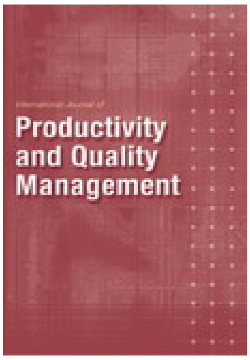

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Abstract: This study addresses ratio data envelopment analysis (DEA-R) models to measure efficiencies of units in a time span covering multi-periods by considering operations of individual periods. In particular, overall and periodic efficiencies can be evaluated simultaneously. The overall efficiency of the proposed model depends on performance of DMUs in all periods. Notably, the proposed model has three main features. First it can identify the pseudo-inefficiency. Second, the proposed overall efficiency measure is depended on all periods. Third, the proposed method is endowed with a high discriminatory power in differentiating the units as efficient and inefficient ones. To expand the present study, a comparison was made between the existing model in the literature and the proposed DEA-R model and efficiency of 22 Taiwanese commercial banks was measured for a period from 2009 to

2011. The three-year results show that overall score of efficiency in the proposed multi-period DEA-R model is greater than or equal to total efficiency of the existing multi-period model.

Keywords: ratio data envelopment analysis; DEA-R; multi-periodic production process; overall efficiency; pseudo-inefficiency.

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1 Introduction

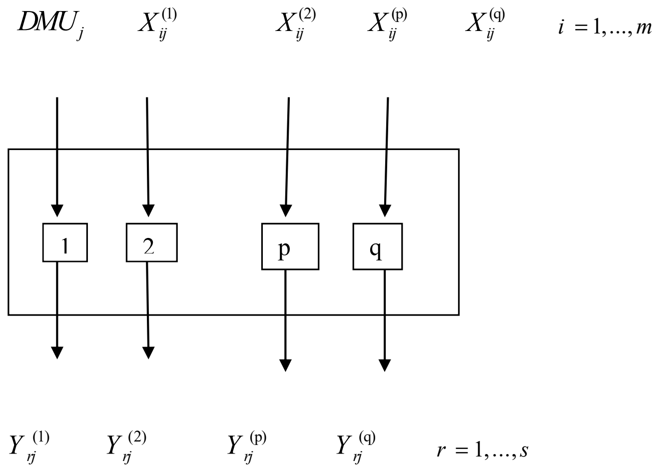
As a non-parametric technique, data envelopment analysis (DEA) uses a ratio of the weighted sum of outputs to the weighted sum of inputs to evaluate the relative efficiency of a set of homogenous decision-making units (DMUs). Specially, the standard DEA models maximises the efficiency provided that this ratio is equal to one or less for any DMU. Although the idea of measuring relative efficiency was introduced by Farrell (1957), the first DEA models, as they are currently conceived, were formulated by Charnes et al. (1978) assuming constant returns to scale and by Banker et al. (1984) assuming variable returns to scale. Later it was found that this technique is applicable in various realm for instance in profit-driven companies such as banks (Paradi and Zhu, 2013; Asmild et al., 2013; Saljoughian et al., 2019) manufacturing companies (Hwang et al., 2013; Sharma et al., 2022; Agarwal and Mehrotra, 2020), hospitals (Chang et al., 2011; Tavana et al, 2021), management context (Niknazar and Bourgault, 2018;

Nazari-Shirkouhi and Keramati, 2017; Yazdi et al., 2018), safety performance (Mozaffari et al., 2021) and retails stores (Assaf, 2011). DEA is a non-parametric technique used to measure performance. Consumption of total inputs and production of total outputs are summed up when the case examination focus is on time periods constituting clearly defined time units, such as, years. In 1999, Nemoto and Goto (1999) presented a dynamic model to evaluate efficiency of a multistage production system. For cases where the period is evaluated, the total consumption inputs and total generated outputs are considered to measure efficiency during each period. Upon the point of departure being the DEA unit-invariant property (Lovell and Pastor, 1995), equal efficiencies will be rendered regarding the total and average calculations from different data types. (Kao and Hwang, 2008; Portela et al. 2012). When using the aggregate data throughout the entire period, efficiency results from an overall measure of the designated time periods of which efficiency remains unknown. In this case, an overall efficient DMU may not be efficient in each period. On the other hand, the possibility of overall inefficiency while the unit is efficient in all periods, may provide clues about pseudo inefficiency. Hence, known period-specific efficiencies would prove informative. Separate calculation of a specific period will contribute to its efficiency identification. Since the discrepancy of peer groups in each period for efficiency measurement can lend a support to different efficiency measurement, the efficiency scores are not comparable among different periods. In this regard, Kao and Liu (2014) discussed the performance measurement in multi-period production process. To measure periodic and overall efficiencies simultaneously, authors have proposed a model of relational network. The main concept in that model concentrate on individual periods operation in efficiency measurement. Interestingly, the overall efficiency is formed by weighted average of the periodic efficiency; these weights are used in evaluation of DMU. Efficiency of twenty-two commercial banks in Taiwan was measured by this model for the period from 2009 to 2011. On the other hand, the overall efficiency of each unit extracted from their proposed approach is between zero and one i.e., (0, 1), hence the approach disables in differentiating the efficient and inefficient ones. However, the relation between the specific-periodic efficiency and the overall efficiency to discriminate the efficient and inefficient units, arise as a question. One way to solve this problem is employing DEA-R. Considering data in the form of ratio enables us to overcome this problem. Recently, Wei et al. (2021) attempted to provide an efficiency measurement in multi-period network DEA model with feedback but they applied a binary heuristic algorithm to obtain the optimal efficiency. DEA-R models were first formulated by Despic et al. (2007) which combine DEA and ratio analysis; since then, such models have been studied and applied by many other researchers. By employing DEA models on ratio-based data, the authors found the relationship between arithmetic mean, geometric mean, and weight in efficiency value. Wei et.al (2011a, 2011b, 2011c) extended the theory of DEA-R models. They focused on relations between traditional DEA models and ratio-based DEA-R models and analysed efficiency of 21 Taiwanese hospitals by using DEA-R models. The authors analysed Pseduo-inefficiency in these units. Liu et al. (2011) studied and verified DEA-R models without using explicit inputs in 15 Chinese research institutes. They presented a different approach focusing on definition of production possibility set and measuring technical efficiency. Based on this axiomatic foundation, they developed the input-oriented DEA-R models with assumption of constant return to scale to evaluate efficiency and super efficiency. Mozaffari et al. (2014) used both DEA and DEA-R models to discuss cost and revenue efficiency and examined the relationship between DEA models without using explicit input; Mozaffari et al. (2014a, 2014b, 2020) specified production possibility set using axioms in CRS technology for DEA-R, and, finally an original algorithm for identification of efficient surfaces in this class of models is proposed. Olesen et al. (2015) demonstrated the problems with ratio data after classifying them, defined a production possibility set and introduced the corresponding models in CRS/VRS and provided a positive answer to the existing debate with regard to the use of DEA models for ratio data. Olesen et al. (2017) also discussed the method by which DEA models are solved with ratio data and introduced a new type of potential ratio (PR) inefficiency. Recently, Kamyab et al. (2021) developed a DEA-R-based CRA models to evaluate commercial banks in a two-stage incentives system. Thirteen commercial banks modeled as two-stage networks were evaluated by the models proposed in two different cases of ratio data. Results suggest that the proposed methodology yields more accurate efficiency scores, thus allowing better discrimination among DMUs. Mozaffari et al. (2022) introduced a DEA-R based approach to take into account the managerial preferences. They presented a multi-objective linear programming (MOLP) model for evaluating efficiency based on defining the production possibility set in the presence of ratio data and to get the corresponding benchmark to each decision-making unit DMU. Using the target setting by manager among the solutions resulting from the MOLP problem, they choose best solution according to the managers' preferences as benchmark. All these researches and others besides their applications in real world reflect the importance of this issue in DEA literature. The paper tends to develop a multi-period production system in order to measure the overall efficiency of a DMU set in a period of time via adopting a DEA-R approach, and evaluating each specific period efficiency. The proposed model can measure the overall and periodic efficiencies simultaneously, and thus identify Pseudo-inefficiency in multi-periodic evaluation. Since, some inputs may not play a role in producing some outputs, based on the weight concepts in multi-periodic system; the model is developed to detect the role of active inputs. The rest of the study is organised as follows. Basic concepts of multi-period production system are reviewed in Section 2. DEA-R is briefly summarised in Section 3; then, the approach for dealing with multi-period system based on DEA-R models is introduced. Section 4 illustrates the applicability of the proposed model with a real numerical example. Conclusion will end the paper.

2 Preliminaries

Evaluating efficiency in multi-period models has attracted considerable attention among researchers. To measure DEA efficiency, assume n *DMUs*; performance of each is determined by production process of m inputs, X_{ij} ($i = 1, \ldots, m$), to yield s outputs, Y_{ij} ($r = 1, \ldots, s$) . As shown in Figure 1, assume a multi-period system composed of q periods, $X_{ij}^{(p)}$ and $Y_{ij}^{(p)}$, where p ($p = 1, \ldots, q$) denotes the corresponding period. Final values of i^{th} input and r^{th} output for DMU_j ($j = 1, \ldots, n$) in all q periods are $X_{ij} = \sum_{p=1}^{q} X_{ij}^{(p)}$ and $Y_{ij} = \sum_{p=1}^{q} X_{ij}^{(p)}$.

Figure 1 The structure of multi period system



Olesen et al. (2017) and Kao (2009), have conducted the standard CCR equation (1) to evaluate the efficiency of a particular period p (p = 1, , q) separately using the data for that period to equation (1). Efficiency of DMU_k is measured by standard DEA equation, also known as CCR, as follows:

$$\begin{aligned}
&E_k^{CCR} = Max \sum_{r=1}^{s} u_r Y_{rk} \\
&s.t. \\
&\sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \leq 0 \quad , \ j = 1,\ldots,n \\
&\sum_{i=1}^{m} v_i X_{io} = 1 \\
&u_r \geq 0, v_i \geq 0 \quad r = 1,\ldots,s \ \ , \ i = 1,\ldots,m.
\end{aligned} \tag{1}$$

This equation is a CRS program, where v_i and u_r are corresponding weights of the i^{th} input and the r^{th} output, respectively. Since overall efficiency of a system can be evaluated by using total input, $X_{ij} = \sum_{p=1}^{q} X_{ij}^{(p)}$ and total output, $Y_{rj} = \sum_{p=1}^{q} Y_{rj}^{(p)}$ in the time period by CCR equation (1) regardless of operations in individual periods, overall efficiency of a unit was calculated in a certain period of time using the Aggregate equation. The equation has the following format:

$$E_k^{AGR} = Min \quad \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$
$$s.t.$$
$$\sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = \theta X_{ik}, \quad i = 1,\dots,m \tag{2}$$
$$\sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = Y_{rk}, \qquad r = 1,\dots,s$$
$$\lambda_j, s_i^-, s_r^+ \ge 0 \quad j = 1,\dots,n, \;\; i = 1,\dots,m, \;\; r = 1,\dots,s$$

Overall efficiency of a DMU is only measured by equation (2) in a single period of time. For measuring the overall efficiency of q periods individually, Park and Park (2009) extended the euation (2) by conceptualising Farrell's technical efficiency.

$$E_k^{PP} = \min \quad \theta - \varepsilon\left(\sum_{p=1}^{q}\sum_{i=1}^{m} s_i^{-(p)} + \sum_{p=1}^{q}\sum_{r=1}^{s} s_r^{+(p)}\right) s.t.$$
$$\sum_{j=1}^{n} \lambda_j^{(p)} X_{ij}^{(p)} + s_i^{-(p)} = \theta X_{ik}^{(p)}, \; p = 1, \dots, q, \; i = 1, \dots, m \tag{3}$$
$$\sum_{j=1}^{n} \lambda_j^{(p)} Y_{rj}^{(p)} - s_r^{+(p)} = Y_{rk}^{(p)}, \; p = 1, \dots, q, \; r = 1, \dots, s$$
$$\lambda_j^{(p)}, \; s_i^{-(p)}, \; s_r^{+(p)} \ge 0, \; p = 1, \dots, q, \; r = 1, \dots, s, \; i = 1, \dots, m, \; j = 1, \dots, n$$

It is noteworthy that the equation (2) is specialisation of the equation (3) with the variable intensity, $\lambda_j^p \,(p = 1,.., q, \; j = 1,.., n)$, for each period as independent process is modelled through the use of slack variables in the constraints. Notably, the equation (3) is adopted from network DEA model proposed by Färe and Grosskopf (2000) for the system shown in Figure 1. Since, these periods are connected with a unique distance measure of, θ equation (3) is called the connected network model. Overall efficiency calculated by the equation (3) is relatively large. That is, an individual period of an overall efficient DMU is the best-performing period; the largest value of θ which is used in constraint $\sum_{j=1}^{n} \lambda_j^{(p)} X_{ij}^{(p)} + s_i^{-(p)} = \theta X_{ik}^{(p)}, \;\; p = 1,\dots q, \;\; i = 1,\dots,m$ is compensated to adjust the slack variable $s_i^{-(p)}$ in corresponding constraint with smaller distance measure. In other words, overall efficiency which is the measure of distance of the best-performing period is adjusted by the effect of slack variables, ε. Considering this effect, a *DMU* is overall efficient if only all the periods are efficient. According to Figure 1, Kao and Liu (2014) has developed the relational network model, provided that each period, which is a part of a network system, invokes a parallel system structure consisting of *q* processes. The model has the following format:

$$
\begin{aligned}
&E_k^{KL} = \max \sum_{r=1}^{s} u_r Y_{rk} \\
&s.t \\
&\sum_{i=1}^{m} v_i X_{ik} = 1 \\
&\sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \le 0,\ j = 1, ..., n \\
&\sum_{r=1}^{s} u_r Y_{rj}{}^{(p)} - \sum_{i=1}^{m} v_i X_{ij}{}^{(p)} \le 0,\ j = 1, ..., n,\ p = 1, ..., q \\
&u_r, v_i \ge \varepsilon, r = 1, ..., s,\ i = 1, ..., m
\end{aligned}
\tag{4}
$$

The main characteristics of the equation (4) can be stated as follows. First, in this equation (4), the weights related to similar factors are identical with respect to the corresponding period. In other words $Y_{rj}^{(p)}$ and $X_{ij}^{(p)}$ of different periods of p (p = 1, 2, …, q) have the same multiplier u_r , and v_i, respectively. Second, overall efficiency is calculated for the multiperiod system by considering both inputs and outputs as well as their corresponding periods by adding the constraints, $\sum_{r=1}^{s} u_r Y_{rj}^{(p)} - \sum_{i=1}^{m} v_i X_{ij}^{(p)} \le 0, (p = 1, ..., q)$ in the equationifically, overall efficiency of the equation (4) will not be greater than the calculated efficiency of the aggregate equation (2), because a third constraint has been added. Since, the sum of constraints of q period, i.e., $\sum_{r=1}^{s} u_r Y_{rj}^{(p)} - \sum_{i=1}^{m} v_i X_{ij}^{(p)} \le 0, (p = 1, ..., q)$ is the system constraint $\sum_{r=1}^{s} u_r Y_{rj}$ $-\sum_{i=1}^{m} v_i X_{ij} \le 0,$ so, the latter can be removed. Applying the optimal solutions $u_r^*, v_i^*,$ the overall efficiency $E_{overall}$ and each period efficiency $E_k^p (p = 1, ..., q)$ are calculated as follows:

$$
E_{overall} = \frac{\sum_{r=1}^{s} u_r^* Y_{rk}}{\sum_{i=1}^{m} v_i^* X_{ik}} = \sum_{r=1}^{s} u_r^* Y_{rk} \tag{5}
$$

$$
E_i^{(p)} = \frac{\sum_{r=1}^{s} u_r^* Y^{(p)}{}_{rk}}{\sum_{i=1}^{m} v_i^* X^{(p)}{}_{ik}},\ p = 1, ..., q \tag{6}
$$

As Kao and Liu (2014) stated , by setting the weight $w^{(p)} = \frac{\sum_{i=1}^{m} v_i^* X_{ij}^{(p)}}{\sum_{i=1}^{m} v_i^* X_{ij}}$, the proportion of the aggregate input consumed in period p in that of all periods, the overall efficiency is the average of the q period efficiencies weighted by $w^{(p)}$, that is, $E_{overall} = \sum_{p=1}^{q} w^{(p)} E_i^{(p)}$.

So, overall efficiency and periodic efficiency can be calculated by the equation (4) in the multiperiod system. However, the existence of multiple solutions is one the concerns in the equation (4) that examined in Kao and Liu (2014). These before mentioned concepts are employed to formulating ratio data in a multi-period production process.

3 Proposed approach for efficiency measurement

In this section the CCR model consists of ratio-based data are stated. Then we introduce the development of this model on multi-period systems as shown in Figure 1.

3.1 DEA-R input oriented CCR model

Again assuming DMUs, let the observed data of inputs and outputs be $X_j = (x_{1j}, \ldots, x_{mj}) > 0$ and $Y_j = (y_{1j}, \ldots, y_{sj}) > 0$ for DMUj(j = 1, …, n). Also assuming the ratios $\frac{x_{ij}}{x_{io}}$ and $\frac{y_{rj}}{y_{ro}}$ are defined. Despic et al. (2007) have introduced their DEA-R efficiency model for evaluating DMU_o by CRS, as follows:

$$\begin{aligned} &\hat{e}_o = Max_{w_{ir}} Min_j \sum\nolimits_{i=1}^{m} \sum\nolimits_{r=1}^{s} w_{ir} \frac{\left(X_{ij} / Y_{rj}\right)}{\left(X_{io} / Y_{ru}\right)} \\ &s.t. \quad \sum\nolimits_{i=1}^{m} \sum\nolimits_{r=1}^{s} w_{ir} = 1 \\ &\qquad w_{ir} \geq 0 \quad i = 1, \ldots, m, r = 1, \ldots, s \end{aligned} \tag{7}$$

The model assumes that x_{io} and y_{ro} are the input and output vectors of under evaluated unit, respectively, and w_{ir} represents the relative weight of i^{th} input and r^{th} output of this vector.

Definition 1 The under evaluated unit (DMU_o) is efficient if and only if the optimal objective function value of CCR-R-I i.e., $\hat{e}_o^* = 1$, otherwise it is inefficient.

It is proved that the efficiency scores (optimal objective function values) given by equation (7) is better than or equal to efficiency scores given by standard CCR model (Mozaffari, 2012).

3.2 DEA-R model in multi period production process

Again let *n* DMUs (DMU_j (j = 1, …, n)) be described by m inputs, X_{ij} (i = 1, …, m), and s outputs, Y_{ij} (r = 1, …, s). Assume a multi-period production process (Figure 1), where the subscript p (p = 1, …, q) represents the corresponding period. In what follows, overall efficiency of a multiperiod production process is calculated by an alternative model based on DEA-R models. To do so, we toy around both DEA-R models and the related models for multi-period systems. DEA-R models can be used to analyse multiperiod systems by measuring overall and periodic efficiencies and achieving the corresponding optimal period weights for each period. As noted before, the value of the i^{th} input and r^{th} output of DMUj (j = 1, …, n) is $X_{ij} = \sum_{p=1}^{q} X_{ij}^{(p)}$ and $Y_{rj} = \sum_{p=1}^{q} Y_{rj}^{(p)}$, respectively, in all q periods. Also, suppose that the summation of the optimal weights of all periods is equal to unity. By these assumptions, the multi-period ratio model (MPR) has the following feature:

$$
\begin{aligned}
&\max \theta \\
&s.t. \sum\nolimits_{i=1}^{m} \sum\nolimits_{r=1}^{s} w_{ir} \frac{\left(x_{ij} / Y_{rj}\right)}{\left(x_{io} / Y_{ru}\right)} \geq \theta && j = 1, \ldots, n \\
&\sum\nolimits_{i=1}^{m} \sum\nolimits_{r=1}^{s} w_{ir}^{(p)} \frac{\left(x_{ij}^{(p)} / y_{rj}^{(p)}\right)}{\left(x_{io}^{(p)} / y_{ro}^{(p)}\right)} \geq \theta && j = 1, \ldots, n,\ j \neq o,\ p = 1, \ldots, p \\
&\sum\nolimits_{i=1}^{m} \sum\nolimits_{r=1}^{s} w_{ir}^{(p)} = 1 && p = 1, \ldots, q \\
&w_{ir} w_{ir}^{(p)}, \theta \geq 0, && i = 1, .., m,\ r = 1, \ldots, s,\ p = 1, \ldots, q
\end{aligned}
\tag{8}
$$

Notably, the optimal objective function i.e., θ^* in the equation (8) only calculates the overall efficiency over the whole period. An up-close exploration of equation (8) shows that it is a combination of both multi-period DEA models and DEA-R. The proposed equation (8) has the following characteristics:

1. In optimality we always have $\theta^* \leq 1$.
2. The equation is always feasible.
3. The efficiency scale in equation (8) determines the evaluation DMU_o with ratio $\frac{X_o}{Y_o}$.
4. For ratio-based data, efficiency scale can only be calculated by DEA-R models rather than DEA models.
5. The equation (8) can easily detect Pseudo-inefficiency of some DMUs. In fact, it is possible to have a DMU that is effective in all periods, but the overall efficiency of this DMU may evaluate it ineffective.

As seen, equation (8) obtains the overall efficiency of the under evaluated DMU. However, the existing of zero-weights in the optimal solutions of equation (8), disregards the effectiveness of all inputs and outputs in computing the overall efficiency. To overcome this problem we propose the following procedure:

1 Apply equation (8) for each period separately and extract the periodic weights $\left(w*_{ir}^{(p)}\right)$ and efficiency (θ*) for each period.

2 Compute $w^{(p)}$ by selecting nonzero weights and multiplying them by corresponding inputs, i.e., $w^{(p)} = \sum_{i=1}^{m} w_{ir}^{*(p)} x_i$

3 Define $w^{*(p)}$ as the optimal periodic weight as $w^{*(p)} = \frac{w^{(p)}}{\sum_{p=1}^{q} w^{(p)}}$.

4 Calculate the overall efficiency of the evaluated DMU as $\bar{\theta}^* = \sum_{p=1}^{q} w^{*(p)} * \theta^{*(p)}$.

$\bar{\theta}^*$ is the overall efficiency score in the multi-period ratio equation (8).

In what follows, it is proved that overall efficiency of the equation (8) is always greater than or equal to overall efficiency of the equation (4).

Theorem 1 The overall efficiency score extracted from the multi-period ratio equation (8) i.e., $\bar{\theta}^*$, is greater than or equal the overall efficiency multi-period equation (4).

Proof: See Appendix.

4 Numerical examples

Empirical analysis shows that the proposed model is applicable. In order to shed a light on the suggested framework, we apply the proposed model on a real case consisting of 22 commercial banks taken from Kao and Liu (2014). Table (1) reports the dataset.

As Table 1 records, three input factors and three output factors were appointed. The first three columns report input variables as labour (IN1), Physical capital (IN2) and Purchased funds (IN3) and the rest columns are output variables demand deposits (OUT1), s-term loans (OUT2) and ml-term loans (OUT3). The data set are recorded over three time periods (2009, 2010, 2011). Employing equations (2), (3) and (4) overall efficiency and the efficiency of each period as well as the optimal weights for each period are calculated. Table 2 lists the results.

In Table 2, the second column reports the efficiency calculated by the aggregate equation (2). Scores of overall efficiency calculated by the connected network equation (3) are listed in the third column. Columns four to seven show the overall and periodic efficiency for three consecutive years by applying relation network equation (4). Weights of the corresponding periods are reported in parentheses.

Table 1 Dataset

DMU		*Year*	*IN1*	*IN2*	*IN3*	*OUT1*	*OUT2*	*OUT3*
1	Chang hwa	2011	8.58	23.51	973.32	303.99	320.61	805.60
		2010	8.67	23.78	933.48	298.78	322.96	723.61
		2009	7.64	24.24	938.09	267.82	263.18	739.54
2	King's town	2011	0.84	2.65	115.01	21.08	25.09	70.52
		2010	0.88	2.90	116.59	18.02	25.95	70.88
		2009	0.92	3.17	119.33	15.89	23.50	71.25
3	Taichung	2011	1.94	3.34	260.14	73.55	82.27	196.63
		2010	1.79	3.23	238.21	64.37	73.48	171.07
		2009	1.68	3.56	217.73	58.65	76.66	139.90
4	Taiwan business	2011	7.00	13.76	786.33	247.20	262.90	677.74
		2010	6.76	14.19	751.26	240.54	232.17	686.72
		2009	6.43	14.51	725.53	238.10	273.47	644.01
5	Kaohsiung	2011	1.25	2.20	155.64	21.72	71.37	95.15
		2010	1.27	2.24	149.68	20.15	72.61	89.06
		2009	1.16	2.28	127.67	18.17	55.94	81.31
6	Cosmos	2011	1.91	6.08	99.52	12.88	32.76	43.62
		2010	1.89	6.16	94.83	13.60	36.63	32.71
		2009	2.20	6.52	92.88	10.30	33.19	38.98
7	Union	2011	2.57	7.97	273.52	47.26	50.66	139.48
		2010	2.43	8.10	246.22	41.73	42.95	145.19
		2009	2.39	8.31	233.58	36.28	29.08	136.22
8	Far eastern	2011	2.99	2.94	313.52	32.00	67.65	206.14
		2010	2.83	2.88	292.52	30.17	63.96	175.10
		2009	2.25	2.24	276.12	29.56	48.07	165.80
9	Ta chong	2011	3.78	2.99	287.15	46.05	65.99	214.37
		2010	3.76	4.66	238.04	51.11	69.73	188.77
		2009	3.44	5.24	238.50	46.66	60.11	171.15
10	En tie	2011	2.03	1.41	223.27	24.09	42.15	156.67
		2010	1.72	2.16	208.01	23.46	40.01	157.23
		2009	1.42	2.79	175.89	19.91	39.64	146.42
11	Hua nan	2011	9.86	28.01	1138.20	474.04	416.39	888.58
		2010	9.45	25.91	1106.80	458.11	425.02	811.31
		2009	9.17	23.11	1094.84	408.93	372.19	716.07
12	Fubon	2011	7.04	10.94	944.66	215.07	243.92	709.48
		2010	6.51	12.07	907.23	265.58	215.67	638.85
		2009	6.17	12.39	887.51	191.89	202.72	624.30

Table 1 Dataset (continued)

DMU		*Year*	*IN1*	*IN2*	*IN3*	*OUT1*	*OUT2*	*OUT3*
13	Cathay	2011	8.22	24.93	1232.79	249.07	264.84	741.54
	2010	7.56	25.46	1113.09	234.74	227.25	677.38	
	2009	7.23	26.43	1059.43	249.63	192.00	629.22	
14	East sun	2011	4.80	15.60	741.16	211.89	148.12	511.01
	2010	4.37	14.19	253.84	194.15	121.80	476.90	
	2009	3.83	13.86	206.14	165.74	90.35	459.00	
15	Yuanta	2011	2.85	2.54	382.03	55.63	73.85	278.40
	2010	2.59	2.62	321.93	59.42	59.52	233.04	
	2009	2.31	2.68	254.72	42.18	44.64	200.75	
16	mega	2011	10.94	14.18	1092.21	488.22	418.62	1037.72
	2010	11.23	14.02	1074.03	470.93	405.56	919.92	
	2009	9.76	15.64	1045.93	427.16	349.10	919.25	
17	Taishin	2011	7.94	17.53	614.69	130.82	132.24	466.17
	2010	6.98	24.33	573.01	131.92	123.48	407.91	
	2009	6.54	24.96	536.50	111.40	109.72	383.79	
18	Shin kong	2011	3.26	6.01	403.70	76.03	88.92	281.22
	2010	2.83	6.23	346.86	65.25	69.96	255.58	
	2009	2.64	6.36	310.47	63.82	47.63	235.01	
19	Sino pac	2011	6.96	9.60	769.00	197.88	156.79	576.29
	2010	6.75	8.95	739.72	186.08	156.14	557.95	
	2009	6.51	9.15	669.00	176.42	146.94	524.88	
20	China trust	2011	17.29	33.96	902.88	126.37	285.69	780.41
	2010	17.01	33.02	871.78	114.28	267.01	729.38	
	2009	14.89	33.84	791.39	116.41	228.07	691.54	
21	first	2011	10.38	25.56	1184.16	416.12	437.39	918.15
	2010	9.60	22.87	1196.79	409.69	409.59	840.06	
	2009	9.22	22.83	1154.62	363.26	317.89	775.09	
22	Taiwan cooperative	2011	13.08	35.39	1753.07	386.84	433.96	1,511.29
	2010	13.00	33.66	1704.62	373.01	338.50	1439.46	
	2009	12.66	33.89	1668.48	342.02	357.91	1428.26	

As seen the periodic efficiency of DMU#10 in 3 periods are calculated as 1.0000, 0.990 and 1.0000, respectively. As stated before, W_{ir} represents the relative weight of i^{th} input and r^{th} output of input and output vector variables. By selecting nonzero weights and then multiplying them by corresponding inputs, i.e.,

w^{2009} $w_{13}x_1 + w_{22}x_2 + w_{23}\,x_2 = 1.7154$

w^{2010} $w_{13}x_1 + w_{22}x_2 + w_{23}x_2 + w_{33}x_3 = 37.6862$

w^{2011} $w_{22}x_2 + w_{23}x_2 = 1.4076$

Table 2 Overall efficiency and periodic efficiency

Banks		Aggregate model (2)	Connected network model (3)	Relation network model(4)			
				Overall	2009	2010	2011
1	Chang Hwa	0.9362	0.9472	0.8981	0.9083 (0.3079)	0.8660 (0.3475)	0.9213 (0.3446)
2	King's Town	0.7809	0.8060	0.7457	0.7022 (0.3179)	0.7692 (0.3386)	0.7628 (0.3435)
3	Taichung	1.0000	1.0000	0.9721	0.9584 (0.3116)	0.9545 (0.3306)	1.0000 (0.3578)
4	Taiwan business	1.0000	0.9988	0.9681	1.0000 (0.3213)	0.9530 (0.3344)	0.9530 (0.3443)
5	Kaohsiung	1.0000	1.0000	0.9731	0.9127 (0.3081)	1.0000 (0.3440)	1.0000 (0.3479)
6	Cosmos	0.7868	0.8113	0.7361	0.7365 (0.3234)	0.7960 (0.3302)	0.6786 (0.3465)
7	Union	0.5304	0.5635	0.5067	0.4690 (0.3223)	0.5332 (0.3292)	0.5159 (0.3486)
8	Far Eastern	0.8887	0.9963	0.7591	0.7691 (0.2991)	0.7115 (0.3416)	0.7960 (0.3593)
9	Ta Chong	0.7997	0.8653	0.7202	0.6177 (0.3477)	0.7174 (0.3304)	0.8337 (0.3219)
10	En Tie	0.9595	0.9997	0.9018	0.9204 (0.3117)	0.9129 (0.3373)	0.8744 (0.3510)
11	Hua Nan	1.0000	1.0000	0.9754	0.9252 (0.3201)	1.0000 (0.3323)	0.9981 (0.3476)
12	Fubon	1.0000	0.9979	0.9680	0.9480 (0.3205)	0.9541 (0.3335)	1.0000 (0.3460)
13	Cathay	0.8538	0.8629	0.8173	0.8010 (0.3142)	0.8180 (0.3285)	0.8311 (0.3573)
14	East Sun	1.0000	1.0000	0.9878	1.0000 (0.2949)	0.9850 (0.3339)	0.9715 (0.3712)
15	Yuanta	1.0000	1.0000	0.9475	0.9058 (0.2948)	0.9265 (0.3346)	0.9996 (0.3705)
16	Mega	1.0000	1.0000	0.9683	0.9644 (0.3097)	0.9409 (0.3491)	1.0000 (0.3412)
17	Taishin	0.6533	0.7865	0.5280	0.5258 (0.3050)	0.5321 (0.3257)	0.5262 (0.3693)
18	Shin Kong	0.8482	0.8615	0.8123	0.7986 (0.3101)	0.8221 (0.3264)	0.8152 (0.3635)
19	Sino Pac	0.9018	0.9433	0.8430	0.8541 (0.3195)	0.8419 (0.3331)	0.8338 (0.3474)
20	China Trust	0.8540	0.8881	0.6259	0.5939 (0.3084)	0.6295 (0.3397)	0.6504 (0.3518)
21	First	0.9592	0.9746	0.9279	0.8674 (0.3159)	0.9608 (0.3291)	0.9511 (0.3550)
22	Taiwan	1.0000	1.0000	0.9818	0.9784 (0.3272)	0.9668 (0.3338)	1.0000 (0.3390)

Table 3 The periodic efficiency and the periodic weights extracted from equation (8) for DMU#10

Bank 10. En Tie	*Periodic efficiency*	*Input and output weights* $w_{ir}^{*(p)}$								
	$\theta^{*(p)}$	*w (1, 1)*	*w (1, 2)*	*w (1, 3)*	*w (2, 1)*	*w (2, 2)*	*w (2, 3)*	*w (3, 1)*	*w (3, 2)*	*w (3, 3)*
2009	1.000	0.000	0.000	0.782	0.000	0.138	0.079	0.000	0.000	0.000
2010	0.990	0.000	0.000	0.656	0.000	0.006	0.163	0.000	0.000	0.174
2011	1.000	0.000	0.000	0.000	0.000	0.952	0.048	0.000	0.000	0.000

Table 4 The overall and periodic efficiency of equation (8)

Multi-period ratio equation					
Banks		*Overall efficiency* $(\bar{\theta}^*)$	*2009*	*2010*	*2011*
1	Chang Hwa	0.9311	0.7900	0.9300	0.9400
			(0.145)	(0.6471)	(0.3382)
2	King's Town	0.7758	0.7300	0.8100	0.7900
			(0.3311)	(0.3395)	(0.3234)
3	Taichung	0.9998	1.0000	1.0000	1.0000
			(0.5156)	(0.2551)	(0.2291)
4	Taiwan bsiness	0.9998	1.0000	1.0000	0.9700
			(0.3182)	(0.134)	(0.5503)
5	Kaohsiung	0.9986	1.0000	1.0000	1.0000
			(0.9341)	(0.009)	(0.056)
6	Cosmos	0.7923	0.8200	0.8000	0.7600
			(0.3232)	(0.3302)	(0.3465)
7	Union	0.5389	0.4900	0.5700	0.5400
			(0.0331)	(0.0336)	(0.9331)
8	Far Eastern	0.8615	1.0000	0.9300	0.8000
			(0.0100)	(0.4639)	(0.5252)
9	Ta Chong	0.7941	0.7300	0.8100	0.8500
			(0.3483)	(0.3439)	(0.3076)
10	En Tie	0.9905	1.0000	0.9900	1.0000
			(0.0419)	(0.9236)	(0.0343)
11	Hua Nan	1.0000	1.0000	1.0000	1.0000
			(0.3219)	(0.3318)	(0.3462)
12	Fubon	1.0000	1.0000	1.0000	1.0000
			(0.3347)	(0.3287)	(0.3368)
13	Cathay	0.8565	0.8400	0.8800	0.8500
			(0.3143)	(0.3286)	(0.357)
14	East Sun	0.9983	1.0000	1.0000	1.0000
			(0.2233)	(0.6064)	(0.1702)
15	Yuanta	1.0000	1.0000	1.0000	1.0000
			(0.3284)	(0.3284)	(0.3432)
16	Mega	1.0000	1.0000	1.0000	1.0000
			(0.3038)	(0.3548)	(0.3413)
17	Taishin	0.7013	0.5600	0.5700	0.7700
			(0.2359)	(0.0923)	(0.6717)
18	Shin Kong	0.8297	0.8300	0.8600	0.8200
			(0.6821)	(0.0877)	(0.2301)
19	Sino Pac	0.9378	0.9500	0.9300	0.8300
			(0.6397)	(0.3152)	(0.0449)
20	China Trust	0.7736	0.7000	0.6900	0.8800
			(0.2801)	(0.2926)	(0.4271)
21	First	0.9859	0.9200	0.9900	0.9900
			(0.0420)	(0.0590)	(0.8983)
22	Taiwan cooperative	1.0000	1.0000	1.0000	1.0000
			(0.3863)	(0.3059)	(0.3078)

The optimal periodic weights are calculated by employing $w^{*(p)} = \dfrac{w^{(p)}}{\sum_{p=1}^{3} w(p)}$.

Using these optimal periodic weights, the overall efficiency of DMU#10 is 0.9905 i.e., $\overline{\theta}^{*} = \sum_{p=1}^{3} w^{*(p)} * \theta^{*(p)}$

The results for other DMUs are reported in Table 4.

Similar to Table 2, the optimal weights for each period are listed below the efficiency of the respective period. Overall efficiency of the proposed Multiperiod ratio equation (8) is reported in the second column of Table (3). Given the optimal weight of each period and multiply it by the corresponding period efficiency finally their summation is called overall efficiency in multi-period ratio form, i.e., $\overline{\theta}^{*} = \sum_{p=1}^{q} w^{*(p)} * \theta^{*(p)}$ represents overall efficiency in multi-period system based on DEA-R models. Looking at the fourth column of Table2 and the second column of Table 4, overall efficiency of multi-period ratio equation (8) is always greater than or equal to that of relation multi-period network equation (4). The rest columns of Table 4 record the periodic efficiencies by employing proposed equation (8). Optimal weights of the corresponding periods are reported in parentheses. The method for calculating the weights is explained in details. Compared with the existing models, the proposed equation (8) supports achieving the periodic optimal weights that have a crucial role in calculating the overall efficiency. Interestingly enough, the overall efficiency with multi-period ratio equation (8) is always smaller than the periodic efficiency calculated for corresponding periods. As a specimen, the fifth unit (DMU#5) is efficient in all periods while its overall efficiency score is 0.9986, indicating that this unit has Pseudo-inefficiency. Similarly, Pseudo-inefficiencies have also occurred in DMUs #3 and #14. On the other hand, according to the second column of Table 4, DMUs #11, #12, #15, #16 and #22 are overall efficient units and the others are inefficient ones. This demonstrated that the proposed approach is responsive and it can differentiate the DMUs discretely.

5 Conclusions

Many studies have measured the efficiency of a set of DMUs in a certain period of time. In a time span consisting of multiple periods of time, overall efficiency is measured by averaging data of all periods. However, a drawback of this aggregated approach is that measurement of efficiency is discarded in operations related to individual periods. In this study, in response to weakness of the existing multiperiod model, a DEA-R model was developed to calculate overall and periodic efficiencies simultaneously. Furthermore, the proposed model can detect the pseudo-inefficiency in multi-period system, and it has a higher accuracy in efficiency measurement. Finally, the proposed approach can discriminate the units as overall efficient and inefficient ones via the proposed overall efficiency measure. At last, an application on 22 Taiwanese commercial banks showed the usefulness of the model. It was also proved that overall efficiency obtained for multiperiod systems which use DEA-R models are higher than or equal to overall efficiency obtained for multiperiod models alone. Introducing a new model which its optimal objective function is the desired overall efficiency of the unit is an interesting challenge for future research.

References

Agarwal, R. and Mehrotra, A. (2020) 'Extracting performance improvement factors using DEA for a robust Indian retail industry', *International Journal of Productivity and Quality Management*, Vol. 30, No. 1, pp.107–125.

Asmild, M., Bogetoft, P. and Hougaard, J.L (2013) 'Rationalizing inefficiency: staff utilization in branches of a large Canadian bank', *Omega International Journal of Management Science*, Vol. 41, No. 1, pp.80–87.

Assaf, A.G., Barros, C. and Sellers-Rubio, R. (2011) 'Efficiency determinants in retail stores: a Bayesian framework', *Omega International Journal of Management Science*, Vol. 9, No. 3, pp.283–292.

Banker, R.D., Charnes, A. and Cooper, W.W. (1984) 'Some models estimating technical and scale inefficiencies in data envelopment analysis', *Management Science*, Vol. 30, No. 9, pp.1078–1092.

Chang, S.J., Hsiao, H.C. and Huang, L.H. (2011) 'Taiwan quality indicator project and hospital productivity growth', *Omega International Journal of Management Science*, Vol. 39, pp.14–22.

Charnes, A., Cooper, W. W. and Rhodes, E. (1978) 'Measuring the efficiency of decision making units', *European Journal of Operational Research*, Vol. 2, No. 6, pp.429–444.

Despic, O., Despic, M. and Paradi, J. C. (2007) 'DEA-R: Ratio-based comparative efficiency model, its mathematical relation to DEA and its use in applications', *Journal of Productivity Analysis*, Vol. 28, No. 1, pp.33–44.

Färe, R. and Grosskopf, S. (2000) 'Network DEA', *Socio-Economic Planning Sciences*, Vol. 34, No. 1, pp.35–49.

Farrell, M. J. (1957) 'The measurement of productivity efficiency', *Journal of The Royal Statistical Society Series A: General*, Vol. 120, No. 3, pp.253–281.

Hwang, S.N., Chen, C.L., Chen, Y., Lee, H.S. and Shen, P.D. (2013) 'Sustainable design performance evaluation with applications in the automobile industry: focusing on inefficiency by undesirable factors', *Omega International Journal of Management Science*, Vol. 41, No. 3, pp.553–558.

Kamyab, P., Mozaffari, M. R., Gerami, J. and Wankei, P. F. (2021), 'Two-stage incentives system for commercial banks based on centralized resource allocation model in DEA-R', *International Journal of Productivity and Performance Management*, Vol. 70, No. 2, pp.427–458.

Kao, C. (2009a) 'Efficiency decomposition in network data envelopment analysis: a relational model', *European Journal of Operational Research*, Vol. 192, No. 3, pp.949-962.

Kao, C. (2009b) 'Efficiency measurement for parallel production systems', *European Journal of Operational Research*, Vol. 196, No. 3, pp.1107–1112.

Kao, C. and Hwang, S.N. (2008) 'Efficiency decomposition in two-stage data envelopment analysis: an application to non-life insurance companies in Taiwan', *European Journal of Operational Research*, Vol. 185, No. 1, pp.418–429.

Kao, C. and Liu, S.T. (2014) 'Multi-period efficiency measurement in data envelopment analysis: the case of Taiwanese commercial banks', *Omega*, Vol. 47, No. 1, pp.90–98.

Liu, W.B., Zhang, D.Q., Meng, W., Li, X.X. and Xu, F.A. (2011) 'Study of DEA models without explicit inputs', *Omega*, Vol. 39, pp.472–480.

Lovell, C.A.K. and Pastor, J.T. (1995) 'Units invariant and translation in variant DEA models', *Operations Research Letters*, Vol. 18, No. 3, pp.147–151.

Mozaffari, M.M., Taghizadeh-Yazdi, M., Nazari-Shirkouhi, S. and Asadzadeh, S.M. (2021). measuring traffic safety culture toward achieving road safety performance: a DEA approach with undesirable inputs-outputs', *Cybernetics and Systems*, Vol., 52, No. 7, pp.601–624.

Mozaffari, M. R., Dadkhah, F., Jablonsky, J., Wanke, P. F. (2020) 'Finding Efficient surface in DEA-R models', *Applied Mathematics and Computation*, Vol. 386, No. 1, p.125497.

Mozaffari, M. R., Gerami, J. and Jablonsky, J. (2014a) 'Relationship between DEA models without explicit inputs and DEA models without explicit inputs and DEA-R models', *Central European Journal of Operation Research*, Vol. 22, No. 1, pp.1–12.

Mozaffari, M.R., Kamyab, P., Jablonsky, J. and Gerami, J. (2014b) 'Cost and revenue efficiency in DEA-R models', *Computers & Industrial Engineering*, Vol. 78, No. 1, pp.188–194.

Mozaffari, M.R., Gerami, J., Wanke, P.F., Kamyab, P and Peyvas, M. (2022) 'Ratio-based data envelopment analysis: An interactive approach to identify benchmark', *Results in Control and Optimization*, Vol. 6, No. 1, p.100081.

Nazari-Shirkouhi, S. and Keramati, A. (2017). Modeling customer satisfaction with new product design using a flexible fuzzy regression-data envelopment analysis algorithm', *Applied Mathematical Modelling*, Vol. 50, No. 1, pp.755–771.

Nemoto, J. and Goto, M. (1999) 'Dynamic data envelopment analysis: Modeling intertemporal behavior of a firm in the presence of productive inefficiencies', *Economics Letters*, Vol. 64, No. 1, pp.51–56.

Niknazar, P. and Bourgault, M. (2018) 'Analysis of axioms and assumptions of data envelopment analysis: application for efficiency measurement in project management contexts', *International Journal of Productivity and Quality Management*, Vol. 23, No. 4, pp.446–472

Olesen, O.B., Petersen, N.C. and Podinovski, V. (2015) 'Efficiency analysis with ratio measures', *European Journal of Operational Research*, Vol. 245, No. 2, pp.446–462.

Olesen, O.B., Petersen, N.C. and Podinovski, V. (2017) 'Efficiency measures and computational approaches for data envelopment analysis models with ratio inputs and outputs', *European Journal of Operational Research*, Vol. 261, No. 2, pp.640–655.

Paradi, J.C. and Zhu, H. (2013) 'A survey on bank branch efficiency and performance research with data envelopment analysis', *Omega International Journal of Management Science*, Vol. 41, No. 1, pp.61–79.

Park, K. S. and Park, K. (2009) 'Measurement to multipored aggregative efficiency', *European Journal of Operational Research*, Vol. 193, No. 4, pp.567–580.

Portela, M.C.S., Camanho, A.S. and Borges, D. (2012) 'Performance assessment of secondary schools: the snapshot of a country taken by DEA', *Journal of the Operational Research Society*, Vol. 63, No. 8, pp.1098–1115.

Saljoughian, M., Shirouyehzad, H., Khajeh, E. and Dabestani, R. (2019) 'Evaluating the efficiency of the commercial banks admired in fortune 500 list; using data envelopment analysis', *International Journal of Productivity and Quality Management*, Vol. 26, No. 1, pp.58–73

Sharma, S.K., Sehgal, S. and Dubey, V. (2022) 'Measuring and benchmarking technical efficiency of indian automobile sector, DOI: 10.1504/IJPQM.2020.10034187

Tavana, M., Nazari-Shirkouhi, S. and Kholghabad, H.F. (2021). An integrated quality and resilience engineering framework in healthcare with Z-number data envelopment analysis. *Health Care Management Science*, Vol. 24, No. 1, pp.768–785

Wei, C.K., Chen, L.C., Li, R.K. and Tsai, C.H. (2011a) 'Using the DEA-R model in the hospital industry to study the pseudo-inefficiency', *Expert Systems with Applications*, Vol. 38, No. 3, pp.2172–2176.

Wei, C.K., Chen, L.C., Li, R.K. and Tsai, C.H. (2011b) 'Exploration of efficiency underestimation of CCR model: Based on medical sectors with DEA-R model', *Expert Systems with Applications*, Vol. 38, No. 4, pp.3155–3160.

Wei, C.K., Chen, L.C., Li, R.K. and Tsai, C.H. (2011c) 'A study of developing an input- oriented ratio-based comparative efficiency model', *Expert Systems with Applications*, Vol. 38, No. 3, pp.2473–2477.

Wei, X.Y., Jun, Z. H., Kai, C., Xuan, Z.Z. and Tian, C.Y. (2021) 'Efficiency measurement in multi-period network DEA model with feedback', *Expert Systems with Applications*, Vol. 175, No. 1, p.114815.

Yazdi, M.R.T., Mozaffari, M.M., Nazari-Shirkouhi, S. and Asadzadeh, S.M. (2018) 'Integrated fuzzy DEA-ANFIS to measure the success effect of human resource spirituality', *Cybernetics and Systems*, Vol. 49, No. 3, pp.151–169.

Appendix

Theorem 1 Overall efficiency of the multi-period ratio equation (8) is greater than or equal to that of the multi-period equation (4).

Proof Efficiency of input-oriented CCR model described by Despic et al. (2007) is calculated by:

$$\theta_o^* = \max_{\substack{\sum_i v_i = 1, v_i \geq 0 \\ \sum_r u_r = 1, u_r \geq 0}} \min \frac{\sum_i v_i \frac{X_{ij}}{X_{io}}}{\sum_r u_r \frac{Y_{rj}}{Y_{ro}}} \tag{9}$$

Efficiency of individual periods is calculated by equation 9 by employing the data obtained for each period, p ($p = 1, \ldots, q$). Since, the constraint $\sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \leq 0$ is redundant is equation (4), so, the periodic efficiency for all periods can be stated as follows:

$$\theta_{o-MP}^* = \max_{\substack{\sum_i v_i = 1, v_i \geq 0 \\ \sum_r u_r = 1, u_r \geq 0}} \min \frac{\sum_i v_i \frac{X_{ij}{}^{(p)}}{X_{io}}}{\sum_r u_r \frac{Y_{rj}{}^{(p)}}{Y_{ro}}} \tag{10}$$

Furthermore, the harmonic CCR efficiency for each period of is calculated employing

$$\overline{\theta_o^*} = \max_{\substack{\sum_i v_i = 1, v_i \geq 0 \\ \sum_r u_r = 1, u_r \geq 0}} \min \sum_i v_i \frac{X_{ij}}{X_{io}} \sum_r u_r \frac{Y_{rj}}{Y_{ro}} \tag{11}$$

and it can be generalized to multi-period systems as

$$\overline{\theta_{o-h}^*} = \max_{\substack{\sum_i v_i = 1, v_i \geq 0 \\ \sum_r u_r = 1, u_r \geq 0}} \min \sum_i v_i \frac{X_{ij}{}^{(p)}}{X_{io}} \sum_r u_r \frac{Y_{rj}{}^{(p)}}{Y_{ro}} \tag{12}$$

Equipped with these propositions, the proof contains two steps:

Step 1 We show that the efficiency of a harmonic multi-periodic CCR is always greater than or equal to that of a multi-periodic CCR. By substituting $X_{ij}^{'(p)}$ and $Y_{rj}^{'(p)}$ with $\frac{X_{ij}^{'(p)}}{X_{io}}, \frac{Y_{rj}^{'(p)}}{Y_{ro}}$ and coefficients $\frac{\sum_r u_r \frac{Y_{rj}^{(p)}}{Y_{ro}}}{\sum_r u_r \frac{Y_{rj}^{(p)}}{Y_{ro}}} = \frac{\sum_r u_r \frac{1}{Y_{rj}^{'(p)}}}{\sum_r u_r \frac{1}{Y_{rj}^{'(p)}}}$, the CCR and Harmonic efficiency formulas can be rewritten as follows:

$$\theta_{O-MP}^{*} = \max_{\substack{\sum_i v_i = 1, v_i \geq 0 \\ \sum_r u_r = 1, u_r \geq 0}} \min \frac{\sum_i v_i X_{ij}^{'(p)} \times \sum_r u_r \frac{1}{Y_{rj}^{'(p)}}}{\sum_r u_r Y_{rj}^{'(p)} \times \sum_r u_r \frac{1}{Y_{rj}^{'(p)}}} \tag{13}$$

$$\overline{\theta}_{O-MP}^{*} = \max_{\substack{\sum_i v_i = 1, v_i \geq 0 \\ \sum_r u_r = 1, u_r \geq 0}} \min \sum_i v_i X_{ij}^{'(p)} \times \sum_r u_r \frac{1}{Y_{rj}^{'(p)}} \tag{14}$$

The difference between $\theta_{o-MP}^{*}, \overline{\theta_{o-MP}^{*}}$ is $\frac{1}{\sum_i u_r Y_{rj}^{\prime(p)} \sum_r u_r \frac{1}{Y_{rj}^{\prime(p)}}}$.

Because $\sum_r u_r Y_{rj}^{\prime\,(p)} \times \sum_r u_r \frac{1}{Y_{rj}^{\prime(p)}} = \sum_{\substack{r=1,\ldots,s \\ t=r+1,\ldots,s}} u_r^{(2)} + u_r u_t \left(\frac{Y_{rj}^{\prime(p)}}{Y_{tj}^{'(p)}} + \frac{Y_{tj}^{\prime(p)}}{Y_{rj}^{\prime(p)}} \right)$

$$\sum_{\substack{r=1,\ldots,s \\ t=r+1,\ldots,s}} u_r^{(2)} + 2u_r u_t - 2u_r u_t + u_r u_t \left(\frac{Y_{rj}^{\prime(p)}}{Y_{tj}^{'(p)}} + \frac{Y_{tj}^{\prime(p)}}{Y_{rj}^{\prime(p)}} \right)$$

$$= \left(\sum_r u_r \right)^2 + \sum_{\substack{r=1,\ldots,s \\ t=r+1,\ldots,s}} \frac{u_r u_t}{Y_{rj}^{\prime(p)} Y_{tj}^{\prime(p)}} \left(Y_{rj}^{\prime(p)} - Y_{tj}^{\prime(p)} \right)^2$$

$$= 1 + \sum_{\substack{r=1,\ldots,s \\ t=r+1,\ldots,s}} \frac{u_r u_t}{Y_{rj}^{\prime(p)} Y_{tj}^{\prime(p)}} \left(Y_{rj}^{\prime(p)} - Y_{tj}^{\prime(p)} \right)^2 \geq 1$$

So the difference between $\theta_{o-MP}^{*}, \overline{\theta_{o-MP}^{*}}$ is $\frac{1}{\sum_i u_r Y_{ij}^{'(p)} \sum_r u_r \frac{1}{Y_{rj}^{'(p)}}}$. Therefore,

$\theta_{o-MP}^{*} \leq \overline{\theta_{O-MP}^{*}}$

Step 2 Overall efficiency of the multi-period ratio model (8) is shown to be always higher than or equal to multiperiod harmonic CCR. The efficiency of Multi-Period Ratio Model (8) can be written as follows:

$$\theta^{*}_{o-MP.Ratio} = \max_{\sum_i \sum_{w'_{ij} \geq 0} w'^{(p)}_{ir} = 1}, \min_j \sum_i \sum_r w^{(p)}_{it} \frac{X'^{(p)}_{ij}}{Y'_{ij}(p)} \tag{15}$$

since the overall efficiency is calculated by the relationship $\sum_{P=1}^{q} w_i^{(p)} \theta^{*\wedge}_{o-MP}$ $= \sum_{p=1}^{q} \frac{\sum_{i=1}^{m} v_i^{*} x_{ik}^{(p)}}{\sum_{i=1}^{m} v_i^{*} x_{ik}}$. In other words, the total efficiency is equal to $\sum_{p=1}^{q} w_i \theta^{*\wedge}_{o-MP}$, since in DEA-Ratio mode, the sum of weights is equal to one, $\sum_{i=1}^{m} w_i = 1$, therefore, in multi-period mode, the sum of optimal weights $w^{(1)} \theta^{*\wedge}_{o-MP_1} + w^{(2)} \theta^{*\wedge}_{o-MP_2} + \ldots + w^{(p)} \theta^{*\wedge}_{o-MP_p}$ is equal to one. In other words, $\sum_{p=1}^{q} \sum_{i=1}^{m} w_i^{(p)} = 1$. Since, $\frac{\wedge}{\theta^{*}_{O-MP}}$ is a special case of θ^{*}_{O-MP}, so we can conclude that $\overline{\theta^{*}_{O-MP}} \leq \theta^{*\wedge}_{o-MP}$ and it completes the proof. As a result, $\theta_{MP-RATIO} \geq \theta_{MP}$.