Hydraulic analogy and visualisation of two-dimensional compressible fluid flows: part 1: theoretical aspects

M. Hafez

Department of Mechanical and Aerospace Engineering,
University of California,
Bainer Hall, One Shields Ave.,
Davis, CA 95616-5294, USA
Email: m.hafez.farahat@gmail.com

Abstract: The principles of the well-known, hydraulic analogy are explained (see Courant and Friedricks, 1948; Loh, 1969). The governing equations of two-dimensional compressible fluid two-dimensional flows in non-dimensional form, based on conservation laws are first discussed. Then, the isentropic flow condition is introduced to produce the isentropic Euler equations. On the other hand, the equations governing surface waves on thin water layers over a flat surface are derived in non-dimensional form, using the assumption of hydrostatic pressure across the water layer, hence the analogy between the two problems is established. The normalised density of the compressible flow corresponds to the normalised height of the thin water layer and the speed of sound corresponds to the speed of surface waves in water, hence, the Mach number corresponds to Froude number. Finally, it is shown that the analogy can be used to visualise supersonic and transonic two-dimensional flow patterns, including shock waves and expansion fans around airfoils and in convergent/divergent nozzles. Also, nonlinear water waves of finite amplitude, in dispersive media, are discussed. In part 2 of this study, water table experiments are presented together with qualitative and quantitative measurement techniques.

Keywords: two dimensional transonic; supersonic flows; shallow water surface waves; hydraulic analogy.

Reference to this paper should be made as follows: Hafez, M. (2018) ‘Hydraulic analogy and visualisation of two-dimensional compressible fluid flows: part 1: theoretical aspects’, *Int. J. Aerodynamics*, Vol. 6, No. 1, pp.41–66.

Biographical notes: M. Hafez has a PhD degree in Aerospace Engineering from University of Southern California (1972). He worked for Flow Research Inc. and as a contractor for NASA Langley, before he joined University of California, Davis, as a Full Professor in 1985.

1 Introduction

Supersonic flows are interesting because of their theoretical and practical aspects. The governing equations are nonlinear and their solutions include shock waves and expansion fans. These phenomena are also important for practical applications in supersonic aerodynamics. Nowadays, supersonic flight is limited to military airplanes and small business jets. Recently, NASA and Boeing are considering supersonic commercial airplanes and problems of sonic boom and wave drag become critical issues again.

The purpose of this paper is to introduce students to an affordable method of visualisation of two-dimensional supersonic flows over airfoils and in nozzles based on the theory of hydraulic analogy. Experiments will help the students to understand the nature of these flows and consequently will affect their analysis, design and optimisation processes. The cost of a supersonic wind tunnel is prohibitive and their maintenance is expensive, therefore most of the students do not see supersonic flow patterns in their schools.

Fortunately, the well-known analogy between two-dimensional supersonic flow and the surface waves on a shallow water layer over a flat surface leads to the simple water table experiments. The details will be discussed below.

This paper consists of four sections: the governing equations of two-dimensional compressible fluid flows, the governing equations of shallow water waves and the principle of hydraulic analogy and practical applications of this analogy.

Finally, some concluding remarks are mentioned.

2 Governing equations of two-dimensional compressible fluid flows

At high Reynolds number, viscous effects are confined to the boundary layers and wakes. In the following, boundary layer is assumed to be thin and attached to the body, hence the flow outside the boundary layer is assumed to be inviscid. To account for the boundary layer effects, viscous/inviscid interaction procedures in terms of displacement thickness can be used (see for example, Chattot and Hafez, 2015).

The standard governing equations for the inviscid flows are based on conservation laws of mass, momentum and energy as well as the perfect gas law, (see for example, Thompson, 1972).

For unsteady two-dimensional flows, the equations in Cartesian coordinates and standard notations are given by:

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$

$$(\rho u)_t + \left(\rho u^2\right)_x + (\rho uv)_y = -P_x$$

$$(\rho v)_t + (\rho uv)_x + \left(\rho v^2\right)_y - P_y$$

$$(\rho H)_t + (\rho uH)_x + (\rho vH)_y = -P_t$$

$$H = h + \frac{1}{2}\left(u^2 + v^2\right)$$

where H is the total enthalpy and h is the specific enthalpy. Here, $h = c_pT$ and c_p is the specific heat under constant pressure.

In the above equations, the density ρ and the velocity components u, v are normalised by the corresponding free stream values ρ_∞ and V_∞, while x, y are normalised by the length l and t by l / V_∞, the pressure P is normalised by $\rho_\infty V_\infty^2$ and the temperature T by V_∞^2 / c_p. The total and specific enthalpies are normalised by V_∞^2. The perfect gas law becomes $P = \rho T\left(\frac{\gamma - 1}{\gamma}\right)$ where $R = C_p - C_v$ and $\gamma = C_p / C_v$, where C_p and C_v are the specific heat under constant pressure and constant volume respectively. The governing equations are written in conservation form and their weak solution admits the Rankine-Hugoniot jump conditions (see Oswatitsch, 1956; Liepmann and Toshko, 1957; Shapiro, 1953; Zucrow and Hoffman, 1976; Landau and Lifshitz, 1987).

In non-conservative form, the equations read:

$$\rho_t + \rho\left(u_x + v_y\right) + u\rho_x + v\rho_y = 0$$

$$\rho u_t + \rho u u_x + \rho v u_y = -P_x$$

$$\rho vt + \rho u v_x + \rho v v_y = -P_y$$

$$\rho H_t + \rho u H_x + \rho v H_y = P_t$$

The characteristic relations for time dependent one space dimensional case and for steady two-space dimensional cases are given in Thompson (1972).

Another model, which will be used in the present study, is the isentropic flow equations, where the energy equation is replaced by the isentropic relations:

$$\frac{P}{P_r} = \left(\frac{\rho}{\rho_r}\right)^\gamma$$

where P_r and ρ_r are reference pressure and density respectively. Using our normalisation, the isentropic relation reads:

$$P = \rho^\gamma / \gamma M_\infty^2$$

where $M_\infty^2 = \frac{V_\infty^2}{a_\infty^2}$ and a_∞^2 is the speed of sound at infinity, $a_\infty^2 = \gamma P_\infty / \rho_\infty$.

A simpler model can be derived based on the assumption that the flow is irrotational. For certain applications, one can use the condition of zero vorticity, namely $u_x = v_y$.

In this case, there exist a potential function such that $u = \Phi_x$ and $u = \Phi_y$. Hence, the governing equations become:

$$\rho_t + \left(\rho\Phi_x\right)_x + \left(\rho\Phi_y\right)_y = 0$$

The momentum equation can be integrated to yield Bernoulli's law for isentropic compressible flows and hence the density can be related to the velocity components by the following relation:

$$\rho = \left[1 - \frac{\gamma - 1}{2}\omega\left(\Phi_x^2 + \Phi_y^2 + 2\Phi_t - 1\right)\right]^{\frac{1}{\gamma - 1}}$$

The non-conservative form of the above equation reads, assuming that the far field is steady.

$$\Phi_{tt} + 2\Phi_x\Phi_{xt} + 2\Phi_y\Phi_{yt} = \left(a2 - \Phi_x^2\right)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \left(a2 - \Phi_y^2\right)\Phi_{yy}$$

where

$$\frac{a^1}{\gamma-1} + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2 + 2\Phi_t^2\right) = \frac{1}{\gamma-1}\frac{1}{M_\infty^2} + \frac{1}{2}$$

For small disturbance approximation φ, the reduced equation becomes

$$M_\infty^2\varphi_{tt} + 2M_\infty^2\varphi_{xt} = \left(1 - M^2\right)\varphi_{xx} + \varphi_{yy}$$

where

$$1 - M^2 = \left(1 - M_\infty^2\right) - (\gamma+1)M_\infty^2\varphi_x.$$

The first term φ_{tt} can be neglected for low reduced frequency cases (see Chattot and Hafez, 2015; Ashley and Landahl, 1965).

2.1 Transonic similitude

For the transonic small disturbance equation, φ and y can be rescaled, so the equation will read,

$$\left(K - \overline{\varphi}_x\right)\overline{\varphi} + \overline{\varphi}_{YY} = 0$$

where $K = \dfrac{1 - M_\infty^2}{\in M_\infty^2(\gamma+1)}$

φ is of order $\in$

and γ is of order $1/\delta$, $\delta = M_\infty\sqrt{\in(\gamma+1)}$

and $\in = \left(\tau/M_\infty\right)^{2/3}(\gamma+1)^{-1/3}$, where τ is the thickness parameter

In this formulation, the effect of γ on the solution is absorbed in the similarity parameter K (Ashley and Landahl, 1965).

2.1.1 Special cases

For convenience, one dimensional steady flow will be considered next, to demonstrate the basic ideas. First, the speed of sound formula is derived followed by the normal and oblique shock jump conditions and the expansion fan relations.

2.2 Speed of sound

Consider a source of noise with disturbance propagating through stand still air with speed u. If the front of the wave has a large radius, a one dimensional model is adequate. To render the problem to a steady one, let the front be fixed and the air goes through it (in the opposite direction).

The conservation of mass in a control volume around the wave front implies

$$\rho u h = constant$$

or $\dfrac{\Delta\rho}{\rho} = -\dfrac{\Delta u}{u}$ since h, the height of the control volume, does not change.

Assuming smooth solution and applying equation of motion to fluid particles yields

$$\rho u \Delta u = -\Delta p$$

where $\Delta\rho$, Δu and Δp represent the variation in density, velocity and pressure, respectively.

Eliminating Δu from the two equations gives:

$$\frac{\Delta p}{\Delta \rho} = u^2$$

Newton assumed the process to be isothermal hence he claimed that the speed of propagation is

$$u^2 = RT.$$

On the other hand, Laplace argued that the smooth, adiabatic process is in fact isentropic, hence

$$\frac{dp}{d\rho} = \frac{\gamma p}{\rho} = \gamma RT$$

2.2.1 Normal shock wave relations

If the source of noise is replaced by an intense explosion, the conservation laws across the front (shock) read:

$$\rho u = \dot{m}$$

$$P + \rho u^2 = \dot{M}$$

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}u^2 = H$$

eliminating ρ from the third equation gives,

$$\frac{\gamma}{\gamma - 1} \cdot \frac{Pu}{\dot{m}} + \frac{1}{2}u^2 = H$$

and eliminating P yields

$$\frac{\gamma}{\gamma - 1}\frac{(\dot{M} - \dot{m}u)}{\dot{m}} + \frac{1}{2}u^2 = H$$

This is a quadratic equation in u, with two roots. After some manipulations, one can show that

$$u_1 u_2 = a^{*2}$$

where

$$\frac{a^{*2}}{\gamma-1}+\frac{1}{2}a^{*2}=H\left(\text{or } H=\frac{\gamma+1}{2(\gamma-1)}a^{*2}\right)$$

a^* is the critical speed of sound, namely (the speed of sound at sonic condition, i.e., $u = a$). Hence, one arrives at Prandtl relation for normalised shocks:

$$M_1^* \cdot M_2^* = 1$$

where $M^* = u/a^*$.

Excluding the trivial solution $M_1^* = M_2^* = 1$, where the flow is smooth and at sonic condition, there are two possibilities

$$M_1^* > 1 \text{ and } M_2^* < 1$$

or

$$M_1^* < 1 \text{ and } M_2^* > 1.$$

Both possibilities satisfy the three conservation laws!

It turns out the second case, is not acceptable based on the second law of thermodynamics since in this case $\Delta S < 0$. Therefore, shocks jump only from supersonic to subsonic flows and not vice versa.

The jump conditions for ρ and P, can be easily obtained in terms of M_1^* as follows:

$$\frac{\rho_2}{\rho_1}=\frac{u_1}{u_2}=\frac{u_1^2}{u_1 u_2}=M_1^{*2}$$

while $P_2 - P_1 = (1-M_1^{*2})$ where pressures are normalised by $\rho_1 u_1^2$ and $\frac{T_2}{T_1}=\frac{P_2}{P_1}\frac{\rho_2}{\rho_1}$.

2.2.2 Mach number and mach angle

The Mach number is defined as the ratio of the particle v to the speed of sound a, i.e., $Ma = v / a$. The significance of Mach number can be explained in several ways:

1 Let, $M^2=\frac{v^2}{a^2}=\frac{v^2}{\gamma RT}=\frac{\left(\frac{2}{\gamma R}\right)}{c_v}\left(\frac{\frac{1}{2}v^2}{c_v T}\right)$.

Hence, M^2 is proportional to the ratio of kinetic energy per unit mass to internal energy per unit mass.

2 Let $M^2=\frac{v^2}{\frac{\gamma P}{\rho}}=\frac{2}{\gamma}\left(\frac{\frac{1}{2}\rho v^2}{P}\right)$.

Hence, M^2 is proportional to the ratio of the dynamic pressure to static pressure.

3 Let, $M^2 = \dfrac{v^2}{\Delta p/\Delta\rho} = \dfrac{\Delta\rho/\rho}{\Delta P/\rho v^2}$.

Hence, M^2 is the ratio of relative change of density to the change of pressure normalised by the dynamic pressure.

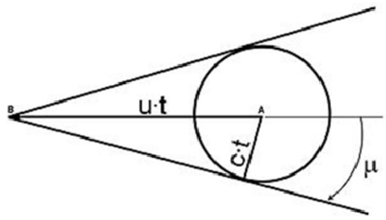
4 Let, $M^2 = \dfrac{v^2}{\Delta P/\Delta\rho} = \dfrac{-\Delta\rho/\rho}{\Delta v/v}$: (since $\rho v\Delta v = -\Delta P$).

Hence, M^2 is proportional to the relative change of density to the relative change of velocity. Therefore, M^2 is an indication of compressibility effects.

For supersonic flow, $Ma > 1$, the particle is faster than the noise. The Mach angle μ (see Figure 1) is given by

$$\sin\mu = \frac{a\cdot t}{v\cdot t} = \frac{1}{Ma} \text{ and } \tan\mu = \frac{1}{\sqrt{Ma^2-1}}$$

Figure 1 Sketch of Mach line and Mach angle



Notice, the relation between M and M^* can be obtained from Bernoulli's law:

$$\frac{a^z}{\gamma-1} + \frac{1}{2}u^2 = \frac{\gamma+1}{2(\gamma-1)}a^{*2}$$

hence,

$$\frac{1}{\gamma-1}\frac{M^{*2}}{M^2} + \frac{1}{2}M^{*2} = \frac{\gamma+1}{2(\gamma-1)}.$$

Notice, the shock relations in isentropic Euler and in potential equations are different from Rankine-Hugoniot relations associated with full Euler equations.

If the conservation of energy is replaced by the isentropic condition, the jump conditions across an isentropic shock are:

$$[\overline{\rho u}] = 0 \text{ and } \left[\frac{1}{\gamma M_r^2}\bar{\rho}^\gamma\right] + \dot{m}_r[\bar{u}] = 0, \text{ or } \frac{1}{\gamma M_r^2}\left[\left(\frac{\dot{m}_r}{\bar{u}}\right)^\gamma\right] + \dot{m}_r[\bar{u}] = 0$$

Hence, the above relation across the isentropic shock replaces Prandtl relation of the standard Euler equation.

The normal shock relations for the full potential and nonlinear small disturbance equations are given in Chattot and Hafez (2015), as

$$\left[\left(\left(1-\frac{\gamma-1}{2}\right)M_r^2\left(u^2-1\right)\right)^{\frac{1}{\gamma-1}}u\right]=0$$

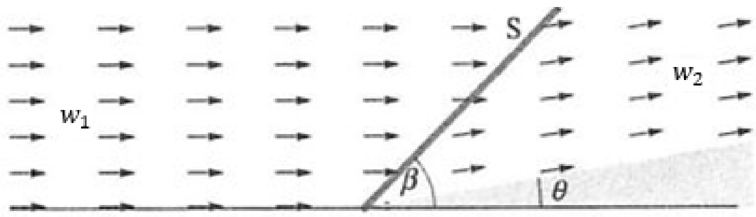
and

$$\left[\left(1-M_\infty^2\right)\varphi_x-\left(\left(\frac{\gamma+1}{2}M_\infty^2\right)\varphi_x^2\right)\right]$$

2.3 Oblique shock relation

Consider supersonic flow over a wedge as in Figure 2 (see Courant and Friedricks, 1948; Chattot and Hafez, 2015; Oswatitsch, 1956; Thompson, 1972).

Figure 2 Sketch of oblique attached shockwave



Let us decompose w_1 into a normal component to the shock u_1 and a tangential component v_1. Similarly, we decompose w_2 into u_2 and v_2. The two normal components u_1 and u_2 are governed by the relations of normal shock waves. The Mach number upstream of the oblique shock is $M_1=\frac{w_1}{a_1}$ and $u_1 = w_1 \sin\beta$, hence,

$$\frac{u_1}{a_1}=M_1\sin\beta,\ \mu\le\beta\le\frac{\pi}{2}.$$

The oblique shock relations are obtained by replacing M_1 in normal shock relations by $M_1 \sin\beta$.

We also notice,

$$M_2=\frac{w_2}{a_2}$$

hence,

$$\frac{u_2}{a_2}=M_2\sin(\beta-\theta).$$

The resulting relations for oblique shocks contains $\sin\beta$ and $\sin(\beta-\theta)$. To determine the shock angle β, we need to impose the conservation of tangential momentum. Taking a control volume along the oblique shock and shrinking its width, the conservation of tangential momentum requires

$$\rho_1 u_1 v_1 = \rho_2 u_2 v_2$$

or

$$v_1 = v_2 = v$$

since,

$$\rho_1 u_1 = \rho_2 u_2$$

The condition on the angle β is obtained from the relation

$$\frac{\tan(\beta-\theta)}{\tan(\beta)} = \frac{\frac{u_2}{v}}{\frac{u_1}{v}} = \frac{u_2}{u_1}$$

It turns out that for a given θ and M_1, there are two solutions β_1 and β_2. Both are attached shocks. There is also a maximum value for θ for attached shocks. For higher values of θ, a bow shock is detached from the wedge. The shock relations can be plotted in the hodograph plane where the coordinates are $\frac{u}{a^*}$ and $\frac{v}{a^*}$ (see Liepmann and Toshko, 1957).

2.4 Prandtl/Meyer expansion

For weak oblique shocks, with small deflection angle θ, the above relations can be reduced to simple expressions (see White, 1986; Liepmann and Toshko, 1957), in particular the linear terms are

$$\frac{w_2 - w_1}{w_1} \approx -\frac{\theta}{\sqrt{M_1^2 - 1}}$$

or in a differential form

$$\frac{dw}{w} = -\frac{d\theta}{\sqrt{M_1^2 - 1}}$$

In fact, supersonic flow over a convex corner can be modelled by a centred wave, where the flow parameters must be constant along rays from the corner.

Integrating the above equation (see White, 1986), gives

$$-\theta + constant = \int \sqrt{M_a^2 - 1}\,\frac{dw}{w}$$

since $W = M_a a$, therefore

$$\frac{dw}{w} = \frac{dM_a}{M_a} + \frac{da}{a}.$$

From Bernoulli's law

$$\frac{a^2}{\gamma-1}+\frac{1}{2}w^2=H$$

$$\frac{2}{\gamma-1}ada+wdw=0$$

or

$$\frac{2}{\gamma-1}\frac{da}{a}+M^2\frac{dw}{w}=0$$

Eliminating $\frac{da}{a}$, one can obtain $\frac{dw}{w}$ in terms of $\frac{dM_a}{M_a}$, hence,

$$d\theta=-\frac{\left(M_a^2-1\right)^{1/2}}{1+\frac{\gamma-1}{2}M_a^2}\frac{dM_a}{M_a}.$$

Introducing $dw = -d\theta$, where $w = \theta$ at $M_a = 1$.

The above relation can be integrated to give an explicit relation for w as a function of M_a.

Prandtl/Meyer expansion fan can be also plotted in the hodograph plane (see Liepmann and Toshko, 1957).

2.5 Steady two-dimensional flows over thin pointed bodies

2.5.1 Shock-expansion theory

Consider steady supersonic flow over a diamond airfoil. The shock-expansion theory is used to calculate the surface pressure as follows.

Assuming oblique shocks are attached at the leading edge, one can calculate the flow downstream of the shocks on the top and bottom surface (they differ if the airfoil is at angle of attack). At the shoulders, there are expansion fans and the flow can be easily calculated all the way to the trailing edge, where shocks or fans can be found depending on the angle of attack.

Obviously the flat plate at angle of attack is a special case.

For a biconvex airfoil, a continuous isentropic flow from the leading edge to the trailing edge replaces the centred fan in the case of a diamond airfoil. One can divide the airfoil in n segments and the continuous flow is approximated by *n* centred waves.

On the other hand, the thin airfoil theory gives the surface pressure coefficients and hence lift and drag, in terms of linearised relations between the deflection angles and the flow speeds, through linearised Bernoulli's relation, hence the surface pressure can be readily calculated.

The above theories are not applicable in case of blunt bodies where bow shock is detached from the bodies. They are also not valid for transonic flows where the flow is locally supersonic somewhere and locally subsonic somewhere else. In these cases, numerical simulations based on the governing equations, are necessary to obtain the flow patterns. The details of the computational methods will be discussed in a separate publication.

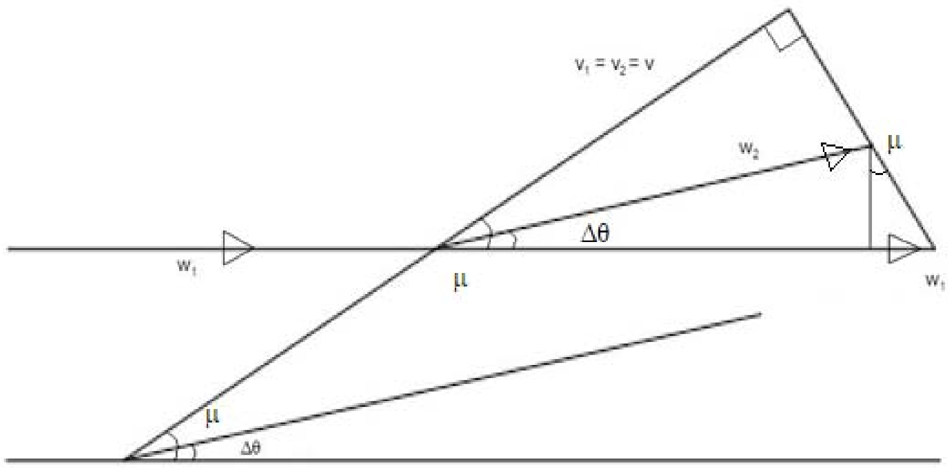
2.5.2 Supersonic thin airfoil theory

Instead of taking the limits of the nonlinear weak oblique shock relations, a simple straightforward approximation from the linearised expression relating the deflection angle to the speed change across a wave is derived in the following.

Consider again supersonic flow over a wedge. The flow will be deflected across the wave and it will be parallel to the surface of the wedge. Notice, from the conservation of the tangential momentum, the tangential velocity does not jump across the shock. Hence, $v_1 = v_2 = v$.

The shock angle β can be approximated by the Mach angle, μ hence (see Figure 3).

Figure 3 Supersonic flow over a wedge



2.5.3 Convergent/divergent nozzles at design and off design conditions

Compressible inviscid smooth flows of a perfect gas inside nozzles can be described by the quasi-one-dimensional equations, where $A = A(x)$ is the cross-section as,

$$\rho u A = Constant$$

or

$$\frac{\Delta\rho}{\rho} + \frac{\Delta u}{u} + \frac{\Delta A}{A}$$

and

$$\rho u \Delta u = -P$$

Assuming isentropic condition

$$\frac{P}{P_r} = \left(\frac{\rho}{\rho_r}\right)^{\gamma}$$

or

$$\Delta P = a^2 \cdot \Delta \rho$$

After some manipulations, one obtains

$$\frac{\Delta u}{u} = -\frac{a^2}{u^2}\frac{\Delta \rho}{\rho}$$

and

$$(1 - M^2)\frac{\Delta u}{u} = -\frac{\Delta A}{A}.$$

The above equations can be integrated step by step starting from the inlet (or the throat) to obtain the behaviour of the solution along the nozzle. Instead, the integral relation of Bernoulli's law may be used, namely

$$c_p T + \frac{1}{2}u^2 = H = c_p T_o$$

where T_o is the stagnation temperature.

Hence,

$$\frac{T^*}{T_o} = \frac{2}{\gamma+1}, \frac{P^*}{P_o}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}, \frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}.$$

In terms of the above relations, the conservation of mass reads:

$$\rho u A = \rho^* u^* A^*,$$

hence,

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\cdot\frac{u^*}{u} = \frac{\frac{\rho^*}{\rho_o}\cdot\frac{a^*}{a_o}}{\frac{\rho}{\rho_o}\cdot\frac{u}{a_o}\cdot\frac{a}{a_o}}$$

Substituting for the ratios in the left hand side their corresponding relations in terms of Mach number, it is clear that $\frac{A}{A^*} = fnc(\mathrm{M})$.

The dependence of A / A^* on M is given in White (1986) and Liepmann and Toshko (1957).

For a given value of A / A^*, there are two values of M corresponding to subsonic and supersonic flows in the convergent and divergent part of the nozzle.

Notice that the mass flow rate per unit area reaches a maximum for a given duct, when it is throat is at sonic condition. Condition for this choking phenomenon can be proved directly from Bernoulli's law as follows:

$$\frac{dp}{\rho} + udu = 0,$$

or

$$\frac{dP}{d\rho},\frac{d\rho}{\rho}+udu=0.$$

Now,

$$d(\rho u)=\rho du+ud\rho$$

$$\frac{d(\rho u)}{du}=\rho-\rho M^2=\rho(1-M_a^2).$$

Since the second derivative can be shown to be negative at $M_a = 1$, then the mass flow rate reaches a maximum at sonic condition.

Thus, at design condition, the nozzle is choked and the flow is supersonic in the divergent part.

The off-design conditions corresponding to different values of the back pressure are normal shock inside the nozzle, oblique shock, or expansion fan at the exit.

See White (1986) and Liepmann and Toshko (1957) for more details.

3 Theory of shallow water surface waves

Consider a shallow layer of water over a flat surface, ignoring the viscous effects and the surface tension, the equations of motion are:

$$\rho_w\frac{Du}{Dt}=-P_x$$

$$\rho_w\frac{Dv}{Dt}=-P_y$$

$$\rho_w\frac{Dw}{Dt}=-P_z-\rho g.$$

Assuming w and $\frac{Dw}{Dt}$ are negligible, the last equation becomes:

$$P_z=-\rho_w g \text{ or } P=\rho_w g(h-z)+P_a$$

where h is the height of the layer, P_a is the atmospheric pressure and ρ_w is the density of water. From this hydrostatic balance, $P_y = -\rho_w g h_y$ and $P_x = -\rho_w g h_x$. Moreover, the conservation of mass can be approximated by:

$$(\rho_w h)_t+(\rho_w hu)_x+(\rho_w hv)_y=0$$

Combining the equations of motion and the conservation of mass, one can obtain the momentum equations in conservation form:

$$(\rho_w hu)_t+(\rho whu^2)_x+(\rho_w huv)_y=-h\rho_w g h_x=-\frac{1}{2}\rho_w g(h^2)_x$$

$$\left(\rho_w hv\right)_t + \left(\rho whuv\right)_y + \left(\rho_w hv^2\right)_y = -h\rho_w gh_y = -\frac{1}{2}\rho_w g\left(h^2\right)_y$$

In non-dimensional form the equations become,

$$h = h/h_\infty\,,\ \bar{u} = u/v_\infty\,,\ \bar{v} = v/v_\infty$$

$$\bar{x} = x/l,\ \bar{y} = y/l, \text{ and } \bar{t} = t/\left(l/v_\infty\right)$$

Where l is a characteristic length,

$$\bar{h}_{\bar{t}} = \left(\bar{u}\bar{h}\right)_{\bar{x}} + \left(\bar{v}\bar{h}\right)_{\bar{y}} = 0$$

$$(\bar{h}\bar{u})_{\bar{t}} + (\bar{h}\bar{u}^2) + \left(\bar{h}\bar{u}\bar{v}\right)_{\bar{y}} = -\frac{1}{2}\frac{gh_\infty}{v_\infty^2}\bar{h}_{\bar{x}}^2$$

$$(\bar{h}\bar{v})_{\bar{t}} + (\bar{h}\bar{u}\bar{v})_{\bar{x}} + \left(\bar{h}\bar{v}^2\right)_{\bar{y}} = -\frac{1}{2}\frac{gh_\infty}{v_\infty^2}\bar{h}_{\bar{y}}^2.$$

Notice the non-dimensional quantity $\frac{gh_\infty}{v_\infty^2}$ is $\frac{1}{F^2}$, where $F = \frac{v_\infty}{\sqrt{gh_\infty}}$ is the Froude number.

Notice the non-dimensional equations in non-conservative form can be written in the form:

$$\frac{D\underline{v}}{Dt} + gh \cdot \Delta h/h = 0.$$

For small disturbances, with $h - h_o << h_o$ and h_o is the height of the undisturbed layer, linearisation of the above equation gives the standard wave equation:

$$\frac{\partial^2 h}{\partial t^2} = gh_o \nabla^2 h.$$

3.1 Special cases

3.1.1 Speed of surface wave in shallow water theory

The conservation of mass in a control volume around the wave front implies:

$$\rho_w uh = constant,$$

where h is the height of the water layer, hence

$$-\frac{\Delta h}{h} = \frac{\Delta u}{u}$$

and the equation of motion gives

$$\rho_w u \Delta u = -\Delta P = -\rho_w g \Delta h.$$

Eliminating Δu gives: $u^2 = gh$.

Hence, the speed of propagation of the surface wave under the above assumptions is $\sqrt{gh}$.

3.1.1.1 Hydraulic jumps

Consider a surface wave on a thin layer over a flat plate. A sudden change of the thickness of the layer may occur at certain conditions. Moving with the discontinuity and considering the relative normal velocity, the following steady relations can be derived:

$$u_1 h_1 = u_2 h_2$$

where subscripts one and two refer to the conditions before and downstream of the jump. The momentum balance using a control volume in Figure 4 and ignoring the friction gives

$$\rho_w u_2^2 h_2 - \rho w u_1^2 h_1 = Force\ due\ to\ pressure$$

Assuming hydrostatic pressure distribution across the water layer:

$$\int_0^{h_1} \rho_w g\left(h_1 - z\right) dz - \int_0^{h_2} \rho_w g\left(h_2 - z\right) dz = \frac{1}{2}\rho_w g\left(h_1^2 - h_2^2\right)$$

hence the jump condition is:

$$u_1^2 h_1 + \frac{1}{2} g h_1^2 = u_2^2 h_2 + \frac{1}{2} g h_1^2.$$

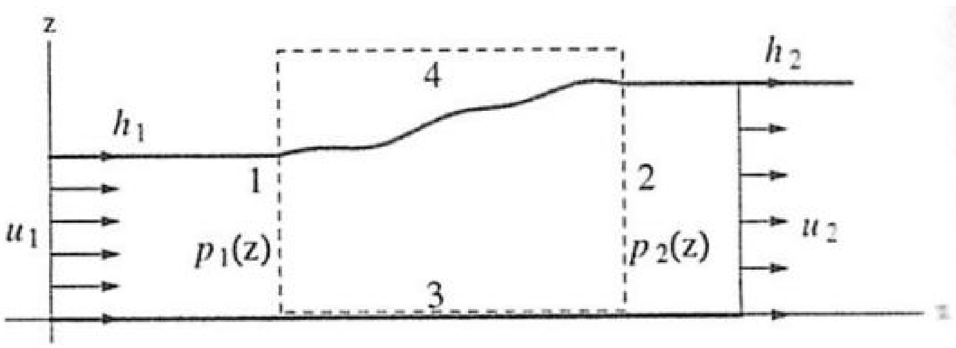
Following Chattot and Hafez (2015) and Thompson (1972), the conservation of mass and momentum together yield

$$\frac{1}{2} g\left(h^3 - h_1^3\right) - h_1 u_1\left(h - h_1\right) = 0.$$

The trivial solution $h = h_1$, i.e., there is no jump is excluded and the other two solutions satisfy the relation

$$\frac{1}{2} g h_2 + \frac{1}{2} g h_1 h + \frac{1}{2} g h_1^2 - u_1 h_1 = 0$$

Figure 4 Sketch of shallow water layer over a flat plate with Hydraulic Jump



The above quadratic equation has two solutions

$$\frac{h}{h_1} = \frac{-1}{2} + \sqrt{\frac{1}{4} + 2F_1^2}$$

for $F_1 \geq 1$ and $h / h_1 \leq 1$.

Notice from conservation of mass $\frac{F_1}{F_2} = \left(\frac{h_2}{h_1}\right)^{3/2}$.

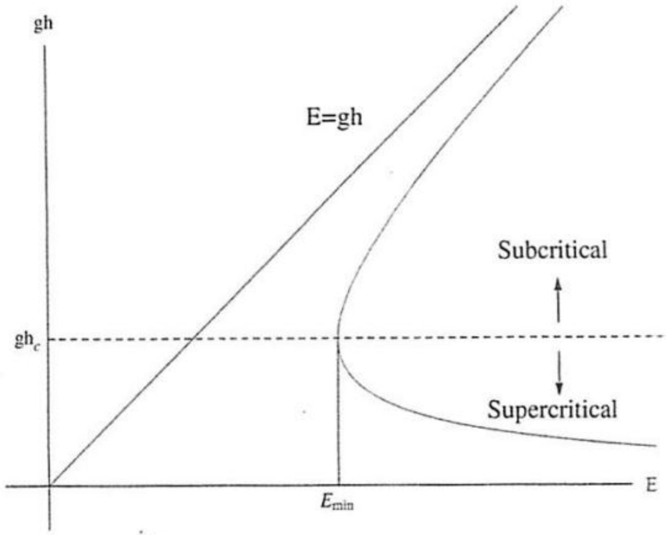
The solution is not physical with negative values of h.

For $F_1 \leq 1$ and $h / h_1 \leq 1$, the solution is excluded based on the second law of thermodynamics, since energy must decrease due to dissipation loss. To calculate the specific energy E for a given discharge Q, let

$$E = gh + \frac{1}{2}u^2 = gh + \frac{1}{2}\frac{Q^2}{2h^2}$$

There are two possible states for the same E and Q (see Figure 5).

Figure 5 Subcritical and supercritical wave speeds



There is a minimum value of E at certain value of h. Setting $\frac{dE}{dH} = 0$, $E_{\min}$ occurs at $g - \frac{Q^2}{h^3} = 0$ or, $h_c = \left(\frac{Q^2}{g}\right)^{1/3}$ and $E_{\min} = gh_c + \frac{1}{2}\frac{Q^2}{2h_c^2}$.

At the critical depth $h = h_c$, $Q^2 = gh_c^3 = c_o^2 h_c^2$ where $c_o = \sqrt{gh_c}$ and $F = 1$.

For $E < E_{min}$, no real solution exists and for $E > E_{min}$ two solutions are possible with $h > h_c$ and $u < c_o$, as well as $h < h_c$ and $u > c_o$. Now, across the jump:

$$E_1 - E_2 = g\frac{(h_2 - h_1)^3}{4h_1h_2}.$$

Hence, there is dissipation loss only if $h_2 > h_1$.

Notice, more consistent analysis would be based on the total enthalpy (and not just potential and kinetic energy), see Chattot and Hafez (2015).

In Chattot and Hafez (2015), the jump relation is written in the form, where

$$[u] = u_2 - u_1 \text{ and } < u > = \frac{u_1 + u_1}{2}.$$

Notice,

$$[u^2] = 2[u] < u >$$

and

$$[u] + gQ\left\langle\frac{1}{u}\right\rangle\left[\frac{1}{u}\right] = 0$$

since

$$\left\langle\frac{Q}{u}\right\rangle =< h > \text{ and } \frac{[u]}{\left[\frac{1}{u}\right]} = -u_1u_2$$

therefore,

$$u_1u_2 = g < h > .$$

3.1.2 Open channel with variable width

Let the width A be a function of x, $A = A(x)$, the conservation of mass becomes

$$\rho_w huA = \text{constant} = \rho_w h_r u_r A_r$$

and the equation of motion for smooth flow reads,

$$\rho_w uu_x = -P_x = -\rho_w gh_x.$$

In non-dimensional form, the equations are

$$\overline{hAu} = 1$$

$$\overline{uu}_{\bar{x}} = -\frac{gh_r}{u_r^2}\cdot\bar{h}_{\bar{x}} = -\frac{1}{F^2}\overline{hx} = -\frac{1}{F^2}\overline{hx}.$$

Multiply the last equation by $\bar{h}$, it becomes,

$$\bar{h}_{\bar{x}}\overline{uu}_{\bar{x}} = -\frac{1}{2}\frac{1}{F^2}(\overline{hx}^2)_{\bar{x}}.$$

Across a discontinuity, the jump condition is

$$\left[h\bar{u}^2 + \frac{1}{2}\frac{1}{F_r^2}\bar{h}^2 \right] = 0.$$

The governing equation for this simple special case is easy to derive from first principles and can be used to describe the flow phenomenon of hydraulic jumps.

3.1.3 One-dimensional unsteady flow

The governing equations are

$$u_t + uu_x + gh_x = 0$$

and

$$h_t + hu_x + uh_x = 0$$

Notice, $c^2 = gh$ and $gh_t = 2cc_t$ and $gh_x = 2cc_x$.

Therefore, the two equations become:

$$u_t + uu_x + 2cc_x = 0$$

$$2c_t + cu_x + 2uc_x = 0.$$

Adding the above two equations gives:

$$(u + 2c)_t + (u + c)(u + 2c)_x = 0$$

while subtracting them gives

$$(u + 2c)_t + (u - c)(u - 2c)_x = 0$$

Therefore (u + 2c) is constant along the curve defined by $\frac{dx}{dt} = u + c,$ while (u – 2c) is constant along the curve defined by $\frac{dx}{dt} = u - c.$

The quantities (u + 2c) and (u – 2c) are the Riemann invariants and the two curves defined by $\frac{dx}{dt} = u \pm c$ are the two characteristics associated with the two equations.

The jump conditions associated with the equations in conservation form (see Figure 6)

$$(hu)_t + (hu^2)_x + \frac{1}{2}g(h^2)_x = 0$$

$$h_t + (hu)_x = 0$$

$$[hu]\left(\frac{dx}{dt}\right)_s - \left[hu^2 + \frac{1}{2}gh^2 \right] = 0$$

$$[h]\left(\frac{dx}{dt}\right)_s - [hu] = 0.$$

The above equations can be rewritten as

$$\left[h\left(u - \left(\frac{dx}{dt}\right)_s + \frac{1}{2}gh^2\right)\right] = 0$$

$$\left[h\left(u - \left(\frac{dx}{dt}\right)_s\right)\right] = 0$$

where $u - \left(\frac{dx}{dt}\right)_s$ is the relative velocity to the jump.

3.1.4 Steady two-dimensional supercritical flow

The governing equations in this case are:

$$huu_x + hvu_y = -\frac{1}{2}g\left(h^2\right)_x$$

$$huv_x + hvv_y = -\frac{1}{2}g\left(h^2\right)_y$$

and $(hu)_x + (vh)_y = 0.$

It can be shown, that with uniform upstream conditions, the flow is everywhere irrotational (as discussed later). Hence, there exists a potential φ, such that $u = \varphi_x$, $v = \varphi_y$ and $\varphi_{xy} = \varphi_{yx}$. The governing equation for φ is

$$\left(c^2 - \varphi_x^2\right)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + \left(c^2 - \varphi_y^2\right)d_{yy} = 0.$$

The characteristics associated with the above equation are well-known; see for example, Chattot and Hafez (2015). Also, the jump conditions of the conservative form $(h\varphi_x)_x + (h\varphi_y)_y$ and $\varphi_{xy} = \varphi_{yx}$ are given by

$$\left[h\varphi_x\right]\cdot\left(\frac{dy}{dx}\right)_x - \left[h\varphi_y\right] = 0 \text{ and } \left[h\varphi_y\right]\cdot\left(\frac{dy}{dx}\right)_s - \left[\varphi_y\right] = 0.$$

The derivation is clear from Figure 6 and Figure 7. For example, conservation of mass across the discontinuity gives $(2\Delta y)(h_1u_1) - h_2u_2(2\Delta y) + h_2v_2(2\Delta x) - h_1v_1(2\Delta x) = 0$ or $[hu]\left(\frac{\Delta y}{\Delta x}\right) - [hv] = 0$ where $\left(\frac{\Delta y}{\Delta x}\right)$ is the slope of the discontinuity. Similarly, from the irrotationality condition:

$$u_2(2\Delta y) + v_2(\Delta y) - u_1(2\Delta x) - v_1(2\Delta x) = 0$$

or $[u]\left(\frac{dy}{dx}\right)_s + [v] = 0.$

Figure 6 Application of Gauss theorem over control volume around the discontinuity

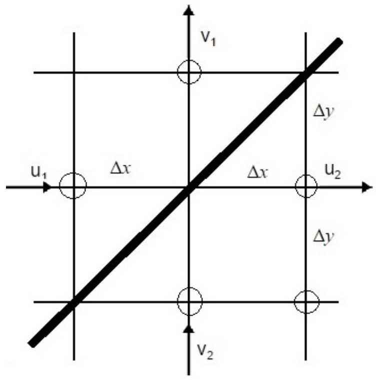
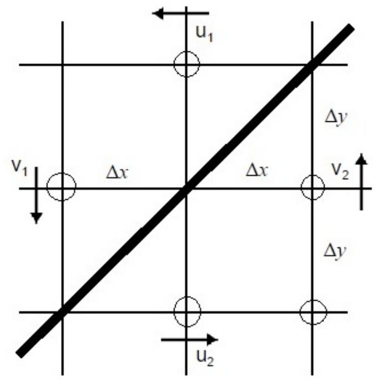


Figure 7 Applications of Stokes theorem over a control volume around the discontinuity



3.1.5 The theory of hydraulic analogy

Comparing the governing equations of shallow water surface waves over a flat surface to the governing equations of two-dimensional compressible fluid flows, assuming isentropic conditions, one can see the following striking analogy.

In non-dimensional forms, the general surface wave equations are:

$$\bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0$$

$$(\bar{h}\bar{u})_{\bar{t}} + (\bar{h}\bar{u}^2)_{\bar{x}} + (\bar{h}\bar{u}\bar{v})_{\bar{y}} = \frac{1}{2}\frac{1}{F_\infty^2}(\bar{h}^2)_{\bar{t}}$$

$$(\bar{h}\bar{u})_{\bar{t}} + (\bar{h}\bar{u}\bar{v})_{\bar{x}} + (\bar{h}\bar{v}^2)_{\bar{y}} = -\frac{1}{2}\frac{1}{F_\infty^2}(\bar{h}^2)_{\bar{y}}.$$

while the ‘isentropic’ Euler equations are:

$$\bar{\rho}_{\bar{t}} + \left(\overline{\rho u}^2\right)_{\bar{x}} + \left(\overline{\rho uv}\right)_{\bar{y}} = 0$$

$$\left(\overline{\rho u}\right)_{\bar{t}} + \left(\overline{\rho u}^2\right)_{\bar{x}} + \left(\overline{\rho uv}\right)_{\bar{y}} = \frac{1}{\gamma}\frac{1}{M_\infty^2}\left(\tilde{P}\right)\bar{x}$$

$$\left(\overline{\rho v}\right)_{\bar{t}} + \left(\overline{\rho uv}\right)_{\bar{x}} + \left(\overline{\rho v}^2\right)_{\bar{y}} = -\frac{1}{\gamma}\frac{1}{M_\infty^2}\left(\tilde{P}\right)_{\bar{y}}.$$

Notice here, $\tilde{P} = \frac{P}{P_\infty}$, while $\bar{P} = \frac{P}{\rho_\infty V_\infty^2} = \frac{P}{P_\infty} \cdot \frac{P_\infty}{\rho_\infty V_\infty^2} = \frac{1}{\gamma M_\infty^2}\tilde{P}.$

The two sets of equations are identical if $\gamma = 2$.

In this case:

M_∞ corresponds to F_∞,

$\tilde{P}$ corresponds to $\bar{h}$,

$\tilde{P}$ corresponds to $\bar{\rho}^2$

$\tilde{T}$ corresponds to $\bar{h}$.

(since for perfect gases, $\tilde{P} = \bar{\rho}\tilde{T}$, where $\tilde{T} = T/T_\infty$).

The analogy is based on the correspondence between speed of the surface wave ($c^2 = gh$) and the speed of sound ($a^2 = \gamma RT$).

The analogy is clear from the non-conservative forms of the equations as well, where

$$\frac{Dv}{Dt} = -gh\frac{\nabla h}{h} \text{ and } \frac{Dv}{Dt} = -a^2\frac{\nabla\rho}{\rho}$$

Notice $\nabla p = a^2\nabla\rho$ (assuming isentropic conditions).

For small disturbances of flows at rest, the two linearised equations become:

$$h_{tt} = c^2\nabla^2 h \text{ and } \rho_{tt} = a^2\nabla^2\rho.$$

The compressible flow relations and their shallow-water versions are:

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

$$\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

$$\frac{P_o}{P} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

and

$$\left(\frac{h_o}{h}\right) = 1 + \frac{\gamma - 1}{2} F^2$$

$$\left(\frac{h_o}{h}\right) = 1 + \frac{\gamma - 1}{2} F^2$$

$$\left(\frac{h_o}{h}\right)^2 = \left(1 + \frac{\gamma - 1}{2} F\right)^2$$

Moreover, flows in nozzles of variable cross-section area correspond to shallow water in open channel with variable width.

Similarly, the analogy is valid also for thin airfoil theory as well as full potential and transonic small disturbance equations including the jump conditions and the expansion fan relations.

The jump conditions for steady and unsteady flows in one and two space dimensions are also similar (under the assumption of isentropic conditions for the compressible flows), since they are the weak solutions of the same non-dimensional equations in conservation forms.

3.1.6 Limitations of hydraulic analogy

When valid, the analogy has many useful applications. However, there are several serious limitations for its validity.

The above equations were derived assuming thin layer of water, over a flat surface, with hydrostatic pressure across the layer.

If the layer of the water is not thin or the bottom surface is not flat, the governing equations must be modified accordingly. For example, the vertical velocity and vertical acceleration, (compared to gravity) may not be negligible.

We assumed the liquid is incompressible with constant viscosity and ignored surface tension. Moreover, the flow is treated as inviscid and the friction at the bottom surface is ignored.

Moreover, the analogy is valid for γ. For air, $\gamma = 1.4$ and there is no gas with $\gamma = 2$! However, for transonic small disturbance theory, the γ- effect is absorbed in the transonic similarity parameter K and the governing equation, with the proper scales, is independent of γ.

Finally, the analogy is only for two-dimensional, isentropic flows. It should be mentioned, that isentropic Euler equations are in general different from the full Euler equations.

The solution of full Euler equations satisfy conservation laws of mass, momentum and energy while entropy jumps (increases) across shocks, while the solution of isentropic Euler equations satisfies conservation of mass, momentum and entropy, while the total enthalpy jumps across shocks. Shocks are not the only discontinuity in the solutions of Euler equations. Across contract discontinuations, pressure, is continuous but tangential velocity jumps. Contact surfaces are produced by intersecting shocks, see Loh (1969). This phenomenon is important in shock dynamics. Across the slip surfaces, there is also entropy discontinuity.

For smooth flows (with uniform upstream conditions) however, mass momentum, energy as well as entropy are conserved for both full and isentropic Euler equations.

For shocks in the isentropic Euler equations, the total enthalpy jumps, not the entropy. Hence, hydraulic analogy is limited to flows with weak and not strong shocks.

To show the differences between the full Euler and isentropic Euler equations, consider the Crocco's relation (see Thompson, 1972; Liepmann and Toshko, 1957).

The momentum equation can be written in the form:

$$\rho \frac{DV}{Dt} = -\nabla P$$

using the vector identity:

$$(V \cdot \nabla)V = (u \times \overline{v}) \times \overline{v} + \frac{\nabla v^2}{2}$$

therefore,

$$\rho\left(\frac{\partial V}{\partial t}\right) + \overline{w} \times V + \frac{\nabla V^2}{2} = -\nabla P.$$

Combining with the thermodynamic relation

$$T\nabla S = \nabla h - \frac{\nabla P}{\rho}$$

one obtains

$$\frac{\partial V}{\partial t} + \omega \times V = T\nabla S - \nabla H$$

where $H = h + \frac{1}{2}V^2$.

Notice, for the isentropic Euler, $\nabla S \equiv 0$.

Taking the curl of the above equation gives

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)V - \omega(\nabla \cdot V) + \nabla T \times \nabla S$$

The conservation of mass equation may be used to eliminate the divergence of velocity term to yield

$$\frac{D(\omega/\rho)}{Dt} = \frac{\omega}{\rho} \cdot \nabla V + \frac{1}{\rho} \nabla T \times \nabla S.$$

For steady two-dimensional isentropic flow, both terms of the right hand side vanish and the result reduces to $\frac{\omega}{\rho}$ is constant on streamlines.

On the other hand for steady two-dimensional isoenergetic flow, the result after manipulations reduces to ω / T is constant on streamlines.

In both cases, vorticity is generated downstream of curved shocks, while in both cases, if the upstream conditions are uniform, for smooth flows (without discontinuity and without closed streamlines), the vorticity vanishes everywhere, i.e., the flow is irrotational and hence can be described by a potential function.

Obviously, hydraulic analogy is not valid when the assumptions of the theory are violated. To summarise, the following approximations are assumed:

1 $\delta h << h_o$: local variations are relatively small.

2 $h_o << \lambda$: water layer is relatively small, where λ is the wave length.

The theory is based on small disturbances. Moreover, the effects of surface tension and finite depth are ignored. These effects are negligible only for relatively long wavelengths and only then the wave speed becomes independent of these effects and $c^2 = gh$ becomes an acceptable result. Still surface tension (capillary waves) can be observed as very short wavelength disturbances. The vertical velocity and acceleration were neglected as well as the viscous force on the horizontal bottom. The last assumption is however questionable (see Thompson, 1972). In water table experiments, the table is usually inclined (slightly), so the component of gravity in the flow direction balances the friction force at the bottom flat surface.

As the wave amplitude increases, the variation of vertical velocity cannot be neglected and the pressure is no longer hydrostatic.

4 Nonlinear dispersive waves with finite amplitude

Boussinesq analysed nonlinear waves in dispersive media where the frequency, ω, depends on the wave number k. His equation reads:

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -c^2 \frac{\nabla \rho}{\rho} + 2\frac{C_o}{\rho_o} \beta \cdot \Delta \nabla \rho$$

and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v = 0,$$

where

$$\frac{c_2}{c_o^2} = \left(\frac{\rho}{\rho_o}\right)^{\gamma - 1}$$

and β is a dispersion parameters.

See Bullough and Caudrey (1980) for details.

The leading terms of the dispersion relation is given by:

$$\omega = c_o k - \beta k^3 + \ldots$$

If the velocity potential φ is introduced, where $v = \nabla\varphi$, Boussinesq equation can be reduced to

$$\varphi_{tt} - c_o^2 \Delta\varphi + \nabla\varphi \nabla\varphi_t + \gamma \Delta\varphi\varphi_t - 2c_o \beta \Delta^2 \varphi = 0.$$

To examine the stationary solution, let

$$\varphi = \varphi(x - V \cdot t)$$

the resulting equation is

$$\left(V^2 - c_o^2\right)\varphi_{xx} - (\gamma + 1)V\varphi_{xx}\varphi_x - 2c_o\beta\varphi_{xxxx} = 0$$

where the velocity of the stationary wave, V, is close to c_o, if the amplitude is small.

Analytical solutions can be obtained by integrating the above equation twice and when $\beta > 0$, a solitary wave is obtained, namely a soliton, while for $\beta < 0$, there is a periodic wave solution.

A different form of the same order of approximation is given in Karpman (1975), namely

$$\varphi_{tt} = c_o^2 \frac{\partial^2}{\partial x^2}\left(\varphi + \frac{3}{2}\frac{\varphi^2}{h} + \frac{h^2}{3}\varphi_{xx}\right)$$

The above equation admits solitary wave solution $\varphi = k\sec h^2\left[\left(\frac{3k}{h^2}\right)^{1/2}(x \pm ct)\right]$, travelling either along the positive or the negative x directions.

This equation can be reduced to a simpler equation with a single direction of propagation by the transformation, $\xi = x - c_o t$ and $\tau = \varepsilon t$. Neglecting ε^2 terms, the following equation after Korweg-deVries' is obtained, namely

$$u_t + uu_x + u_{xxx} = 0$$

which has the 'soliton' solution $u = 12\eta^2 sech^2[\eta(x - 4\eta^2 t)]$ where its speed is $4\eta^2$.

For more details about solitons, see Drazin and Johnson (1980).

For steady two-dimensional flows around a thin body in a dispersive media, the governing equation, in terms of a small disturbance potential is given by:

$$\left[\left(1 - M_\infty^2\right) - (\gamma + 1)M_\infty^2\varphi_x\right]\varphi_{xx} + \varphi_{yy} + \beta\varphi_{xxxx} = 0.$$

Numerical solutions of the above equations for supersonic flows in dispersive media are discussed in (Thompson, 1972).

5 Flow visualisation of compressible fluid flows

In shallow water theory, waves propagate at speed given by $c = \sqrt{gh},$ where $g = 9.8$ m/s^2. Thus for $h = 2$ mm, $c \cong 0.2$ m/s which is feasible for real-time observation, see Loh (1969). In fact, hydraulic jump can be seen in a kitchen sink.

Hydraulic jumps can be seen in flows from reservoir or through variable-width channel with transition from subcritical ($F < 1$) to supercritical ($F > 1$) flow. Other examples are tidal bores and front of powder avalanche in snow.

The analogy with shock waves in compressible fluid flow is striking.

Visualisation of supersonic flows with shock waves and expansion fans, in supersonic wind tunnels, is very expensive. On the other hand, using hydraulic analogy, such visualisation is affordable, at least for weak shocks, where the analogy is valid and where potential flow theory can be used.

In part 2, water table experiments will be presented to show how to obtain good results, qualitatively as well as quantitatively. There are two types of water tables, one with the model fixed and water flows around it for external flow simulation or inside the model as in nozzles for internal flows. The second type is to move the model in a thin layer of water. In both cases, the surface waves on the water layer can be easily seen. Measurement techniques for the height of the water layer will be discussed in part 2. Pictures can be analysed to obtain quantitative results.

The limitations of the analogy should be examined before using the data of water table experiments in analysis and design of supersonic and transonic airfoils and nozzles.

6 Concluding remarks

Compressible fluid flow theories are first reviewed, followed by shallow water are surface wave formulation. The hydraulic analogy is explained and its limitations are discussed. Nonlinear dispersive waves of finite amplitude are also studied. Flow visualisation of compressible fluid flows based on the analogy with shallow water surface waves are indeed feasible and can be a useful tool (when it is valid) in industry and for educational purposes. In part 2 of this work, practical aspects of water table experiments are presented.

Acknowledgements

This work is supported by NSF grant, ICORP-Award Number 1644559.

References

Ashley, H. and Landahl, M. (1965) *Aerodynamics of Wings and Bodies*, Addison-Wesley, Massachusetts.

Bullough, R.K. and Caudrey, P.J. (1980) *Solitons*, Springer, New York.

Chattot, J.J. and Hafez, M.M. (2015) *Theoretical and Applied Aerodynamics*, Springer, New York.

Courant, R. and Friedricks, K.O. (1948) *Supersonic Flow and Shock Waves*, Wiley, New York.

Drazin, P.G and Johnson, R.S. (1980) *Solitons: An Introduction*, Cambridge, New York.

Karpman, V.I. (1975) *Nonlinear Waves in Dispersive Meida*, Pergamon Press, New York.

Landau, L.D. and Lifshitz, E.M. (1987) *Fluid Mechanics*, 2nd Ed., Pergamon Press, New York.

Liepmann, H. and Toshko, A. (1957) *Elements of Gas Dynamics*, Wiley, New York.

Loh, W. (Ed.) (1969) *Modern Developments in Gas Dynamics*, Plenum Press, New York.

Oswatitsch, K. (1956) *Gas Dynamics*, Academic Press, New York.

Shapiro, A. (1953) *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Vol. 1/2, The Ronald Press Company, New York.

Thompson, P. (1972) *Compressible Fluid Dynamics*, McGraw Hill, New York.

White, F.M. (1986) *Fluid Mechanics*, Wiley, New York.

Wintersein, R., Hafez, M. and Brewer, J. (1993) 'Numerical simulation of supersonic flow around a thin body in a dispersive medium' AIAA 93-0773, AIAA paper presented in *AIAA Conference*.

Zucrow, M. and Hoffman, J. (1976) *Gas Dynamics*, Vols. 1/2, Wiley, New York.