THE RELATIONSHIP BETWEEN THE PRICE INTEREST RATE RISK AND THE HOLDING PERIOD RETURN RISK IN A BOND INVESTMENT

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The interest rate risk of bond investments is divided into the price interest rate risk and the holding period return risk. The former can be measured by Macaulay's duration. However, there is no instrument for the latter. This paper derives a measurement for the latter based on an elasticity similar to the duration and provides a mathematical relationship between the two risks. The relationship derived is very useful for understanding the interest rate risk of bond investments. It also allows an easy derivation of the Fisher-Weil immunization strategy without sophisticated mathematical proof.

In 1938, Macaulay found a measurement for the price interest rate risk and called it "duration." In 1939, Hicks also independently derived a similar measurement. Thereafter, as reviewed by Weil (1973), many scholars used the duration concept to explore conditions under which the interest rate risk of bond investments can be reduced or immunized.

The bond holding period return is said to be "immunized" if the actual holding period return which will be realized by holding the bonds up to a given investment horizon is greater than or equal to the return expected from the interest rates prevailing at the time of the bond investment. A historical paper in the area of the bond immunization was published in 1971 by Fisher-Weil. They derived the famous immunization theorem (hereafter Fisher-Weil immunization strategy) based on the duration concept and showed that under certain conditions such as a flat yield curve and parallel interest rate changes, the holding period return is

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immunized when the bond portfolio is formed such that the duration of the portfolio is equal to the investment horizon.

Other scholars reexamined the immunization strategy. For example, Bierwag and Kaufman (1977), Bierwag (1977), Cooper (1977), Ingersoll, Skelton, and Weil (1978), Cox, Ingersoll, and Ross (1979), and Khang (1979), examined conditions under which Macaulay's duration represents the price interest rate risk and/or proposed several different forms of duration which can represent the price interest rate risk under specific assumptions along with immunization strategies. Gultekin and Rogalski (1984) performed empirical tests on the different forms of the duration measures including Macaulay's original duration and found that none of these duration measures showed any superiority in explaining the interest rate risk.

The Macaulay's duration concept has also been used to control the interest rate risk of financial institutions. For example, Samuelson (1945), Redington (1952), Wallas (1960), and Cronin (1995) showed that the interest rate risk of financial institutions can be immunized by equating the duration of the assets and liabilities.

MEASUREMENT OF THE PRICE AND REINVESTMENT RISKS

The present value of a default-risk-free bond, V_o, can be defined as:

$$V_o = \sum_{t=1}^{n} \frac{C_t}{R^t}, \qquad (1)$$

where n = the maturity of the bond, C_t = default-risk-fre annual coupon at time t (the coupon + the face value when t = n), $R^t = (1+r)t$, and r = the nominal risk-free interest rate. Then, it is well known that the price interest rate risk can be measured in terms of the elasticity of the present value with respect to the interest rate factor (1+r) such that:

$$R^{\epsilon}V_o = \lim_{\Delta R \to 0} [(\frac{\Delta V_o}{V_o})]/[(\frac{\Delta R}{R})] = \frac{dV_o}{dR}\frac{R}{V_o}$$

$$= \frac{R}{V_o} \sum_{t=1}^{n} \frac{-tR^{t-1}C_t}{R^{2t}}$$

$$= -\sum_{t=1}^{n} t \frac{(C_t)/R^t}{V_o}, \qquad (2)$$

$$= -D_p$$

where D_p = Macaulay's duration. Note that Macaulay's duration is the same as the absolute value of the elasticity. Since the duration is an elasticity of the present value, it represents the price interest rate risk. It should be noted that if the bond is not default-risk free, the elasticity derived should include an additional term showing the impact of the change in interest rate on the coupon payments in addition to Macaulay's duration (Haugen and Wichern, 1975).

Just as the price interest rate risk of a bond can be measured in terms of an elasticity of the present value with respect to the interest rate factor (1+r), the holding period return risk of a bond can also be similarly measured in terms of an elasticity of the terminal value of the bond with respect to the same interest rate factor (1+r).

If the interim coupon incomes from a coupon bond are reinvested and compounded at the interest rate r, the terminal value, V_n, to be realized by holding the bond until its maturity, can be written as:

$$V_n = \sum_{t=1}^{n} C_t R^{n-t}, \tag{3}$$

The elasticity of the terminal value with respect to the interest rate factor (1 + r) can be derived as:

$$R^{\epsilon}V_n = \lim_{\Delta R \to 0} \left[\left(\frac{\Delta V_n}{V_n}\right) / \left(\frac{\Delta R}{R}\right)\right] = \frac{dV_n}{dR}\,\frac{R}{V_n}$$

$$= \frac{R}{V_n} \sum_{t=1}^{n} [(n-t)\, C_t R^{n-t-1}]$$

$$= n - \sum_{t=1}^{n} \frac{C_t R^{n-t}}{V_n} \quad \sum_{t=1}^{n} t\, \frac{C_t R^{n-t}}{V_n}$$

$$= n - \sum_{t=1}^{n} t\, \frac{(C_t R^{n-t})/R^n}{V_n/R^n}$$

$$= n - \sum_{t=1}^{n} t\, \frac{C_t/R^t}{V_o}$$

$$= n - D_p, \tag{4}$$

Therefore,

$$D_r + D_p = n, \tag{5}$$

where $D_r = R^e V_n$. Hereafter, D_p and D_r will be called "price duration" and "reinvestment duration" respectively. The relationship between the two risks is immediately available from equation (5) without any additional derivation. The implications of equation (5) can be summarized as follows:

a. If the investment horizon is the same as the bond maturity, the combined interest rate risk (the price and the holding period return risks) of a bond is always equal to the bond maturity, the investment horizon.

b. Once one risk (the price risk or the reinvestment risk) is known, the other risk (the reinvestment risk or the price risk) can be found by subtracting the known risk from the maturity of the bond.

c. The reduction or elimination of one risk can be achieved at the expense of increasing the other risk by exactly the same amount. The two risks exactly offset each other, leaving the combined interest rate risk at the same amount. It is not possible to reduce the combined duration below the investment horizon.

d. Consequently, controlling the interest rate risk of bond investments should be dealt with a matter of selecting and maintaining a desirable combination of the two risks depending on an investor's expectation on future interest rates and risk attitude.

The price and reinvestment durations are derived based on the present and terminal values which are discounted and compounded by a single discount rate. Since future interest rates are not likely to remain constant (flat yield curve), the present value, terminal value, and the durations computed by a single discount rate can not precisely represent the intended time value of money and also the price and reinvestment risks. For the true time value of money and accurate price and reinvestment risks, actual future interest rates at which the future cash flows can be reinvested should be used for the computation. However, it is practically impossible to accurately estimate such future reinvestment rates. Consequently, in practice, there is not much choice but to use the single discount rate for the durations. Nevertheless, it should be noted that just as the single discount rate is used for bond valuation, the two durations based on a single discount rate are still useful instruments for measuring and understanding overall interest rate risk of bond investments.

THE FISHER-WEIL IMMUNIZATION STRATEGY

The Fisher-Weil immunization theorem which was derived with a sophisticated mathematical proof can be easily obtained from the equation for the reinvestment duration, equation (4). It should be noted that Fisher-Weil found that for the immunization of the holding period return risk, the price duration of a bond or a bond portfolio should be equal to the investment horizon.

According to equation (4), in order to immunize the holding period return risk, that is; in order to have zero reinvestment duration, one should choose a bond such that its price duration is equal to the maturity of the bond. This condition can be satisfied only with zero coupon bonds because the price duration of a coupon bond is always less than the bond maturity. However, it should be noted that equation (4) was derived based on the assumption that the bond will be held until its maturity. In the case when bonds are not held up to the maturity, equation (4) should be accordingly adjusted.

The adjustment can be shown as follows. Let n be the maturity of a bond and h be the investment horizon. When n > h, Vnh, the terminal value of a n-year-maturity bond at the end of investment horizon can be written as:

$$V_{nh} = \sum_{t=1}^{n} C_t R^{(h-t)} + \sum_{t=h+1}^{n} C_t R^{-(t-h)} , \qquad (6)$$

$$V_{nh} = \sum_{t=1}^{n} C_t R^{(h-t)} , \qquad (7)$$

The first part of the right side of equation (6) represents the total compound (terminal) value at the end of period h of those coupon incomes to be received up to the investment horizon. The second part of the equation represents the present value at the end of period h of the cash flows after the investment horizon.

Now, the elasticity of V_{nh} with respect to the interest factor R can be derived as:

$$R^{\epsilon}V_{nh} = \frac{dV_{nh}}{dR} \frac{R}{V_{nh}}$$

$$= \frac{R}{V_{nh}} \sum_{t=1}^{n} [(h-t)\, C_t R^{h-t-1}]$$

$$= \sum_{t=1}^{n} (h-t) \frac{C_t R^{(h-t)}}{V_{nh}}$$

$$= h - \sum_{t=1}^{n} t \frac{C_t R^{(h-t)} R^{(n-h)}}{V_{nh} R^{(n-h)}}$$

$$= h - \sum_{t=1}^{n} t \frac{C_t R^{(n-t)}}{V_n}$$

$$= h - \sum_{t=1}^{n} t \frac{C_t R^{(n-t)} R^{-n}}{V_n R^{-n}}$$

$$= h - \sum_{t=1}^{n} t \frac{C_t R^{-t}}{V_o}$$

$$= h - D_{pn}, \qquad (8)$$

Therefore,

$$D_{pn} + D_{mh} = h\,, \qquad (9)$$

Note that $V_n = \sum_{t=1}^{n} C_t R^{n-t}$, $V_o = \sum_{t=1}^{n} \frac{C_t}{R^t}$, D_{pn} = price duration of a n-year-maturity bond, $D_{mh} = R^{\epsilon} V_{nh}$ = reinvestment duration of a n-year-maturity bond when it is held up to h.

According to equation (8), the holding period return risk becomes zero when the price duration of the bond, D_{pn} is equal to h, the investment horizon. This is the same immunization condition derived by Fisher-Weil. However, contrary to the complex mathematical derivation of Fisher and Weil, the immunization condition can be easily obtained from the equation for the reinvestment duration without any rigorous mathematical proof. Furthermore, the equation for the reinvestment duration provides the following implications:

First, according to equation (9), the total combined interest rate risk of the price and reinvestment risks is always equal to the investment horizon, h, again implying that the reduction or elimination of the reinvestment risk cannot be achieved without increasing the price interest rate risk by the same amount. Therefore, the Fisher-Weil immunization strategy eliminate the holding period return risk by taking the maximum price risk.

Second, the reinvestment duration, contrary to the price duration, varies depending on the length of the investment horizon. Such variation of the reinvestment duration can be easily observed from the following example. Consider a twenty year maturity bond whose price duration is twelve. If the investment horizon is 15 years, we can see from equation (9) that the reinvestment duration becomes +3. On the other hand, if the investment horizon is 10 years, the reinvestment duration of the same bond becomes -2, a negative number. The negative reinvestment duration implies that changes in the first term of the right side of equation (6) due to changes in interest rates is less than the corresponding changes in the second term. Consequently, if reinvestment duration is zero, the changes in the first and second terms should be the same. The variation of the reinvestment risk can also be intuitively recognized by considering the following. If a long term bond is used for short term investment horizon, the reinvestment risk of the coupon incomes are limited to that short period. On the other hand, if the investment horizon is extended to the bond maturity, the reinvestment risk becomes high because all the coupons received before the maturity should be reinvested.

In the case of a portfolio of more than one bond, the Fisher-Weil immunization condition can be satisfied more easily than in the case of the a single coupon bond. This is because the (price or reinvestment) durations of a bond portfolio are determined by the weighted average of durations of component bonds and therefore, the reinvestment durations of all component bonds are not necessarily equal to zero. Bonds with positive and negative reinvestment durations (9) can be combined to make the portfolio reinvestment duration equal to zero. On the contrary, in the case of a single coupon bond investment, it is not easy to find a bond which can satisfy the immunization condition that the price duration is equal to a given investment horizon.

SUMMARY AND CONCLUSION

The interest rate risk of bond investments is divided into the price risk and the holding period return risk. The measurement of both risks is essential for an accurate understanding of the overall interest rate risk of bond investment and investment strategies controlling the interest rate risk. The price interest rate risk can be measured by Macaulay's duration, but there is no instrument available which can measure the holding period return risk.

The paper provides an equation for the holding period return risk. Just as the price interest rate risk can be measured in terms of the elasticity of the present value with respect to the interest rate factor (1+r), the holding period return risk can be measured in terms of the elasticity of the compound (terminal) value with respect to the interest rate factor. The derivation of both risks based on the elasticity with respect to the same interest rate factor reveals a mathematical relationship between the two risks. The most important implication of the relationship is that the reduction or elimination of one risk can be achieved at the expense of increasing the other risks by exactly the same amount, leaving the combined interest rate risk always at the same amount. Consequently, controlling the interest rate risk of bond investments should be dealt with a matter of selecting and maintaining a desirable combination of the two risks depending on an investor's expectation on future interest rates and risk attitude. The measurement of the both risks are also useful in that bond investment strategies such as Fisher Weil Immunization theorem can be easily derived and properly evaluated.

Both the Macaulay's price and the reinvestment durations are based on a single discount rate. Consequently, there are limitations in its application. Nevertheless, just as, in practice, a single discount rate is used for bond valuations, the two durations computed based on a single discount rate are still useful instruments in measuring and understanding the overall interest rate risk in bond investments.

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