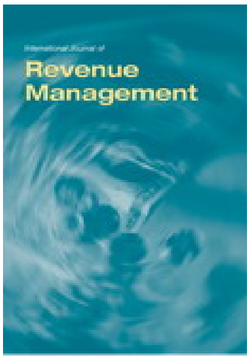

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Nash equilibrium computation in airline frequency game

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Abstract: In many situations, airlines compete for frequency of services which they offer along flight legs due to the limited capacity of airports. In this paper, we study a frequency game in which two airlines offer services along two flight legs. Demand on one leg is considered to be low, while on the other leg demand is considered to be high. We determine the Nash equilibrium strategies of the flight frequency of each of the airlines along each of these legs. In developing countries, there is a need to introduce and promote services along the low demand leg to achieve better connectivity. Our objective is to increase airline connectivity and hence this study aims at analysing the policy of frequency restrictions as a measure to increase flight operations on low demand legs. We study the impact of such frequency restrictions on Nash equilibrium strategies. Our results show that these restrictions increase the number of flight operations on the low demand leg.

Keywords: frequency competition; Nash equilibrium; regional airline connectivity.

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1 Introduction

After the Airline Deregulation Act of 1978, airlines had the freedom to choose the flight schedule and prices for routes (Talluri and Van Ryzin, 2004). With this liberalisation, the frequency of flights continuously increased however, the infrastructure at airports did not increase proportionally. This gave rise to the problem of airport congestion (Levine, 1969). According to the Worldwide Airport Slot Guidelines (WASG), airports can be classified into three categories. The airports where current infrastructure is adequate to meet the demand come under level 1 category. The airports where congestion occurs occasionally come under level 2 category. The airports where infrastructure is inadequate to meet demand come under level 3 category. The legs that either start or end with the level 3 airport will not have enough capacity to fulfil the request of all airlines. Thus, there is a high level of competition among the airlines on these legs. Further, the airlines want to operate more flights on these legs to satisfy more passenger demand and eventually acquire more market share. Given the demand on a leg, the market share depends on the frequency of all airlines which offer services on the particular leg. O'Connor (2001) and Belobaba (2009) described S-curve or sigmoidal relationship between the market and frequency share. This relationship is widely used to model competition on highly competitive routes. In this research, we study such frequency competition between two airlines, both of which operate on two legs; one leg is a low demand leg and the other a high demand leg.

Post deregulation, airlines mostly started operating on profitable routes and the remote locations were deprived of air services. A few countries adopted policies to serve remote areas with air services. The USA launched the 'Essential Air Services Programme' in 1978 to ensure the basic level of air services to small communities. In 2013 Turkey also started providing subsidies to the air carriers who offer services to remote regions (Uzgör and Şengür, 2022). In India, to maintain the air connectivity to remote locations Government of India launched regional connectivity scheme (RCS) in 2016.The features of the RCS scheme are as follows:

- RCS is applicable on routes of length between 200 to 800 km with no lower limit set for hilly, remote, island and security sensitive regions.
- The central government will provide concessions to the tune of 2% excise on value added tax (VAT) and service tax at one by ten rate and liberal code sharing for RCS airports.
- For balanced regional growth, allocations will be spread equitably across five regions – North, West, South, East and North East with a cap of 25%.
- Market-based reverse bidding mechanism to determine least viability gap funding (VGF) to select the airline operator with the right to match to the initial proposer. VGF will be reduced if passenger load factor remains high and will be discontinued after three years when a route becomes self-sustainable (Airports Authority of India, 2022).

These were the initial features of the RCS in which the government mainly provides concession on taxes to encourage airlines to operate on new routes. As on date, 70 airports and 439 routes have been operationalised under the RCS scheme. However, the slow progress of the scheme has been a concern for various stakeholders. According to Shroff (2022), as of December 2021, out of the 948 routes awarded under the RCS scheme in the past five years, only 403 routes connecting 65 unserved and underserved airports are currently operational. Several factors have contributed to the partial success of the RCS scheme, including the impact of the pandemic on demand, lack of airport infrastructure development, lack of funding, and unfavourable weather conditions. Deol (2021) also highlighted the slow progress of the RCS scheme, citing similar issues such as inadequate infrastructure, pandemic effects, and funding challenges. Hence, in this research we aim to address the issue of slow progress by analysing the potential impact of hard constraints on the Nash equilibrium of a frequency game, with the goal of improving the sustainability and efficiency of the RCS scheme.

2 Literature review

Two types of airline games that are exhaustively addressed in the literature are seat allocation game and pricing game. In the seat allocation game, airlines decide their optimal seat inventory allocation among fare classes, while seat allocation decisions of one airline affect the demands of other airlines (Netessine and Shumsky, 2005). Littlewood (1972) gave a pioneer model to give seat allocation between two classes. Belobaba and Wilson (1997), Li et al. (2007), and Netessine and Shumsky (2005) are some authors who addressed the seat allocation game in the airline industry. Li et al. (2008) considered a seat allocation game between two airlines under different fare structure. They considered both simultaneous and sequential move games and proved the existence of pure strategy Nash equilibrium.

Bertrand-Edgeworth model is a primary model of price competition. Feng and Xiao (2006), Chew et al. (2009) and Cizaire and Belobaba (2013) are some of the authors who studied pricing and seat allocation optimisation in the airline industry without competition. Many authors also studied simultaneous pricing and seat allocation competition among airlines and provided Nash equilibrium strategies of pricing and seat allocation (Côté et al., 2003; Raza and Akgunduz, 2008). Zhao and Atkins (2008) studied simultaneous pricing and inventory competition in the retail industry, where N newsvendors compete with each other and decide their optimal price and inventory.

Apart from pricing and seat allocation decision, a critical decision in airline scheduling is frequency planning, i.e., the number of flights to be operated on each route. Due to the limited number of slots in congested airports, airlines also compete to decide their optimal strategies in a frequency game. Dobson and Lederer (1993) studied both flight schedules and fare choices of airlines in a competitive environment. Mazumdar and Ramachandran (2014) and Garg and Venkataraman (2020) were some of the authors who considered pricing and seat inventory games among airlines. Adler (2001) studied an extensive form game on fares, frequencies, and aircraft sizes and provided Nash equilibrium results for four airports and two airlines network. Subsequently, Adler (2005) considered a two-stage extensive form game for three airlines with two hubs, who decide hub locations, price, frequencies, and aircraft sizes while competing with each other. Vaze and Barnhart (2012) modelled a frequency game between two airlines. They show that a pure-strategy Nash equilibrium may not always exist. Hence, they formulated the problem as a mixed strategy Nash equilibrium game and provided a dynamic programming-based algorithm to find approximate Nash equilibrium. Wang et al. (2022) studied the frequency game with flow balance constraint to prevent dramatic frequency changes across the network. They formulated the problem as a mixed-strategy Nash equilibrium game.

In 2016, Government of India launched the National Civil Aviation Policy. One of the main objectives of this policy was to make airlines more accessible and affordable to the public. RCS is one of the primary schemes of this policy. The objective of the government is to encourage airlines to operate flights to remote locations (Iyer and

Thomas, 2020). This motivates us to analyse a policy which enables airlines to provide more services on low demand legs. In this study, we describe and analyse such a model.

Fageda et al. (2018) studied policies applied worldwide to improve air connectivity. They categorised these policies into four groups: route-based policies, passenger-based policies, airline-based policies, and airport-based policies.

Route-based policies are widely implemented by applying public service obligations (PSOs) in specific routes. PSOs are the contract between the government and airlines to maintain specific service levels on remote routes. USA, Australia and some countries of the European Union implemented PSOs to improve air connectivity. In India, before the RCS, the policy that took care of air connectivity was the route dispersal guidelines (RDG 1994). However, these guidelines do not make sure the operation of airlines on each remote route. Hence in this study, we analyse the PSOs on specific routes.

Airlines decide the number of flights to be operated on each leg in which they offer services. Such frequency planning of an airline is often restricted by the number of slots available at an airport. So, the airlines compete with each other to get more number of slots. The concept of relationship between market share and frequency share among *n* airlines has been described by Belobaba (2009) through the S-curve and is described below.

$$MS_i = \frac{FS_i^{\alpha}}{\sum_{j=1}^{n} FS_j^{\alpha}} \tag{1}$$

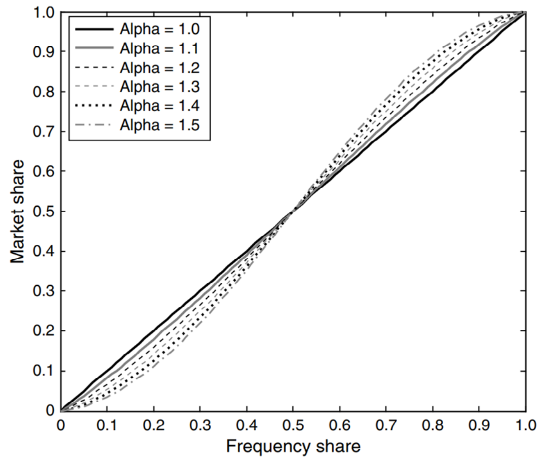
Here MS_i is the market share of airline i on a leg and FS_i is the frequency share of airline i. α is the model parameter which defines the curvature of the market share and frequency share relationship. Thus, the market share of an airline depends on its own frequency (number of flights it operates on a leg) as well as the frequency of the competitors. The relationship between market share and frequency share for different values of α is shown in Figure 1. In our study, we use this concept to model the market share of airlines.

2.1 Problem statement

Frequency game has not been well addressed in the literature compared to pricing and seat allocation games. In this study, we consider two airlines which compete for frequency along two flight legs. Demand on one leg is considered to be low, and the demand on the other leg is considered to be high. For such a frequency game, we determine the Nash equilibrium strategies of the airlines in the two legs. The strategies are the number of flight services each airline offers along the low and high demand legs. To promote flight services along low demand legs, the government can impose restrictions on the frequency of flights on the low and high demand legs. Hence, we also study the impact of such restrictions on the Nash equilibrium of the frequency game.

The rest of the paper is organised as follows. In Section 3, we develop the model of frequency competition between two airlines. In Section 4, we study the developed model under restrictions on low and high demand legs. In Section 5, we perform numerical analysis to compare the Nash equilibrium strategies of the models discussed in Sections 3 and 4. In Section 6, we conclude our study and discuss the scope for future work.

Figure 1 Relation between market share and frequency share for different values of α



Source: Graph from Vaze and Barnhart (2012)

3 Unrestricted airline frequency game (model 1)

In this section, we develop a model of frequency competition between two airlines that offer their services along two legs. We derive the expression for Nash equilibrium strategies.

3.1 Problem setting

We consider two airlines competing on two legs, low demand leg (represented by *L*) and high demand leg (represented by *H*). We assume that the aggregate revenue of each airline on each leg is known on the basis of the previous history. For flights scheduled for a particular time period, players set approximately equal prices for seats on a particular leg. Buzzell et al. (1975) have established a linear relationship between market share and return on investment (ROI). Thus, we represent the market share of an airline on a particular leg by the ratio of its expected revenue to the total expected revenue along the particular leg. The market share of airline 1 on the high demand leg is defined as:

$$MS_{H1} = \frac{R_{H1}}{R_{H1} + R_{H2}} \tag{2}$$

Here R_{H1} and R_{H2} are the expected revenue of the airlines 1 and 2 respectively on the high demand leg. Similarly, the market share of airline 2 on high demand leg is obtained. On the low demand leg, the concept of market share and frequency share relationship becomes less relevant, hence we do not consider it here.

We consider two airlines which compete for the frequency with which each leg will be operated. The decision variables of each airline are the number of flights they operate on each leg. The airlines play a simultaneous move game and decide their Nash equilibrium strategies. The Nash equilibrium strategy for airline 1 is number of flights it offers on the low and high demand legs and is denoted by (s_{L1}, s_{H1}). Similarly, the Nash equilibrium strategies of airline 2 are the number of flights it offers on the low and high demand legs and is denoted by (s_{L2}, s_{H2}). The parameters and the decision variables of model 1 are summarised below.

3.1.1 Parameters

R_{ij} average revenue of airline j (j = 1, 2) on leg i ($i = L, H$)

f_i fleet size of airline i (i = 1, 2).

3.1.2 Decision variables

s_{ij} the number of flights operated by j (j = 1, 2) on leg i ($i = L, H$).

We make the following assumptions in this model.

1. It is more profitable for airlines to operate on high demand leg as compared to low demand legs, i.e., $R_{H1} > R_{L1}$, $R_{H2} > R_{L1}$.
2. On low demand legs airlines do not regularly operate. Hence the concept of market share and frequency share relationship is not applicable for low demand legs.

3.2 Optimisation problem

The two airlines find their Nash equilibrium strategies by first solving their respective optimisation model while treating decision variable of the other airline as fixed. The local maxima so obtained are solved simultaneously to obtain their Nash equilibrium strategies.

We observe that the market share relationship between revenue and frequency of airlines is obtained [from (1) and (2)] as below.

$$MS_{H1} = \frac{R_{H1}}{R_{H1} + R_{H2}} = \frac{s_{H1}^{\alpha}}{s_{H1}^{\alpha} + s_{H2}^{\alpha}}$$

$$\Rightarrow R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

Optimisation model for airline 1:

$$\max R_{L1} s_{L1} + R_{H1} s_{H1}$$

Subject to

$$s_{L1} + s_{H1} \leq f_1$$

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

$$s_{L1}, s_{H1} \geq 0$$

Optimisation model for airline 2:

$$\max R_{L2} s_{L2} + R_{H2} s_{H2}$$

Subject to

$$s_{L2} + s_{H2} \le f_2$$

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

$$s_{L2}, s_{H2} \ge 0$$

In these models the objective function is the weighted sum of flight frequency on low and high demand legs with weights representing expected revenue along the particular leg. Hence the objective function represents the airlines' expected revenue. The first constraint denotes the fleet size restriction. The second constraint relates the market and frequency share.

We obtain the local maxima for each of the airlines which satisfy first order necessary and sufficiency conditions. The local maxima of two airlines are simultaneously solved to obtain two Nash equilibria under various conditions. These are described in Propositions 1 and 2.

3.2.1 Proposition 1

The Nash equilibrium strategies of airlines 1 and 2 are given by

$$s_{L1} = 0, s_{H1} = f_1$$

$$s_{L2} = f_2 - f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}}, s_{H2} = f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}}$$

whenever the following constraint is satisfied.

$$f_2 - f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}} \ge 0$$

3.2.2 Proposition 2

The Nash equilibrium strategies of airlines 1 and 2 are given by

$$s_{L1} = f_1 - f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha}, s_{H1} = f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha}$$

$$s_{L2} = 0, s_{H2} = f_2$$

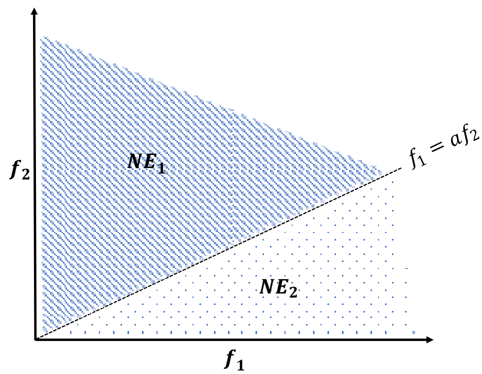
whenever the following constraint is satisfied.

$$f_1 - f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha} \ge 0$$

Proof of Propositions 1 and 2 is given in Appendix A.1.

We represent the Nash equilibrium obtained from Propositions 1 and 2 as NE_1 and NE_2 respectively. These two Nash equilibriums NE_1 and NE_2 give the maximum number of flights that can be operated by both the airlines based on market-frequency share relationship and the maximum fleet size of airlines. In Figure 2, the region of existence of Nash equilibrium NE_1 and NE_2 are shown. Here we assumed $a = \left(\frac{R_{H1}}{R_{H2}}\right)^{1/\alpha} \ge 1$ which means that airline 1 has more market share. Figure 2 shows that for any value of fleet sizes only one Nash equilibrium exits.

Figure 2 Regions of existence of Nash equilibrium NE_1 and NE_2 (see online version for colours)



4 Restricted airline frequency game (model 2)

In this model we impose restrictions on the frequency of services to be offered on the low and high demand leg. To increase the flights services offered on low demand leg we include the following two constraints.

$$s_{L1} + s_{L2} \geq S_L$$

$$s_{H1} + s_{H2} \leq S_H$$

We next analyse the impact of these constraints on the Nash equilibrium strategies of the two airlines. To find the Nash equilibrium, we first find the local maxima of each airline and then simultaneously solve them. The optimisation model of airline 1 is described below.

$$\max R_{L1}s_{L1} + R_{H1}s_{H1}$$

Subject to:

$$s_{L1} + s_{H1} \leq f_1$$

$$R_{H1}s_{H2}^{\alpha} = R_{H2}s_{H1}^{\alpha}$$

$$s_{L1} + s_{L2} \geq S_L$$

$$s_{H1} + s_{H2} \leq S_H$$

$$s_{L1}, s_{H1} \geq 0$$

Similarly, the optimisation model of airline 2 is written. We obtain the local maxima for each of the airlines which satisfy first order necessary and sufficiency conditions. The local maxima of the two airlines are simultaneously solved to obtain the Nash equilibria.

These Nash equilibria under various conditions are explained in Propositions 3, 4, 5, and 6.

4.1 Proposition 3

The Nash equilibrium strategies of airlines 1 and 2 are given by

$$s_{L1} = 0,\ s_{H1} = f_1$$

$$s_{L2} = f_2 - f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}},\ s_{H2} = f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}}$$

whenever the following constraints are satisfied.

$$f_1 \left(1 + R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}}\right) \le S_H$$

$$f_2 - f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}} \ge S_L$$

4.2 Proposition 4

The Nash equilibrium strategies of airlines 1 and 2 are given by

$$s_{L1} = f_1 - \frac{R_{H1}^{\frac{1}{\alpha}} (f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}},\ s_{H1} = \frac{R_{H1}^{\frac{1}{\alpha}} (f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

$$s_{L2} = -f_1 + \frac{R_{H1}^{\frac{1}{\alpha}} (f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}} + S_L,\ s_{H2} = \frac{R_{H2}^{\frac{1}{\alpha}} (f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

whenever the following constraints are satisfied.

$$f_1 + f_2 \ge S_L$$

$$f_1 + f_2 \le S_H + S_L$$

$$f_1 R_{H2}^{\frac{1}{\alpha}} + (S_L - f_2) R_{H1}^{\frac{1}{\alpha}} \ge 0$$

$$f_2 R_{H1}^{\frac{1}{\alpha}} + (S_L - f_1) R_{H2}^{\frac{1}{\alpha}} \ge 0$$

4.3 Proposition 5

The Nash equilibrium strategies of airlines 1 and 2 are given by

$$s_{L1} = f_1 - \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}},\ s_{H1} = \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

$$s_{L2} = f_2 - S_H + \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}},\ s_{H2} = S_H - \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

whenever the following constraints are satisfied.

$$f_1 + f_2 \geq S_L + S_H$$

$$f_1 \left(R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}} \right) \geq R_{H1}^{\frac{1}{\alpha}} S_H$$

$$f_2 \left(R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}} \right) \geq R_{H2}^{\frac{1}{\alpha}} S_H$$

4.4 Proposition 6

The Nash equilibrium strategies of airlines 1 and 2 are given by

$$s_{L1} = f_1 - f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha},\ s_{H1} = f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha}$$

$$s_{H2} = f_2,\ s_{L2} = 0$$

whenever the following constraints are satisfied.

$$f_2 \left(1 + R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha} \right) \leq S_H$$

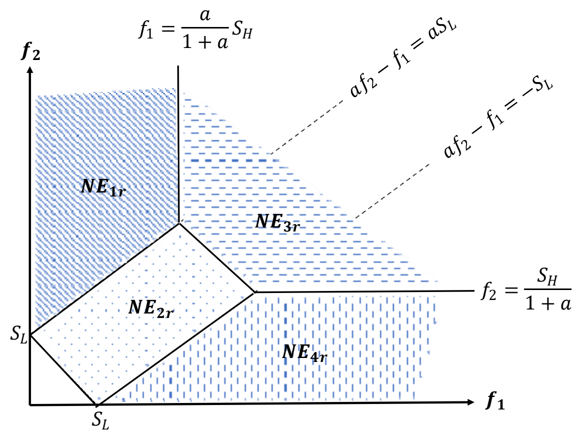
$$f_1 - f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha} \geq S_L$$

Proof of Propositions 3, 4, 5, and 6 is given in Appendix A.2.

We represent the Nash equilibrium obtained from Propositions 3, 4, 5, and 6 as NE_{1r}, NE_{2r}, NE_{3r} and NE_{4r} respectively. In Nash equilibrium NE_{1r} and NE_{4r} one airline operates on high demand leg only and the other on both the legs while in Nash equilibrium NE_{2r} and NE_{3r} both airlines provide services on both type of legs. The Nash equilibrium NE_{2r} exists when the total fleet size of the airlines is between S_L and $S_H + S_L$ and the NE_{4r} exists when the total fleet size of the airlines is greater than $S_H + S_L$.

To depict the region of existence of Nash equilibriums under restrictions we assume $a = \left(\frac{R_{H1}}{R_{H2}} \right)^{1/\alpha} \geq 1$. Figure 3 shows that all Nash equilibriums are non-overlapping and $f_1 + f_2 < S_L$ is the only region where Nash equilibrium does not exist.

Figure 3 Region of existence of Nash equilibrium NE_{1r}, NE_{2r}, NE_{3r} and NE_{4r} (see online version for colours)



4.5 *Proposition 7*

In the restricted model more flights are offered along the low demand leg compared to the unrestricted model.

Proof is provided in Appendix A.3.

5 Numerical analysis

Next, we compare Nash equilibrium strategies of models 1 and 2 under similar conditions through numerical illustrations.

We take $R_{L1} = 2 * 10^5$, $R_{L2} = 6 * 10^5$, $R_{H1} = 4 * 10^5$, $R_{H2} = 15 * 10^5$, $f_1 = 13$, $f_2 = 20$, $\alpha = 1.5$. We analyse the Nash equilibrium strategies for various values of S_L and S_H (Table 1). We observe that in the restricted model there are more number of flights on the low demand leg.

Table 1 Nash equilibrium strategies of the frequency game with variation in S_L and S_H

S_L	S_H	*Nash equilibrium under restriction (model 2)*					*Nash equilibrium without restriction (model 1)*					*% revenue loss*
		NE	*SL1*	*SH1*	*SL2*	*SH2*	*NE*	*SL1*	*SH1*	*SL2*	*SH2*	
9	25	NE_{2r}	5.969	7.03	3.03	16.96	NE_2	4.71	8.28	0	20	8.72
8	24	NE_{3r}	5.969	7.03	3.03	16.96	NE_2	4.71	8.28	0	20	8.72
8	23	NE_{3r}	6.26	6.737	3.73	16.26	NE_2	4.71	8.28	0	20	10.74

Next, we reduce the gap between R_{H1} and R_{H2} to study the Nash equilibrium strategies when the airlines have similar market share. We take $R_{L1} = 2 * 10^5$, $R_{L2} = 6 * 10^5$, $S_L = 9$, $S_H = 25$, $f_1 = 13$, $f_2 = 20$, $\alpha = 1.5$. We analyse the Nash equilibrium strategies by varying R_{H1} and R_{H2} (Table 2). We observe that when the market shares of the two airlines are almost equal, they operate equal number of flights on the high demand leg and the loss of revenue is very less compared to the uneven market share scenario.

Table 2 Nash equilibrium strategies of the frequency game with variation in R_{H1} and R_{H2}

R_{H1} (*10^5)	R_{H2} (*10^5)	Nash equilibrium under restriction (model 2)					Nash equilibrium without restriction (model 1)					% revenue loss
		NE	s_{L1}	s_{H1}	s_{L2}	s_{H2}	NE	s_{L1}	s_{H1}	s_{L2}	s_{H2}	
5	14	NE_{2r}	4.96	8.03	4.03	15.96	NE_2	2.93	10.06	0	20	11.44
6	13	NE_{2r}	4.02	8.97	4.97	15.02	NE_2	1.05	11.94	0	20	14.02
7	12	NE_{2r}	3.13	9.86	5.86	14.13	NE_1	0	13	1.37	18	11.17
8	11	NE_{2r}	2.26	10.73	6.73	13.26	NE_1	0	13	3.92	16.07	9.10
9	10	NE_{2r}	1.42	11.57	7.57	12.42	NE_1	0	13	6.05	13.94	5.50
9.5	9.5	NE_{2r}	1	12	8	12	NE_1	0	13	7	13	3.80

Next to analyse cases where Nash equilibrium NE_{1r} and NE_{4r} exist, we take $R_{L1} = 2 * 10^5$, $R_{L2} = 6 * 10^5$, $R_{H1} = 4 * 10^5$, $R_{H2} = 15 * 10^5$, $\alpha = 1.5$. The numerical analysis for two sets of S_L, S_H values are provided in Table 3. We observe that both restricted and unrestricted model give the same Nash equilibrium strategies and there is no revenue loss due to restriction.

Table 3 Nash equilibrium strategies under Nash equilibrium NE_{1r} and NE_{4r}

f_1	f_2	Nash equilibrium under restriction (model 2)					Nash equilibrium without restriction (model 1)					% revenue loss
		NE	s_{L1}	s_{H1}	s_{L2}	s_{H2}	NE	s_{L1}	s_{H1}	s_{L2}	s_{H2}	
$S_L = 4$, $S_H = 15$												
4	14	NE_{1r}	0	4	4.34	9.65	NE_1	0	4	4.34	9.65	0
9	10	NE_{4r}	4.85	4.14	0	10	NE_2	4.85	4.14	0	10	0
$S_L = 9$, $S_H = 25$												
7	26	NE_{1r}	0	7	9.10	16.89	NE_1	0	7	9.10	16.89	0
17	17	NE_{4r}	9.95	7.04	0	17	NE_2	9.95	7.04	0	17	0

It is thus observed that there are some situations where restrictions do not affect the Nash equilibrium strategies of airlines. Hence there is need to vary S_L and S_H values, or there should be another policy to restrict airlines to increase operations to remote locations.

The findings from numerical analysis indicate that if restrictions on minimum operation on low demand legs are imposed, there can be a maximum revenue loss of 10%–15%. These restrictions cause a revenue loss of around 4% when both airlines have equal market share. Hence, we can conclude that in a market where airlines have equal market share, restricting the minimum number of flights on low demand legs is a good policy to increase airline connectivity.

6 Conclusions

In this research, we have identified the problem of frequency competition on high demand legs and developed the frequency competition game between two airlines, each of which operates on two legs, one is a low demand leg, and the other is a high demand leg. The exact Nash equilibrium strategies of the frequencies on low and high demand legs have been derived. The model has been analysed by imposing frequency restrictions to increase airline connectivity to remote locations. Our study shows that such restrictions along the flight legs increases the frequency along the low demand legs and improves the effectiveness of the RCS scheme in addressing issues of connectivity in low demand sectors. The numerical analysis indicates that these restrictions work efficiently for an equal market share situation.

We have identified areas for future research which can extend the model presented in this study. One potential extension is to consider the flight frequencies as integer values rather than continuous variables, as it is more realistic in practice. Incorporating the cost of operations into the objective function could also provide a more comprehensive understanding of the impact on Nash equilibrium strategies. Additionally, examining demand patterns through additional constraints and studying the frequency competition game in a network of multiple airlines would be valuable areas of investigation. Design and analysis of new policies shall help in better understanding of their implications on airline connectivity to remote locations.

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Appendix

A.1 Proof of Propositions 1 and 2

The Lagrangian of optimisation problem of airline 1 (model 1) is represented as L_1 and given by:

$$L_1 = R_{L1}s_{L1} + R_{H1}s_{H1} + \lambda_1\left(s_{L1} + s_{H1} - f_1\right) + \lambda_2\left(R_{H1}s_{H2}^{\alpha} - R_{H2}s_{H1}^{\alpha}\right) - \lambda_3 s_{L1} - \lambda_4 s_{H1}$$

Here λ_1, λ_2, λ_3, λ_4 are the Lagrange multipliers of the constraints in the optimisation model of airline 1 in the respective order in which they appear. The KKT conditions are described below.

$$R_{L1} + \lambda_1 - \lambda_3 = 0$$

$$R_{H1} + \lambda_1 - \lambda_2 R_{H2} s_{H1}^{\alpha-1} - \lambda_4 = 0$$

$$\lambda_1 \left(s_{L1} + s_{H1} - f_1 \right) = 0$$

$$\lambda_3 s_{L1} = 0$$

$$\lambda_4 s_{H1} = 0$$

$$s_{L1} + s_{H1} \le f_1$$

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

$$s_{L1}, s_{H1} \ge 0$$

$$\lambda_1, \lambda_3, \lambda_4 \le 0$$

The solution of the above KKT conditions give rise to the desired two local maxima considering decision variable of airline 2 as fixed, which we describe next.

Local maxima 1:

In this case $\lambda_4 = 0$, $\lambda_1, \lambda_3 \ne 0$, and local maximum strategy of airline 1 is given by:

$$s_{L1} = 0, s_{H1} = f_1$$

whenever the following constraint is satisfied.

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

Local maxima 2:

In this case $\lambda_3, \lambda_4 = 0$, $\lambda_1 \ne 0$, and local maximum strategy of airline 1 is given by:

$$s_{H1} = \left(\frac{R_{H1}}{R_{H2}} \right)^{1/\alpha} s_{H2}$$

$$s_{L1} = f_1 - \left(\frac{R_{H1}}{R_{H2}} \right)^{1/\alpha} s_{H2}$$

whenever the following constraint is satisfied.

$$f_1 - \left(\frac{R_{H1}}{R_{H2}} \right)^{\frac{1}{\alpha}} s_{H2} \ge 0$$

In a similar way we find the local maxima of airline 2. The Nash equilibrium is obtained by solving local maxima of airline 1 and 2 simultaneously.

Solving local maxima 1 of airline 1 and local maxima 2 of airline 2 simultaneously give the Nash equilibrium strategies.

$$s_{L1} = 0, s_{H1} = f_1$$

$$s_{L2} = f_2 - f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}}, s_{H2} = f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}}$$

whenever the following constraint is satisfied.

$$f_2 - f_1 R_{H1}^{-1/\alpha} R_{H2}^{\frac{1}{\alpha}} \geq 0$$

This proves Proposition 1.

Solving local maxima 2 of airline 1 and local maxima 1 of airline 2 give the Nash equilibrium strategies.

$$s_{L1} = f_1 - f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha}, s_{H1} = f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha}$$

$$s_{L2} = 0, s_{H2} = f_2$$

whenever the following constraint is satisfied.

$$f_1 - f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/\alpha} \geq 0$$

This proves Proposition 2.

A.2 Proof of Propositions 3–6

The Lagrangian of the optimisation problem of airline 1 (model 2) is represented as L_2 and given by:

$$\begin{aligned} L_2 = {} & R_{L1}s_{L1} + R_{H1}s_{H1} + \lambda_1 (s_{L1} + s_{H1} - f_1) + \lambda_2 (R_{H1}s_{H2}^{\alpha} - R_{H2}s_{H1}^{\alpha}) + \lambda_3 (S_L - s_{L1} - s_{L2}) \\ & + \lambda_4 (s_{H1} + s_{H2} - S_H) - \lambda_5 s_{L1} - \lambda_6 s_{H1} \end{aligned}$$

Here λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 are the Lagrange multipliers of the constraints in the optimisation model of airline 1 in the respective order in which they appear. The KKT conditions are described below.

$$R_{L1} + \lambda_1 - \lambda_3 - \lambda_5 = 0$$

$$R_{H1} + \lambda_1 - \lambda_2 R_{H2} s_{H1}^{\alpha-1} + \lambda_4 - \lambda_6 = 0$$

$$\lambda_1 (s_{L1} + s_{H1} - f_1) = 0$$

$$\lambda_3 (S_L - s_{L1} - s_{L2}) = 0$$

$$\lambda_4 (s_{H1} + s_{H2} - S_H) = 0$$

$$\lambda_5 s_{L1} = 0$$

$$\lambda_6 s_{H1} = 0$$

$$s_{L1} + s_{H1} \le f_1$$

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

$$s_{L1} + s_{L2} \ge S_L$$

$$s_{H1} + s_{H2} \le S_H$$

$$s_{L1}, s_{H1} \ge 0$$

$$\lambda_1, \lambda_3, \lambda_4 \le 0$$

The solution of the above KKT conditions give rise to four desired local maxima considering decision variables of airline 2 as fixed, which we describe next.

Local maxima 1:

In this case, $\lambda_5, \lambda_6, \lambda_4 = 0$, $\lambda_1, \lambda_3 \ne 0$, and local maximum strategy of airline 1 is given by:

$$s_{L1} = S_L - s_{L2}$$

$$s_{H1} = f_1 + s_{L2} - S_L$$

whenever the following constraints are satisfied.

$$0 \le S_L - s_{L2} \le f_1$$

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

$$s_{L2} + s_{H2} \le S_H + S_L - f_1$$

Local maxima 2:

In this case, $\lambda_5, \lambda_6, \lambda_3 = 0$, $\lambda_1, \lambda_4 \ne 0$, and local maximum strategy of airline 1 is given by

$$s_{H1} = S_H - s_{H2}$$

$$s_{L1} = f_1 + s_{H2} - S_H$$

whenever the following constraints are satisfied.

$$0 \le S_H - s_{H2} \le f_1$$

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

$$s_{L2} + s_{H2} \ge S_H + S_L - f_1$$

Local maxima 3:

In this case, $\lambda_5, \lambda_6, \lambda_3, \lambda_4 = 0$, $\lambda_1 \ne 0$, and local maximum strategy of airline 1 is given by

$$s_{H1} = \left(\frac{R_{H1}}{R_{H2}}\right)^{1/\alpha} s_{H2}$$

$$s_{L1} = f_1 - \left(\frac{R_{H1}}{R_{H2}}\right)^{1/\alpha} s_{H2}$$

whenever the following constraints are satisfied.

$$f_1 - \left(\frac{R_{H1}}{R_{H2}}\right)^{\frac{1}{\alpha}} s_{H2} \geq 0$$

$$s_{L1} + s_{L2} \geq S_L$$

$$s_{H1} + s_{H2} \leq S_H$$

Local maxima 4:

In this case, $\lambda_3, \lambda_4, \lambda_5 = 0$, $\lambda_1, \lambda_5 \neq 0$, $s_{L1} = 0$ and the local maximum strategy of airline 1 is given by

$$s_{L1} = 0$$

$$S_{H1} = f_1$$

whenever the following constraints are satisfied.

$$R_{H1} s_{H2}^{\alpha} = R_{H2} s_{H1}^{\alpha}$$

$$s_{L1} + s_{L2} \geq S_L$$

$$s_{H1} + s_{H2} \leq S_H$$

In a similar way we find the local maxima of airline 2. The Nash equilibrium is obtained by solving local maxima of airline 1 and 2 simultaneously.

Solving local maxima 4 of airline 1 and local maxima 3 of airline 2 simultaneously give the Nash equilibrium strategies.

$$s_{L1} = 0, s_{H1} = f_1$$

$$s_{L1} = f_2 - f_1 R_{H1}^{-1/a} R_{H2}^{\frac{1}{\alpha}}, s_{H2} = f_1 R_{H1}^{-1/a} R_{H2}^{\frac{1}{\alpha}}$$

whenever the following conditions are satisfied.

$$f_1 \left(1 + R_{H1}^{-1/a} R_{H2}^{\frac{1}{\alpha}}\right) \leq S_H$$

$$f_2 - f_1 R_{H1}^{-1/a} R_{H2}^{\frac{1}{\alpha}} \geq S_L$$

This proves Proposition 3.

Solving local maxima 1 of airline 1 and local maxima 1 of airline 2 simultaneously give the Nash equilibrium strategies.

$$s_{L1} = f_1 - \frac{R_{H1}^{\frac{1}{\alpha}}(f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}, \; s_{H1} = \frac{R_{H1}^{\frac{1}{\alpha}}(f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

$$s_{L2} = -f_1 + \frac{R_{H1}^{\frac{1}{\alpha}}(f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}} + S_L, \; s_{H2} = \frac{R_{H2}^{\frac{1}{\alpha}}(f_1 + f_2 - S_L)}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

whenever the following conditions are satisfied.

$$f_1 + f_2 \geq S_L$$

$$f_1 + f_2 \leq S_H + S_L$$

$$f_1 R_{H2}^{\frac{1}{\alpha}} + (S_L - f_2) R_{H1}^{\frac{1}{\alpha}} \geq 0$$

$$f_2 R_{H1}^{\frac{1}{\alpha}} + (S_L - f_1) R_{H2}^{\frac{1}{\alpha}} \geq 0$$

This proves the Proposition 4.

Solving local maxima 2 of airline 1 and local maxima 2 of airline 2 simultaneously give the Nash equilibrium strategies.

$$s_{L1} = f_1 - \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}, \; s_{H1} = \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

$$s_{L2} = f_2 - S_H + \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}, \; s_{H2} = S_H - \frac{R_{H1}^{\frac{1}{\alpha}} S_H}{R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}}}$$

whenever the following constraints are satisfied.

$$f_1 + f_2 \geq S_L + S_H$$

$$f_1 \left(R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}} \right) \geq R_{H1}^{\frac{1}{\alpha}} S_H$$

$$f_2 \left(R_{H1}^{\frac{1}{\alpha}} + R_{H2}^{\frac{1}{\alpha}} \right) \geq R_{H2}^{\frac{1}{\alpha}} S_H$$

This proves Proposition 5.

Solving local maxima 3 of airline 1 and local maxima 4 of airline 2 simultaneously give the Nash equilibrium strategies.

$$s_{L1} = f_1 - f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/a}, \; s_{H1} = f_2 R_{H1}^{\frac{1}{\alpha}} R_{H2}^{-1/a}$$

$$s_{H2} = f_2, \; s_{L2} = 0$$

whenever the following constraints are satisfied.

$$f_2\left(1+R_{H1}^{\frac{1}{\alpha}}R_{H2}^{-1/a}\right)\le S_H$$

$$f_1-f_2R_{H1}^{\frac{1}{\alpha}}R_{H2}^{-1/a}\ge S_L$$

This proves Proposition 6.

Note: other combinations of local maxima are not feasible, so we only have four Nash equilibria.

A.3 Proof of Proposition 7

We show that in each of the Nash equilibrium strategies NE_{1r}, NE_{2r}, NE_{3r}, and NE_{4r} at least as many flights are scheduled on the low demand leg as in the strategies corresponding to NE_1 or NE_2.

NE_{1r}:

Constraints corresponding to Nash equilibrium NE_{1r} (Proposition 3) in model 2 (restricted model) implies that NE_1 will be the Nash equilibrium in model 1 (unrestricted model). In these Nash equilibrium strategies, the total number of flights offered by the airlines on the low demand leg, i.e., $s_{L1}+s_{L2}$ are the same.

NE_{2r}:

Let $\frac{R_{H2}^{\frac{1}{\alpha}}}{R_{H1}^{\frac{1}{\alpha}}}=a.$ The corresponding constraints of Proposition 4 become:

$$-S_L\le af_1-f_2\le aS_L$$

If $-S_L\le af_1-f_2\le 0$ then NE_1 is the Nash equilibrium of the unrestricted model. In NE_1 the total number of flights on the low demand leg equals $f_2-af_1\le S_L$. However, in the restricted model total S_L number of flights is offered on low demand leg.

If $0\le af_1-f_2\le aS_L$ then NE_2 is the Nash equilibrium of the unrestricted model. In NE_2 the total number of flights on the low demand leg equals $f_1-\frac{f_2}{a}\le S_L$. However, in the restricted model total S_L number of flights is offered on low demand leg.

NE_{3r}:

From the constraints of Proposition 5, we can write:

$f_1=\frac{S_H}{1+a}+\Delta_1$ where Δ_1 is the surplus variable of the constraint.

$f_2 = \frac{aS_H}{1+a} + \Delta_2$ where Δ_2 is the surplus variable of the constraint.

If $af_1 - f_2 \geq 0$, NE_2 will be the Nash equilibrium of the unrestricted model. The total number of flights offered by airlines along the low demand leg are given by:

$$f_1 - \frac{f_2}{a} = \Delta_1 - \frac{\Delta_2}{a}$$

If $af_1 - f_2 \leq 0$, NE_1 will be the Nash equilibrium of the unrestricted model. The total number of flights offered by airlines along the low demand leg is given by:

$$f_2 - af_1 = \Delta_2 - a\Delta_1$$

However, the total number of flights offered by airlines along the low demand leg corresponding to Nash equilibrium NE_{3r} strategies is:

$$f_1 + f_2 - S_H = \Delta_1 + \Delta_2$$

NE_{4r}:

Constraints corresponding to Nash equilibrium NE_{4r} in the restricted model imply that NE_2 will be the Nash equilibrium in the unrestricted model. In these Nash equilibrium strategies, the total number of flights offered by the airlines on the low demand leg, i.e., $s_{L1} + s_{L2}$ are the same.