

Similarly, we can have pairwise comparison values for other alternatives, and they are shown in Table 3 for the attributes C_1, C_2, C_3 and C_4 .

Let the variable corresponding to the comparison index 0.114 is $s_{0.114}$. As said in Section 4, the need is to find an associated linguistic term $s(0.114) \in S$ conforming to the variable $s_{0.114}$ and a linked value in the domain of $s(0.114)$.

Table 3 Pairwise comparisons of alternatives

C_1	A_1	A_2	A_3	A_4	C_2	A_1	A_2	A_3	A_4
A_1		0.289	0.057	0.039	A_1		0.548	0.1	0.016
A_2	0.166		0	0	A_2	0		0	0
A_3	0.463	0.548		0.114	A_3	0.322	0.68		0.089
A_4	0.404	0.516	0.068		A_4	0.388	0.768	0.243	
C_3	A_1	A_2	A_3	A_4	C_4	A_1	A_2	A_3	A_4
A_1		0	0	0.017	A_1		0.05	0.466	0.366
A_2	0.549		0.177	0.168	A_2	0.332		0.6	0.565
A_3	0.46	0.1		0.154	A_3	0	0		0.151
A_4	0.573	0.232	0.28		A_4	0.067	0.016	0.2	

Assume $P(A_i > A_k) = a_{ik}^1$. Let $s_{a_{ik}^1}$ is the variable corresponding to the comparison index a_{ik}^1 . The BIP model shown below not only identifies the linguistic term $s(a_{ik}^1) \in S$ conforming to a_{ik}^1 but also associates a_{ik}^1 to a value in the domain interval of $s(a_{ik}^1)$. For example, $P(A_1 > A_2) = a_{12}^1 = 0.289$. Let the variable corresponding to 0.289 is $s_{0.289}$. From Table 3, we have $s_{0.289} \in [s_{1L}, s_{1R}]$, $s_{0.289} \in [s_{2L}, s_{2R}]$, and $s_{0.289} \in [s_{3L}, s_{3R}]$. As explained in Section 4, we need to select only one linguistic term out of s_1, s_2 and s_3 and a value for $s_{0.289}$ in the domain of the selected linguistic term. This is shown in first three constraints

of the BIP model. Similarly, other constraints are articulated for other comparison indices.

$$\text{Max CI(PC}_1) = 1 - \frac{\left(\left| s_{0.289} + s_{0.23} - s_{0.057} - 0.5 \right| + \left| s_{0.289} + s_{0.24} - s_{0.039} - 0.5 \right| \right. \\ \left. + \left| s_{0.23} + s_{0.114} - s_{0.24} - 0.5 \right| + \left| s_{0.057} + s_{0.114} - s_{0.039} - 0.5 \right| \right)}{4(4-1)(4-2)}$$

s.t.

$$0 \leq s_{0.289} \leq 0.4y_1 + 10(1 - y_1)$$

$$0.1 \leq s_{0.289} \leq 0.7y_2 + 10(1 - y_2)$$

$$0.2 \leq s_{0.289} \leq 0.8y_3 + 10(1 - y_3)$$

$$y_1 + y_2 + y_3 = 1$$

$$s_{0.289} + s_{0.166} = 1$$

$$0 \leq s_{0.23} \leq 0.1y_4 + 10(1 - y_4)$$

$$0 \leq s_{0.23} \leq 0.4y_5 + 10(1 - y_5)$$

$$y_4 + y_5 = 1$$

$$s_{0.23} + s_{0.548} = 1$$

$$0 \leq s_{0.057} \leq 0.1y_6 + 10(1 - y_6)$$

$$0 \leq s_{0.057} \leq 0.4y_7 + 10(1 - y_7)$$

$$y_6 + y_7 = 1$$

$$s_{0.057} + s_{0.463} = 1$$

$$0 \leq s_{0.039} \leq 0.1y_8 + 10(1 - y_8)$$

$$0 \leq s_{0.039} \leq 0.4y_9 + 10(1 - y_9)$$

$$y_8 + y_9 = 1$$

$$s_{0.039} + s_{0.404} = 1$$

$$0 \leq s_{0.24} \leq 0.1y_{10} + 10(1 - y_{10})$$

$$0 \leq s_{0.24} \leq 0.4y_{11} + 10(1 - y_{11})$$

$$y_{10} - y_{11} = 1$$

$$s_{0.24} + s_{0.516} = 1$$

$$0 \leq s_{0.114} \leq 0.4y_{12} + 10(1 - y_{12})$$

$$0.1 \leq s_{0.114} \leq 0.7y_{13} + 10(1 - y_{13})$$

$$y_{12} + y_{13} = 1$$

$$s_{0.114} + s_{0.068} = 1$$

y_i are binary.

The solution of BIP that verifies the consistency of the pairwise comparisons of alternatives in attribute C_1 (as given in Table 3) is shown in Table 4 with consistency index CI (PC_1) as 100%. The solutions for other attributes including their consistency level can be obtained similarly and are shown in Table 4. It may be noted that the inconsistency level in all the comparison matrices is found to be zero indicating full consistency in the buyer’s preferences of the alternatives. Note that in Table 4, we have the transformed version of the pairwise comparison indices after verifying the consistency measure in the pairwise comparison matrices.

Using equation (5.1.2), the priority vectors of the alternatives in each attribute are derived as intervals and are shown in Table 5.

Table 4 Pairwise comparisons of alternatives with consistency

C_1	A_1	A_2	A_3	A_4	C_2	A_1	A_2	A_3	A_4
A_1		0.60	0.36	0.4	A_1		0.60	0.4	0.4
A_2	0.4		0.26	0.29	A_2	0.4		0.29	0.29
A_3	0.64	0.74		0.53	A_3	0.6	0.71		0.5
A_4	0.6	0.71	0.47		A_4	0.6	0.71	0.5	
C_3	A_1	A_2	A_3	A_4	C_4	A_1	A_2	A_3	A_4
A_1		0.3	0.4	0.2	A_1		0.4	0.71	0.8
A_2	0.7		0.6	0.4	A_2	0.6		0.81	0.9
A_3	0.6	0.4		0.3	A_3	0.29	0.19		0.59
A_4	0.8	0.6	0.7		A_4	0.2	0.1	0.41	

Table 5 Priority vectors of alternatives

	C_1	C_2	C_3	C_4
A_1	[0.15, 0.2]	[0.18, 0.2]	[0, 0]	[0.2, 0.6]
A_2	[0, 0]	[0, 0]	[0.3, 0.4]	[0.4, 0.8]
A_3	[0.43, 0.48]	[0.39, 0.42]	[0.1, 0.2]	[0, 0.18]
A_4	[0.37, 0.42]	[0.39, 0.42]	[0.5, 0.6]	[0, 0]

Using equation (5.2.3) and quadratic programming, we have calculated the following attribute weights.

$$w_1 = 0.291, w_2 = 0.309, w_3 = 0.254, w_4 = 0.146$$

By equations (5.3.1) and (5.3.2), and the attribute weights, we have aggregated pairwise comparison matrices and the aggregated matrix obtained is shown below:

$$\begin{matrix}
 & A_1 & A_2 & A_3 & A_4 \\
 A_1 & \left[\begin{matrix} 0 & 0.596 & 0.400 & 0.356 \end{matrix} \right] \\
 A_2 & \left[\begin{matrix} 0.403 & 0 & 0.374 & 0.294 \end{matrix} \right] \\
 A_3 & \left[\begin{matrix} 0.6 & 0.625 & 0 & 0.448 \end{matrix} \right] \\
 A_4 & \left[\begin{matrix} 0.644 & 0.705 & 0.552 & 0 \end{matrix} \right]
 \end{matrix}$$

Using equation (5.3.3), the non-dominance degree of each alternative is derived and is shown below:

$$\text{Non-Dom}(A_1) = 1 - \text{Max}(0, 0.403, 0.6, 0.644) = 0.356$$

Similarly, we have $\text{Non-Dom}(A_2) = 0.295$, $\text{Non-Dom}(A_3) = 0.448$, $\text{Non-Dom}(A_4) = 0.552$.

Thus, we have the alternative ranking as:

$$A_4 > A_3 > A_1 > A_2$$

From the above example, we found that the car A_4 is most preferred and A_2 is the least preferred car.

7 Comparison with existing similar papers

In this section, our focus is to highlight the advantages of our work in comparison with similar works that are defined under the HFLTS environment. Our procedure is compared mainly on the works given in Liao et al. (2020) and Sellak et al. (2018). Certain shortcomings in these works are identified and required improvements are made in our method.

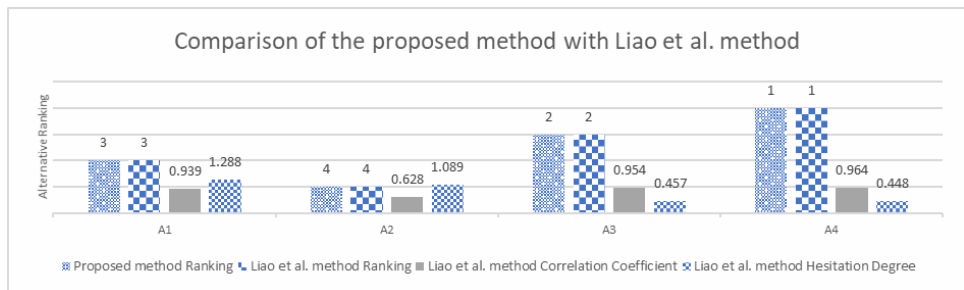
7.1 Comparison with Liao et al. (2020)

Using the methodology of Liao et al. (2020) and taking the data from Table 1, we have the ranking of the alternatives ($A_4 > A_3 > A_1 > A_2$) that is same as of the proposed method shown in Table 6. Graphically, the ranking is shown in Figure 8.

Table 6 Comparison of ranking of proposed method with Liao et al. method

	Proposed method			Liao et al. method		
	Ranking	Correlation coefficient	Hesitation degree	Ranking	Correlation coefficient	Hesitation degree
A_1	3	NA	NA	3	0.939	1.288
A_2	4			4	0.628	1.089
A_3	2			2	0.954	0.457
A_4	1			1	0.964	0.448

Figure 8 Comparison of proposed method with Liao et al. method (see online version for colours)



From the computational results and Figure 8, the alternative ranking obtained in the proposed work matches to that of the ranking given in Liao et al. (2020). The superiority of the proposed methodology is verified as the low degrees of hesitation (in the sense of lower and upper value of the correlation coefficient) correspond to the most preferred alternatives (A_4 and A_3). Though the degrees of hesitation are not in right match with the ranking of alternatives A_1 and A_2 , their difference is insignificant.

In the above method, the mean and hesitance degree are derived to determine the correlation between HFLTS. Further, the correlation coefficient between the alternatives and the ideal solution is derived and the alternative with the highest coefficient is taken as the best alternative. Other alternatives are ranked accordingly. This is a novel methodology for ranking of the alternatives in MCDM depending on the closeness of the alternatives to the ideal solution. However, the drawback is that the derivation of mean, variance, and correlation coefficients is calculated using the indices of the linguistic terms only and the semantics of the linguistic terms are completely left out. This may, to some extent, work if the linguistic terms are pre-specified and are uniformly defined. In the case of non-uniform or arbitrarily defined linguistic terms, the sole use of linguistic term indices, and their use in the ranking of alternatives may lead to wrong results. Therefore, the incorporation of semantics of linguistic terms to interpret the real views of decision-maker is essential in the ranking of the alternatives. The proposed model considers the semantics and removes the above deficiencies.

7.2 Comparison with Sellak et al. (2018)

We have compared our work with the work given in Sellak et al. (2018). Computationally, the proposed work cannot be compared with this work as in our work, the alternative assessment in HFLTS consists of non-uniform, non-regular, unbalanced, and not from any pre-specified linguistic term set whereas in Sellak et al. (2018), the linguistic terms are from a regular, uniform, and pre-specified linguistic term set. Moreover, our work considers both normal and non-normal fuzzy numbers. Secondly, the work in Sellak et al. (2018) uses clustering methodology of MCDM that is mostly based on distance/similarity measures, which is beyond the scope of the proposed work. However, based on the theoretical concepts, we have compared our work with Sellak et al. (2018) and the shortcomings therein are shown below:

- a The linguistic terms are predefined from a given linguistic term set and the HFLEs are always an ordered set of consecutive terms.
- b The degree of uncertainty, central value, and later the scores of HFE are derived using either cardinality or the indices of the linguistic terms only. The semantics of the linguistic terms are completely ignored while deriving the above values.
- c The work compares the alternatives pairwise and the preference relations between a pair of alternatives are based on the score values and the certainty factor of the HFEs. The shortcoming is the use of linguistic term indices (not semantics) to determine the above factors.
- d In the work, the weights to the criteria are taken subjectively for calculating the scores of the alternatives in each attribute.

Our work has taken all the above-mentioned factors into account and obtained a viable solution to MADM problems under the HFLTS environment. Point a is resolved by taking the non-uniform linguistic terms of experts. Points b and c are taken care of by considering the semantics of the linguistic terms in our work. The issue in point d is addressed by deriving the attribute weights by the aggregation of priority vectors of the alternatives with respect to the attributes.

Thus, our paper addresses all the above issues and removes the deficiencies mentioned above.

8 Conclusions

In this paper, we have introduced a new procedure for solving MADM problems under HFLTS where the attribute values are in the form of HFLTS with non-uniform and arbitrarily defined linguistic terms. The most important characteristic of the proposed work is that it is capable of handling any type of linguistic term set, not necessarily only with the pre-defined linguistic terms with their prefixed semantics. Compared with the existing procedures of MADM with HFLTS, to our knowledge, this specific aspect of non-uniform linguistic terms is not taken into account in any of the existing methodologies. Further, our methodology has introduced a pairwise comparison procedure that takes the comparison of HFLTS using the methods of the ranking of L-R fuzzy numbers based on the concept of existence and the specific ordering procedure of HFS. While comparing a pair of linguistic terms, the extent to which a linguistic term is maximally higher or minimally lower is calculated and incorporated as entries in the pairwise comparison matrix. We also provide the weighted value of the pairwise comparison matrix in each attribute after deriving the weights of the attributes. The weights of the attributes are derived using quadratic programming by aggregating the priority vectors of the alternatives that are in interval numbers. We applied the concept of dominance/non-dominance in the aggregated pairwise comparison matrix to derive the degrees of non-dominance of the alternatives and the alternative with the highest non-dominance degree is ranked as the most preferred alternative.

8.1 *Scope for future research*

In many practical situations, interdependence amongst the attributes in MADM cannot be ruled out. Therefore, it is worth to verify the association amongst the attributes when the attributes values are in HFLTS with appropriate semantic measures in linguistic terms. This can be taken as a future research. The other direction of future research could be to employ intuitionistic fuzzy sets with non-uniform linguistic terms instead of HFLTS. Our procedure derives the dominance of one alternative over the other. However, the partial dominance when they are assessed in intuitionistic fuzzy linguistic terms is a scope for future research. The quadratic programming is used in our work to aggregate the priority vectors corresponding to each attribute to arrive at a preference structure amongst the alternatives. The priority vector with entries as HFLTS and its aggregation over the attributes may be a further aid to the decision maker to arrive at a realistic decision. This may be considered as a scope for future research.

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