INTEGER PROGRAMMING AND HEURISTIC APPROACHES TO THE MINIMUM DIVERSITY PROBLEM

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The minimum diversity problem is concerned with selecting a set of elements from some larger collection such that those chosen exhibit the smallest variety of characteristics. In this paper, we propose a framework within which the concept of diversity can be defined and the diversity of a set of elements can be measured. We then formulate a number of integer programs to model the minimum diversity problem and its variation. We also show the theoretical complexity of the problem and provide specialized methods for efficiently obtaining approximate solutions, which are submitted to empirical analysis.

Consider a set of elements (e.g., items to be produced in a job shop) and some of their attributes (e.g., size, shape, and type of processing). For every element in the set, each of its attributes can be in one of several possible states (e.g., large, medium, or small; rectangular or triangular; grinding, cutting, machining, or painting). The objective of the minimum diversity problem is to select a predetermined number of elements from the set that are most similar in that they exhibit the lowest degree of difference in attribute states.

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The concept of minimizing diversity is useful in many real-life applications. In cellular layout (Bedworth, Henderson, & Wolfe, 1991), for instance, component parts to be produced are grouped into families with similar processing requirements by using the technique of group technology (Burbidge, 1975). A set of machines are then dedicated to each of the part families to form a manufacturing cell to realize such benefits as reduced setup time, decreased material handling cost, faster throughput time, and lower work-in-process inventory level. In capital budgeting, portfolio selection from a set of interrelated investment proposals involves minimizing the overall risk, which is measured by the covariance of the security returns, at a given level of expected return (Laughhunn, 1970). In industry, workers are frequently placed in teams on the basis of compatibility in terms of similarity in sociometric choice. This has resulted in increased job satisfaction and lower turnover rate (Van Zelst, 1952). Finally, as another example, consider the scenario in which a college student majoring in comparative history is interested in identifying, say, three cultures in the world that are most similar to each other. It is obvious that this may be considered as a minimum diversity problem.

These examples, which provide only a sampling, clearly show the prevalence of the need to account for diversity minimization in practical settings. Nevertheless, the minimum diversity problem is largely ignored in management science and related literature. The main focus of this paper is on building a broad foundation which allows the wide variety of diversity minimization issues to be viewed within a common framework. We will present several integer programming formulations to model the minimum diversity problem, including a special "minimax" variation. Also included is an examination of the connections of the problem to the class of NP-complete problems. Finally, we propose heuristic algorithms for minimizing diversity and conduct computational tests that reveal the value of these special methods.

LITERATURE REVIEW

In recent years, diversity analysis has been utilized in empirical studies in major functional areas of business (Jones, 1987; Nayak & Gastworth, 1989; Varadarajan & Ramanujam, 1987). Typically, an entropy-based index is calculated and used to compare the degrees of diversification of several systems with respect to some variable of interest (e.g., income distribution), or to examine the change in the degree of diversification of an economic system over time. In most cases, the treatment is descriptive in nature; no mathematical model has been described that includes diversity minimization as a general goal.

In a sense, statistical clustering (Johnson & Wichern, 1988) offers a limited approach to minimizing diversity in the presence of representivity requirements since a form of diversity is created by selecting elements from different clusters that have been identified. However, as pointed out by Kuo, Glover, & Dhir (1993b), this method suffers from several drawbacks. Dhir, Glover, & Kuo (1993) show how the minimum diversity concept can be applied to the selection of dependent engineering projects, where the risk of committing to two proposals is measured by the covariance between their returns. These, along with the study by Glover, Kuo, and Dhir (1994), account for the sparse research on diversity minimization in the existing literature.

In contrast, there has been a growing interest in the maximum diversity problem over the past few years. A succinct review of the latest developments in this area can be found in Kuo, Glover, & Dhir (1993a). More recently, Weitzman (1992) proposes a unified diversity theory using a function which satisfies a basic dynamic programming equation. Solow, Polasky, and Broadus (1993) formulate optimal policies for preserving biological diversity by taking economic factors into account. Glover, Kuo, and Dhir (1995) present an interesting application of the maximum diversity model for the conversation of crane species based on genetic distinctiveness.

In summary, while the subject of maximizing diversity has been researched from different perspectives, there have been few studies on its minimization counterpart. This observation provides ample motivation for the study of the minimum diversity problem.

MATHEMATICAL MODELING

We begin with a rudimentary formulation of the minimum diversity problem that will subsequently be refined and elaborated. Consider a set of elements $s_1, s_2, \ldots, s_n$ and some of their common attributes. Each of the attributes can be in one of several possible states. Our goal is to select m of the n elements so that the diversity of the subset is minimized.

In order to characterize the diversity of the elements in the subset to be chosen, we stipulate the existence of a value d_{ij} for each pair of elements s_i and s_j representing a distance measure over them. This measure can be the Euclidean distance, the Manhattan distance, or something in the nature of an affinity relationship (Francis, McGinnis, & White, 1992). For the purpose of our initial discussion, we will assume that these values are nonnegative though this assumption will be relaxed later.

According to these conventions, we introduce zero-one variables x_i, where $x_i = 1$ indicates that s_i is selected and $x_i = 0$ if it is not, $1 \le I \le n$. Our starting formulation of the minimum diversity problem is a quadratic zero-one integer program shown as follows.

Model (M1)

Minimize $Z = \sum_{I=1}^{n-1} \sum_{j=I+1}^{n} d_{ij}x_ix_j$

subject to $\sum_{I=1}^{n} x_i = m$

$x_i = 0 \text{ or } 1,\ 1 \leq I \leq n$

EQUIVALENT LINEAR MODEL AND VARIATION

The nonlinearity of the objective function in (M1) is inconvenient for most of the existing integer programming solution methods. Although a number of quadratic algorithms have been devised (see, for example, Laughhunn, 1970; Lawler & Bell, 1966; Taha, 1972a; 1972b), they are useful only for small problems and have not found widespread applications in the real world. A more efficient approach is to derive a linear equivalent of the nonlinear model and solve it. Techniques for converting polynomial zero-one programs into linear zero-one formulations are available in literature. Particularly, it can be readily established that (M1) can be transformed into (M2) below (Glover & Woolsey, 1974).

Model (M2)

Minimize $Z = \sum_{I=1}^{n-1} \sum_{j=I+1}^{n} d_{ij}y_{ij}$

subject to $\sum_{I=1}^{n} x_i = m$

$x_i + x_j - y_{ij} \leq 1,\ 1 \leq I < j \leq n$ (1)
$-x_i + y_{ij} \leq 0,\ 1 \leq I < j \leq n$ (2)
$-x_j + y_{ij} \leq 0,\ 1 \leq I < j \leq n$ (3)
$y_{ij} \geq 0,\ 1 \leq I < j \leq n$ (4)
$x_i = 0 \text{ or } 1,\ 1 \leq I \leq n$ (5)

A few observations about (M2) are in order. First, note that $y_{ij} \leq 1, I = 1, 2, ..., n - 1; j = I + 1, I + 2, ..., n$. Moreover, the y_{ij} variables will automatically receive zero-one values whenever the x_i variables are assigned such values. Constraints (1) through (4) also validly model y_{ij} as the product of x_i and x_j in the case where one of them is continuous, relaxing the integer requirements on these variables in (5). Finally, if all $d_{ij} \geq 0$ then (2) and (3) can be disregarded since they will be nonbinding at an optimal solution.

While it is possible to improve further on (M2) using the results of Glover (1975; 1984), the more efficient formulations will not be presented here since the present one is sufficient for treating the minimum diversity problem. Instead, we give an additional model based on an objective of minimizing the maximum diversity among the elements to be selected. In the following "minimax" model, M represents an extremely large positive number.

Model (M3)

Minimize $Z = w$

subject to $\sum_{I=1}^{n} x_i = m$

$$(M + d_{ij})y_{ij} - w \leq M,\ 1 \leq I < j \leq n$$
$$x_i + x_j - y_{ij} \leq 1,\ 1 \leq I < j \leq n$$
$$-x_i + y_{ij} \leq 0,\ 1 \leq I < j \leq n$$
$$-x_j + y_{ij} \leq 0,\ 1 \leq I < j \leq n$$
$$y_{ij} \geq 0,\ 1 \leq I < j \leq n$$
$$x_i = 0 \text{ or } 1,\ 1 \leq I \leq n$$

Although w is unrestricted in the above formulation, the constraint $w \geq 0$ may be included to facilitate the solution of (M3) if all d_{ij} are nonnegative since this is a minimization problem.

AN EXAMPLE

In this section we provide an illustrative example based on (M2), which is the simplest minimum diversity model. Several distance measures have been in use in facility layout and location and, depending on the area of application, one measure may be preferred to the others (Klastorian, 1985). The Euclidean distance is chosen here because it is one of the most commonly used measures (Francis, McGinnis, & White, 1992).

Consider a set of elements $S = \{s_i: I \in N\}$ with the index set $N = \{1, 2, ..., n\}$. Each element s_i contains r attributes whose states are given by s_{ik}, $k \in R = \{1, 2, ..., r\}$. Hence, $\mathbf{s}_i = (s_{ik}: k \in R)$ is the vector of attribute states of s_i. By the Euclidean distance measure, the diversity of a subset containing two elements s_i and s_j is defined as

$$d_{ij} = [\sum_{k=1}^{r} (s_{ik} - s_{jk})^2]^{1/2} \tag{6}$$

From this, we may define the diversity of a set of elements, Z, as the sum of the Euclidean distances between each distinct pair of elements, i.e.,

$$Z = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [\sum_{k=1}^{r} (s_{ik} - s_{jk})^2]^{1/2}$$

Now, suppose a manager wishes to select four engineers to work on a certain project as a group. He has rated each of the eight engineers in his department on a seven-point differential scale with respect to five personality attributes: considerate, open, pleasant, supportive, and trustworthy. The results are summarized in Table 1. The manager's goal is to identify the four engineers who are most compatible with each other in that the overall personality difference among them is as small as possible.

Table 1

Evaluation of Engineers with Respect to Relevant Attributes

No.	Name	Considerate	Open	Pleasant	Supportive	Trustworthy
1	Alan	4	2	5	5	4
2	Beth	3	4	5	1	7
3	Carl	6	6	6	7	1
4	Dave	5	5	4	4	1
5	Eileen	7	6	5	5	6
6	Frank	7	3	3	3	1
7	Greg	3	5	1	3	3
8	Hal	4	5	5	3	4

In order to formulate a minimum diversity model for the problem, we need to compute the personality distance between each pair of the engineers using the formula described in (6). For example, the Euclidean distance between Beth (No. 2) and Frank (No. 6) is $d_{26} = [(7 - 3)^2 + (3 - 4)^2 + (3 - 5)^2 + (3 - 1)^2 + (1 - 7)^2]^{1/2} \approx 7.8102$. The resulting symmetric personality distance matrix is shown in Table 2.

Table 2

Personality Distance Matrix

	1	2	3	4	5	6	7	8
1	0.0000	5.4772	5.8310	4.5826	5.3852	5.1962	5.5678	3.6056
2	5.4772	0.0000	9.2736	7.1414	6.0828	7.8102	6.0828	3.8730
3	5.8310	9.2736	0.0000	3.8730	5.5678	5.9161	7.4162	5.5678
4	4.5826	7.1414	3.8730	0.0000	5.6569	3.1623	4.2426	3.4641
5	5.3852	6.0828	5.5678	5.6569	0.0000	6.4807	6.7823	4.2426
6	5.1962	7.8102	5.9161	3.1623	6.4807	0.0000	5.2915	5.0990
7	5.5678	6.0828	7.4162	4.2426	6.7823	5.2915	0.0000	4.2426
8	3.6056	3.8730	5.5678	3.4641	4.2426	5.0990	4.2426	0.0000

It follows that the problem of determining the four most compatible engineers can be modeled as the following mixed integer program based upon (M2).

Model (M4)

Minimize $Z = 5.4772y_{12} + 5.8310y_{13} + \ldots + 4.2426y_{78}$

subject to

$$
\begin{aligned}
& x_1 + x_2 + \ldots + x_8 = 4 \\
& x_1 + x_2 - y_{12} \leq 1 \\
& \vdots \\
& x_7 + x_8 - y_{78} \leq 1 \\
& -x_1 + y_{12} \leq 0 \\
& \vdots \\
& -x_7 + y_{78} \leq 0 \\
& -x_2 + y_{12} \leq 0 \\
& \vdots \\
& -x_8 + y_{78} \leq 0 \\
& y_{12}, y_{13}, \ldots, y_{78} \geq 0 \\
& x_1, x_2, \ldots, x_8 = 0 \text{ or } 1
\end{aligned}
$$

Using QSB+ (Chang & Sullivan, 1991) to solve (M4), we obtain an optimal solution given by $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*) = (1, 0, 0, 1, 0, 1, 0, 1)$ with a total diversity of $Z^* = 25.1098$. In other words, to minimize the total personality difference of the team of engineers, the manager should select Alan, Dave, Frank, and Hal. If it is desirable to include at least one female engineer in the group, the constraint $x_2 + x_5 \geq 1$ may be added to (M4). The solution to this new problem indicates that the following four will be chosen with a slightly higher overall diversity of $Z^* = 26.9370$: Alan, Dave, Eileen, and Hal. If, alternatively, it is required that Carl be included in the group due to his previous experience with similar projects, the constraint $x_3 = 1$ may be incorporated into (M4). The optimal solution shows that the team will then consist of Alan, Carl, Dave, and Hal and the total diversity becomes $Z^* = 26.9241$. It is easy to see that other modeling considerations can be accommodated in a similar way.

HEURISTIC ALGORITHMS

Because the example problem in the previous section is quite small, there is a good chance of reaching a verified optimal solution. We will show, however, that the general problem of minimizing diversity is NP-hard (Garey and Johnson, 1979); namely, it is sufficiently difficult that no existing procedures can be expected to solve the problem within a reasonable amount of time as its size grows. A theoretical basis of such an argument is presented below (Kuo, Glover, & Dhir, 1993a).

Theorem 1. The maximization version of (M1) is NP-hard, both with and without restricting the d_{ij} coefficients to nonnegative values.

Since, given the same set of constraints, minimizing an objective function is equivalent to maximizing the negative of the objective function, the following result follows immediately:

Corollary 1. The problem of (M1) is NP-hard.

As the minimum diversity problem is intractable, attempts to solve it by general-purpose integer programming codes may encounter significant difficulties. Consequently, it is important to identify special-purpose heuristics capable of obtaining high quality solutions efficiently for handling these problems in practice. Toward that end, we will propose two constructive heuristics and two destructive heuristics for minimizing diversity.

As before, $S = \{s_i: I \in N\}$ is a set of elements with the index set $N = \{1, 2, ..., n\}$. Each element s_i contains r attributes whose states are given by s_{ik}, $k \in R = \{1, 2, ..., R\}$, and $\mathbf{s}_i = (s_{ik}: k \in R)$ is the vector of the attribute states of s_i. The problem is to select m of the n elements in S with the minimum possible diversity. Alternatively, we may express this as selecting an m-element subset M of N and the set of elements chosen may be represented by S_Selected = {s_i: I ∈ M}.

Each of the two constructive heuristics begins with S_Selected empty, and then progressively transfers elements from S to S_Selected. On the other hand, each of the two destructive heuristics begins with S_Selected = S, and then progressively eliminates elements from S_Selected. In both cases, the procedure terminates when S_Selected contains exactly m elements. Relative to the index sets N and M, we note that transferring an element from S to S_Selected corresponds to transferring an index from N to M, while eliminating an element from S_Selected corresponds to deleting an index from M. The algorithms terminate when |M| = m. Moreover, the final solution assigns values to the zero-one variables in the vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ of the preceding models according to the following rule: $x_i = 1$ if I ∈ M and $x_i = 0$ otherwise.

Some additional notation is required to characterize the heuristics. Let X = {s_i: I ∈ I} represent either S or S_Selected (depending on whether I = N or I = M) and define s_center(X) to be the center of gravity of X; that is, s_center(X) = ($\sum s_i$)/|X|. We denote the operation of deleting or adding a selected index I^* to I (hence deleting or adding a selected element s_i^* to X) by I = I - I^* or I = I + I^*, respectively. We also allow the distance d_{ij} between elements s_i and s_j to be represented by d(s_i, s_j); hence the distance between s_i and s_center = s_center(X) is expressed as d(s_i, s_center). Finally, we define the composite between s_i and X as D(s_i, X) = $\sum$ (d(s_i, s_j), j ∈ I - {I}). Observe that this composite distance stipulates that I is removed from {I}, and it is the sum of the distances between s_i and all the remaining elements in X. Of course, if $s_i \notin$ X then I = I - {I}.

The four heuristic algorithms for the minimum diversity problem follow:

Constructive Heuristic 1 (C1)

Step 0. Let S_Selected be empty and s_center = s_center(S).
Step 1. If |M| = m, terminate. Otherwise, identify I^* such that d(s_i^*, s_center) = Min(d(s_i, s_center): I ∈ N).
Step 2. Set N = N - I^*, M = M + I^*, and s_center = s_center(S_Selected). Then return to Step 1.

Destructive Heuristic 1 (D1)

Step 0. Let S_Selected = S and s_center = s_center(S_Selected).
Step 1. If |M| = m, terminate. Otherwise, identify I^* such that d(s_i^*, s_center) = Max(d(s_i, s_center): I ∈ M).

Step 2. Set M = M - I^* and return to Step 1.

Constructive Heuristic 2 (C2)

Step 0. Let S_Selected and M be empty and X = S.
Step 1. If |M| = m, terminate. Otherwise, identify I^* such that $D(s_i^*, X) = Min(D(s_i, X): I \in N)$.
Step 2. Set N = N - I^* and M = M + I^* and let X = S_Selected. Then return to Step 1.

Destructive Heuristic 2 (D2)

Step 0. Let S_Selected = S.
Step 1. If |M| = m, terminate. Otherwise, identify I^* such that $D(s_i^*, S\_Selected) = Max(D(s_i, S\_Selected): I \in M)$.
Step 2. Set M = M - I^* and return to Step 1.

In order to help understand the heuristics proposed above, we apply C1 to solve the problem of selecting four engineers discussed previously. A summary of the results from implementing the algorithm is included in Table 3. The heuristic solution suggests that the project team be formed by Alan, Dave, Frank, and Hal with a total diversity of 25.1098, which happens to be the optimal choice. As a matter of fact, it can be demonstrated that each of D1, C2, and D2 will also provide the same solution to this particular minimum diversity problem.

Table 3

Summary of Results from Implementing C1

Iteration	I^*	$d(s_i^*, s\_center)$	M	N	s-center
0			Ø	{1, 2, 3, 4, 5, 6, 7, 8}	(4.88, 4.50, 4.25, 3.88, 3.38)
1	8	1.65	{8}	{1, 2, 3, 4, 5, 6, 7}	(4.00, 5.00, 5.00, 3.00, 4.00)
2	4	3.46	{4, 8}	{1, 2, 3, 5, 6, 7}	(4.50, 5.00, 4.50, 3.50, 2.50)
3	1	3.74	{1, 4, 8}	{2, 3, 5, 6, 7}	(4.33, 4.00, 4.67, 4.00, 3.00)
4	6	3.99	{1, 4, 6, 8}	{2, 3, 5, 7}	

A COMPUTATIONAL STUDY

The heuristics presented in the previous section are all very efficient to execute. In fact, it has been shown that the theoretical complexities of C1, D1, C2, and D2 are of orders O(mnr),

$O(n^2r)$, $O(nm^2)$, and $O(n^3)$, respectively (Kuo, Glover, & Dhir, 1993a). In order to evaluate them more completely, each of the four procedures was empirically tested on a sample of 60 minimum diversity problems. We sought the optimal solutions as benchmarks by applying the branch-and-bound method to the test problems based on (M2).

The problems were randomly generated by computer, with the total number of elements considered (n), the percent selection rate (100m/n), and the number of relevant attributes possessed by each element (r) ranging, respectively, from 10 to 30, from 15 to 75, and from 2 to 21. For each n-m combination, five random examples were solved by each of the four algorithms. The deviation of the heuristic solution from the optimal solution was computed, and the average deviation (d) of the five individual deviations was then calculated. The results have been summarized in Table 4.

RESULTS AND DISCUSSION

We see from Table 4 that the first constructive heuristic (C1), though intuitive, turns out to be the least efficient of those tested. More specifically, C1 admits a largest individual deviation of 123.61%, but the majority (73%) of the heuristic solutions are within 95% of optimality. In other words, the occurrence of large individual deviation is very rare. The value d can be as large as 26.59% although the overall average deviation is 7.48%. Moreover, about 45% of the solutions obtained by C1 are optimal.

Using the first destructive heuristic (D1), the largest individual deviation is significantly smaller, at 23.00%. Thus the worst solution obtained over all runs is still 77% of optimality. The d value is never greater than 5.37%, and the overall average deviation from optimality is a mere 1.78%. In addition, D1 yields an optimal solution about 68% of the time.

The second constructive heuristic (C2) nearly rivals the performance of C1. The largest individual deviation is 123.61%, which is not as good as that with D1, and d reaches a maximum value of 26.59%. On the other hand, the overall average deviation is 7.13%, which indicates that in the large the average deviation from optimality is less than 8%. Furthermore, about 47% of the heuristic solutions obtained through C2 are in fact optimal.

Finally, the second destructive heuristic, D2, appears to perform somewhat similarly to D1. The largest individual deviation under D2 is 23.00%. The d value does not exceed 6.73%, and the overall average deviation is quite impressive for this procedure also, at 1.76%. Moreover, about 67% of the time the solution given by D2 is an optimal solution.

To gain fuller insight into the behaviors of the approximate methods, the average deviations are plotted against the percent selection rates for each of the four heuristics. An examination of

Figures 1, 2, 3, and 4 reveals that, with a few exceptions, d generally decreases as 100m/n increases.

When similar graphs are plotted for sets of elements of different sizes, we see from Figures 5, 6, and 7 that C1 and C2 give the worst solutions in the majority of the example problems, whereas D1 and D2 appear to be comparable. These observations are consistent with the results presented in Table 4.

CONCLUSION

In this paper, we have argued that the diversity minimization problem has widespread applications as well as practical importance. Our concern has been to build a broad foundation that allows the rich variety of diversity minimization problems to be viewed within a common framework. As a basis for this, we have formulated a number of interrelated integer programming models. We have also presented an illustrative example showing how our formulations can be flexibly modified to handle additional constraints.

We have shown that the theoretical complexity of the minimum diversity problem places it in the realm of NP-hard problems. This observation motivates the development of special procedures in order to generate solutions of high quality efficiently, since standard integer programming packages cannot be expected to fulfill this need. We therefore have developed four heuristic approaches using simple constructive and destructive moves. Applied to a testbed of 60 problems of varying dimensions, two of the approximate algorithms gave solutions whose average deviation from optimality was approximately 7%, while the other two rendered solutions with an average deviation of less than 2%. From an empirical standpoint, this experimentation represents only a beginning, but it suggests that problems from the diversity minimization class can be handled effectively by appropriately designed heuristics.

In taking these first steps, we concede that a great deal remains to be done. The diversity minimization problem poses a challenge to researchers and practitioners to develop an extended repertoire of models and solution approaches to deal with important variants. The scope of this area and the need to address the problem directly call for further examination of its fundamental elements in future investigations.

Table 4

Summary of Empirical Results

n	m	d = (h - 0)/0 * 100%			
		C1	D1	C2	D2
10	2	26.59	1.61	26.59	1.61
	3	4.11	5.37	4.11	6.73
	4	4.88	0.73	4.88	0.73
	6	2.27	0.38	0.17	0.38
	8	0.14	0.00	0.14	0.00
15	2	12.59	1.02	12.59	4.02
	4	4.63	3.12	2.26	2.40
	6	0.37	0.82	0.51	0.82
	9	3.11	0.42	3.11	0.08
	11	0.00	0.00	0.00	0.00
30	5	24.24	2.51	24.24	1.33
	8	6.89	2.42	6.97	3.02
Largest individual deviation		123.61	23.00	123.61	23.00
Overall average deviation		7.48	1.78	7.13	1.76
% of time solution is optimal		45	68	47	67
% of time solution is within 95% of optimum		73	90	77	90

0 : optimal total diversity
h: heuristic total diversity
d: average percent deviation from optimality based on 5 random examples

Figure 1

Empirical Performance of C1

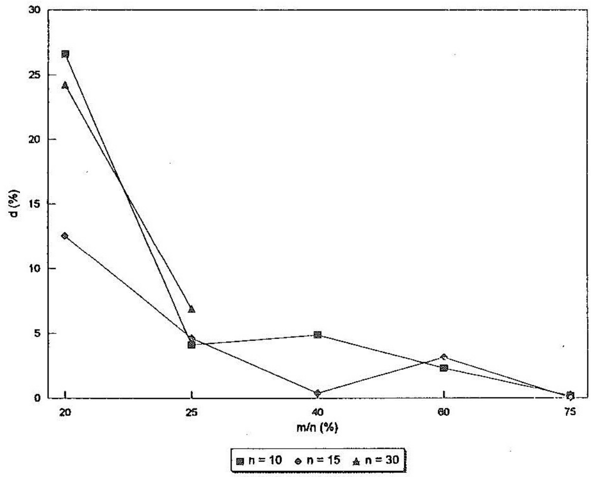


Figure 2

Empirical Performance of D1

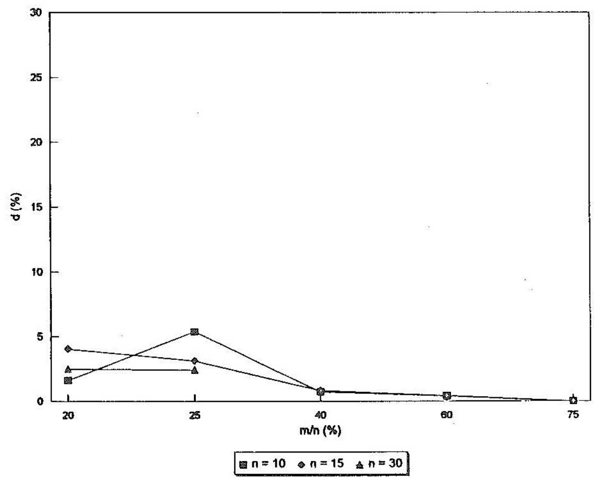


Figure 3

Empirical Performance of C2

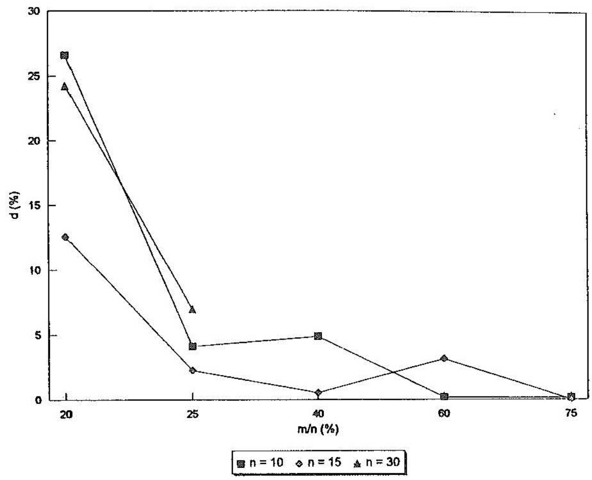


Figure 4

Empirical Performance of D2

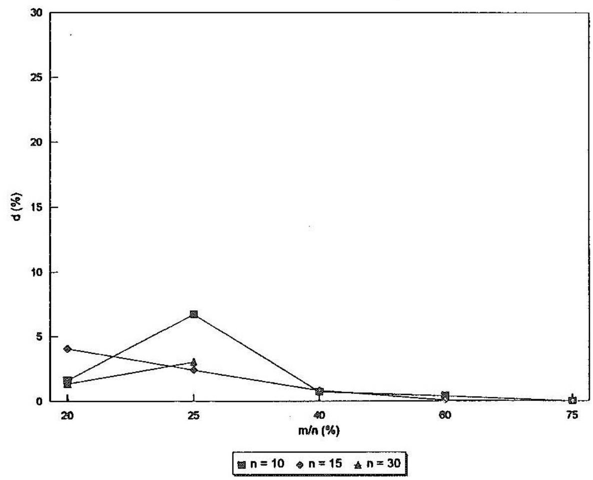


Figure 5

Empirical Performance of Four Heuristics with n = 10

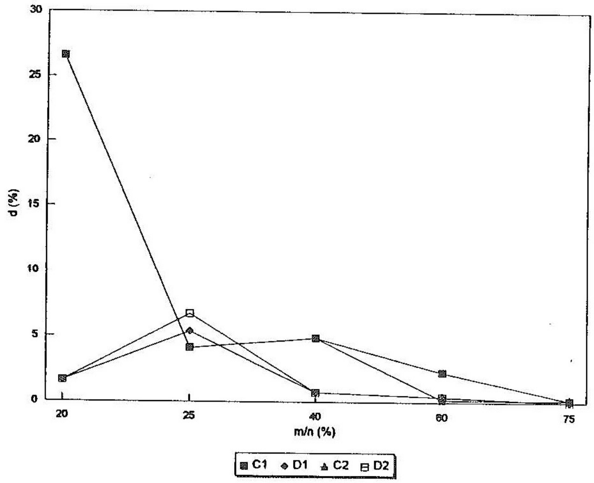


Figure 6

Empirical Performance of Four Heuristics with n = 15

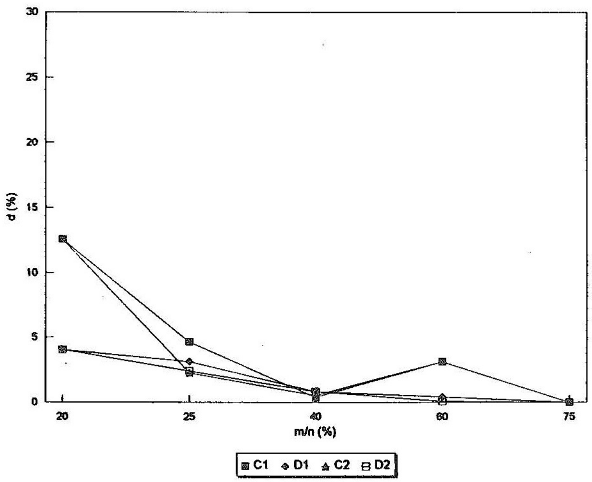
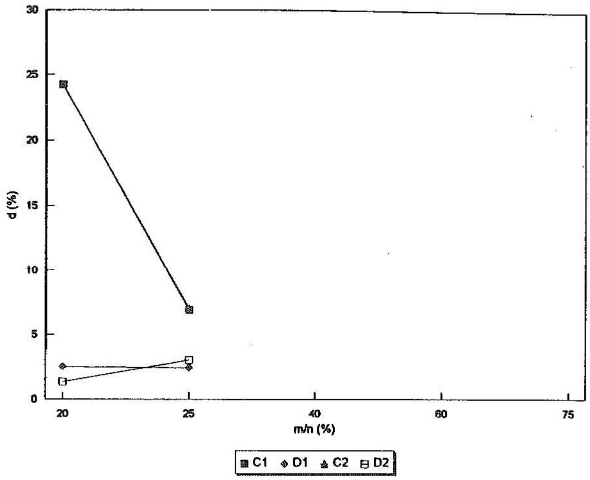


Figure 7

Empirical Performance of Four Heuristics with n = 30



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