THE OPTIMAL ORDER QUANTITY PROBLEM WITH QUANTITY DISCOUNTS: A MIXED BIVALENT INTEGER FORMULATION

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A bivalent mixed integer formulation of the economic order quantity problem is presented here. This model accommodates varying demand over the several periods of a planning horizon. It seeks to minimize the total of cost of goods, fixed ordering costs and inventory carrying costs over the horizon. The model incorporates quantity discounts that are made possible if a purchase amount is large enough. The model determines the amount of inventory to acquire in each period of the time horizon. It establishes the cost of goods by determining the purchase quantity in each period to match the corresponding unit price.

The lot size problem has been very notable in operations management literature for years. The familiar square root formula appears in every traditional operations management text. It is well known that this formula arises from some very restrictive conditions. One of those is the assumption of constant demand over time. Another is that the cost of the goods is constant, and thus can be ignored.

An extension permits inclusion of quantity discounts. Here, several different total cost functions are nested vertically and the true total cost will jump from one cost curve to the next at the price breaks. The optimal order quantity is easily seen to occur either at the minimum point of one of the cost functions or at a price break.

When demand is not constant over time the problem gets much more challenging. It is common in this situation to divide the time continuum into consecutive periods, such as months. A separate inventory acquisition decision arises in each time period. The objective is to minimize the total of cost of goods, fixed costs of placing orders and inventory carrying costs. The cost of goods will remain constant and immaterial if the unit cost is always fixed. Wagner and Whitin formulated this problem in 1958 and demonstrated a convergent search routine that

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yields an optimal ordering schedule. A similar construct was also presented by Manne in 1958. Dynamic programming has been used extensively to analyze extended versions of this problem. The work by Denardo (1982) is an excellent source for illustrating the dynamic programming approach. A very recent work using dynamic programming is that of Aggarwal and Park (1993), who use Monge arrays to bring about dramatic computational efficiencies. The problem is similar to that of economic lot sizing and production scheduling. Zangwill demonstrated the network nature of the production/inventory scheduling problem as early as 1966, 1968 and 1969.

The quantity discount problem for a single item was considered by Widrick (1985). His focus was on the determination of the pricing of the goods. Extensions to the problem of multiple items have recently been analyzed by Katz, Sadrian and Tendick (1994), Pirkul and Aras (1985) and by Sadrian and Yoon (1994). As pointed out by Rosenthal, Zydiak and Chaudry (1995), these formulations are quite intractible. The goal programming approach was presented by Weber and Current (1993). They also considered the case of multiple vendors. This approach was further extended by Rosenthal, Zydiak and Chaudry (1995) to bundling of purchases from multiple vendors when the vendors offer price breaks on one item when quantities of other items are sufficiently large.

The problem considered here is one of determining the optimal order lot size of a single good under conditions of variable demand, discrete time periods and the availability of quantity discounts. A mixed integer programming model of the problem will be formulated and the analysis will permit consideration of limits upon order quantities.

THE MODEL

This problem arose from an inquiry made to the author by a small manufacturer. They produce a frozen dairy product that has a highly seasonal demand pattern. Summer sales are quite high and December - January holiday season sales are strong. Demand is relatively low in the spring and fall. Raw materials are costly to store. Quantity discounts on raw material purchases are available. They seek to determine an optimal ordering schedule for raw material acquisitions. The cost function to minimize includes cost of goods, fixed costs of placing orders and inventory carrying costs. They desire to make monthly purchase decisions over a limited planning horizon. While this manufacturer desired to establish monthly decision points, it is generally necessary that the duration of a single decision period should be so short that product demand can be taken to be constant and deterministic over the course of the period.

The variables and parameters of the problem are:

N = number of time periods in the planning horizon

J = number of price levels available through discounting

D_i = demand in time period I

INV_i = inventory at end of time period I

X_{ji} = number of units to purchase at price level j in period I

h = unit inventory holding cost per time period

I_{ji} = $\{0,1\}$ is a bivalent variable and equals 1 if price level j is used for purchases

C_j = upper limit on quantity available at price j

F = fixed cost of placing an order

C_{ji} = cost per unit at price level j in period I

J_i = $\{0,1\}$ and equals 1 if a purchase is made in time period I, equals 0 otherwise

M = maximum allowable purchase quantity in any period

Z_1 = total cost of purchasing goods

Z_2 = total ordering cost

Z_3 = total inventory carrying cost

The mixed integer model is given as:

minimize: $Z = Z_1 + Z_2 + Z_3$ (1)

subject to:

$$\sum_{j=1}^{J} \sum_{I=1}^{N} C_{ji} X_{ji} = Z_1 \qquad (2)$$

$$F \sum_{j=1} \sum_{I=1} I_{ji} = Z_2 \qquad (3)$$

$$h \sum_{I=1}^{N} INV_i \qquad = Z_3 \qquad (4)$$

$$INV_{i-1} + X_{1i} + X_{2i} + \dots + X_{Ji} - INV_i \qquad = D_i \qquad (5)$$
$I = 1,2,\dots,N$

$$X_{1i} \leq C_1 I_{1i}$$

$$(C_1 + 1) I_{2i} \leq X_{2i} \leq C_2 I_{2i}$$

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$$(C_{J-2} + 1) I_{(J-1)I} \leq X_{(J-1)I} \leq C_{(J-1)} I_{(J-1)I}$$

$$(C_{J-1} + 1) I_{Ji} \leq X_{Ji} \qquad (6)$$
$I = 1, 2,\dots,N-1$

$$X_{Ji} \leq I_{Ji} M \qquad (7)$$

$$I_{1i} + I_{2i} + \dots + I_{Ji} = J_i \quad I = 1,2,\dots,N \qquad (8)$$

all X_{ji}, INV_i, Z_1, Z_2, $Z_3 \geq 0$
all $I_{ji} = \{0,1\}$
all $J_i = \{0,1\}$

The objective function in (1) seeks the minimization of the sum of cost of goods plus ordering costs plus inventory carrying cost. Constraints of (2), (3) and (4) establish these three parts as Z_1, Z_2 and Z_3. The constraints of (5) declare that ending inventory in any time period I is obtained as the sum of beginning inventory plus new acquisitions minus demand. The quantity intervals for the price breaks are established with constraint set (6). The purchase amount X_{ji} is forced to be either zero or in the interval from $C_{j-1} + 1$ to C_j. Note that if I_{ji} is zero, then the purchase amount X_{ji} must be zero as well. However, if $I_{ji} = 1$ then the constraints require that X_{ji} be between $C_{j-1} + 1$ and C_j. The maximum allowable acquisition in any period is set by the constraints of (7).

The constraints of (8) permit the use of at most one price level j in any time period I. The model will thereby force the selection of the time periods during which purchases will be made, the number of units to purchase and the appropriate corresponding unit price level. There will be JN of the X_{ji} and I_{ji} variables and J of the J_i and INV_i variables, for a total of 2JN + 2J operational variables in the formulation.

AN EXAMPLE

The parameter values here are totally unrelated to the operating environment of the above manufacturer. They are artificial and used only to illustrate the structure of the model. Let the time periods be eight consecutive months. Note that the standard time unit used in the model must be sufficiently small so that there do not exist any significant fluctuations in demand over the course of a single interval. For the purpose of this illustration the requirement means that there should be no significant intra month demand fluctuations. This condition would be easily satisfied if, for example, there were just a single product delivery to be made at the end of the month.

Take the inventory carrying cost to be h = $1.20 per unit per month and the fixed ordering cost to be F = $30. Further, let there be three price levels available in any month. If the purchase quantity is 50 units or less the unit price will be $3. If the order quantity is between 51 and 75 units the purchase price will be $2.80. If the purchase quantity is at least 76 units, then the unit price is $2.50.

Let the 8 monthly demands be for 63, 46, 36, 32, 37, 54, 67 and 78 units. The formulation of this problem is presented in Table 1. For convenience, the three components of the objective function were established as constraints equal to Z_1, Z_2 and Z_3, respectively. The objective function is thereby simply given as the minimization of $Z_1 + Z_2 + Z_3$. Inventory at the beginning of the eight months is taken to be zero. Maximum permitted inventory purchases in any month are limited to the amount M = 1000.

The optimal solution to this formulation is seen in Table 2. The minimum total cost is $1374.80. It is composed of the three parts $Z_1 + Z_2 + Z_3$. The total cost of goods, Z_1, is $1085.60. The total ordering cost is Z_2 = $180.00. The total inventory carrying cost is Z_3 = $109.20. The purchase month variables J1, J2, J4 branches there will also be linear programming optimization for the set values of the bivalent variables. As an illustration, this example problem required 12,684 iterations.

Table 1

The Mixed Integer Formulation

MIN Z1 + Z2 + Z3
SUBJECT TO
2) - Z1 + 3 X0101 + 2.8 X0201 + 2.5 X0301 + 3 X0102 + 2.8 X0202
+ 2.5 X0302 + 3 X0103 + 2.8 X0203 + 2.5 X0303 + 3 X0104 + 2.8 X0204
+ 2.5 X0304 + 3 X0105 + 2.8 X0205 + 2.5 X0305 + 3 X0106 + 2.8 X0206
+ 2.5 X0306 + 3 X0107 + 2.8 X0207 + 2.5 X0307 + 3 X0108 + 2.8 X0208
+ 2.5 X0308 = 0
3) J1 + J2 + J3 + J4 + J5 + J6 + J7 + J8 - JZ = 0
4) - Z2 + 30 JZ = 0
5) - Z3 + 1.2 INV1 + 1.2 INV2 + 1.2 INV3 + 1.2 INV4 + 1.2 INV5
+ 1.2 INV6 + 1.2 INV7 + 1.2 INV8 = 0
6) X0101 + X0201 + X0301 - INV1 = 63
7) X0102 + X0202 + X0302 + INV1 - INV2 = 46
8) X0103 + X0203 + X0303 + INV2 - INV3 = 36
9) X0104 + X0204 + X0304 + INV3 - INV4 = 32
10) X0105 + X0205 + X0305 + INV4 - INV5 = 37
11) X0106 + X0206 + X0306 + INV5 - INV6 = 54
12) X0107 + X0207 + X0307 + INV6 - INV7 = 67
13) X0108 + X0208 + X0308 + INV7 - INV8 = 78
14) - 50 I0101 + X0101 <= 0
15) - 51 I0201 + X0201 >= 0
16) - 75 I0201 + X0201 <= 0
17) - 76 I0301 + X0301 >= 0
18) - 1000 I0301 + X0301 <= 0
19) - 50 I0102 + X0102 <= 0
20) - 51 I0202 + X0202 >= 0
21) - 75 I0202 + X0202 <= 0
22) - 76 I0302 + X0302 >= 0
23) - 1000 I0302 + X0302 <= 0
24) - 50 I0103 + X0103 <= 0
25) - 51 I0203 + X0203 >= 0
26) - 75 I0203 + X0203 <= 0
27) - 76 I0303 + X0303 >= 0
28) - 1000 I0303 + X0303 <= 0
29) - 50 I0104 + X0104 <= 0
30) - 51 I0204 + X0204 >= 0
31) - 75 I0204 + X0204 <= 0
32) - 76 I0304 + X0304 >= 0
33) - 1000 I0304 + X0304 <= 0
34) - 50 I0105 + X0105 <= 0
35) - 51 I0205 + X0205 >= 0
36) - 75 I0205 + X0205 <= 0
37) - 76 I0305 + X0305 >= 0
38) - 1000 I0305 + X0305 <= 0
39) - 50 I0106 + X0106 <= 0
40) - 51 I0206 + X0206 >= 0
41) - 75 I0206 + X0206 <= 0

42) - 76 I0306 + X0306 >= 0
43) - 1000 I0306 + X0306 <= 0
44) - 50 I0107 + X0107 <= 0
45) - 51 I0207 + X0207 >= 0
46) - 75 I0207 + X0207 <= 0
47) - 76 I0307 + X0307 >= 0
48) - 1000 I0307 + X0307 <= 0
49) - 50 I0108 + X0108 <= 0
50) - 51 I0208 + X0208 >= 0
51) - 75 I0208 + X0208 <= 0
52) - 76 I0308 + X0308 >= 0
53) - 1000 I0308 + X0308 <= 0
54) - J1 + I0101 + I0201 + I0301 = 0
55) - J2 + I0102 + I0202 + I0302 = 0
56) - J3 + I0103 + I0203 + I0303 = 0
57) - J4 + I0104 + I0204 + I0304 = 0
58) - J5 + I0105 + I0205 + I0305 = 0
59) - J6 + I0106 + I0206 + I0306 = 0
60) - J7 + I0107 + I0207 + I0307 = 0
61) - J8 + I0108 + I0208 + I0308 = 0

END

Also, J1, J2, ..., J8, I0101, I0201, ..., I0308 = {0,1}

Table 2
The Optimal Solution to the Example Problem

OBJECTIVE FUNCTION VALUE

1) 1374.8000

VARIABLE	VALUE	REDUCED COST
J1	1.000000	30.000000
J2	1.000000	30.000000
J3	.000000	30.000000
J4	1.000000	30.000000
J5	.000000	30.000000
J6	1.000000	30.000000
J7	1.000000	30.000000
J8	1.000000	30.000000
I0101	.000000	.000000
I0201	1.000000	.000000
I0301	.000000	-299.999900
I0102	.000000	.000000
I0202	.000000	15.300000
I0302	1.000000	.000000
I0103	.000000	-35.000000
I0203	.000000	-67.500010
I0303	.000000	-1200.000000
I0104	.000000	.000000
I0204	.000000	.000000
I0304	1.000000	250.800000
I0105	.000000	.000000
I0205	.000000	.000000
I0305	.000000	.000000
I0106	.000000	.000000
I0206	1.000000	61.200000
I0306	.000000	.000000
I0107	.000000	.000000
I0207	1.000000	.000000
I0307	.000000	-299.999900
I0108	.000000	.000000
I0208	.000000	.000000
I0308	1.000000	.000000
Z1	1085.600000	.000000
Z2	180.000000	.000000
Z3	109.200000	.000000
X0101	.000000	.200000
X0201	63.000000	.000000
X0301	.000000	.000000
X0102	.000000	.500000
X0202	.000000	.000000
X0302	82.000000	.000000
X0103	.000000	.000000
X0203	.000000	.000000
X0303	.000000	.000000
X0104	.000000	3.800000

X0204	.000000	3.600000
X0304	76.000000	.000000
X0105	.000000	2.600000
X0205	.000000	2.400000
X0305	.000000	2.100000
X0106	.000000	1.400000
X0206	51.000000	.000000
X0306	.000000	.900000
X0107	.000000	.200000
X0207	63.000000	.000000
X0307	.000000	.000000
X0108	.000000	.500000
X0208	.000000	.300000
X0308	78.000000	.000000
JZ	6.000000	.000000
INV1	.000000	1.500000
INV2	36.000000	.000000
INV3	.000000	5.700000
INV4	44.000000	.000000
INV5	7.000000	.000000
INV6	4.000000	.000000
INV7	.000000	1.500000
INV8	.000000	3.700000

NO. ITERATIONS = 12684
BRANCHES = 1079 DETERM. = 1.000E 0

CONCLUSION

A mixed integer formulation of the lot size problem has been presented here. It determines the optimal inventory purchase decision when there are both uneven demands over time and price breaks available from making large orders. The objective function seeks to minimize the sum of cost of goods plus fixed ordering cost plus inventory carrying cost. One set of constraints guarantees that demand will be met through carrying and replenishing inventory. Another set forces new purchase amounts to conform to the intervals that fit the price breaks. An example problem was worked to a successful conclusion. It is noted that computer solution of this problem may require a large amount of numerical calculations because of the presence of the bivalent variables. However, the example problem presented here was solved in just a few minutes on a desktop microcomputer with a commercially available software package.

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