Adaptive backstepping control for a class of MIMO uncertain underactuated systems with input constraints

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Abstract: This paper presents a backstepping methodology-based semi generic adaptive controller scheme for a class of multi-input multi-output (MIMO) uncertain underactuated systems in presence of actuator constraints. To develop a feasible controller scheme for multi-input multi-output underactuated systems, $(n - p + 1)$ dimensions of the n dimensional configuration space are stabilised by using one dimension of the input space. This control term is developed by applying hierarchical methodology whereas as remaining $p - 1$ input dimensions are assigned as dedicated control terms to solve the control problem of remaining dimensions of the configuration space. Backstepping technique is used to develop the classical control terms whereas wavelet networks are used to approximate the uncertain dynamics as well as the nonlinear effects of actuator saturation. The proposed scheme, thus, relaxes the constraint of completely and accurately known system dynamics as well as the requirement of measuring the clipped portion of the control effort. A robust control term is used to attenuate the approximation error to a prescribed level. Uniform ultimate boundedness (UUB) stability of the closed loop system is verified in the Lyapunov sense. Simulation results illustrate the effectiveness of theoretical development.

Keywords: underactuated systems; hierarchical control structure; backstepping control; wavelet neural network; actuator saturation.

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1 Introduction

Underactuated systems refer a particular class of nonlinear systems where the dimensions of input space are less than the dimensions of configuration space (Spong, 1997). Simultaneous stabilisation of all the configuration variables with lesser actuations often results in a complicated controller design as the classical control schemes meant for fully actuated systems are not applicable to this class. Depending upon the physical structure of the underactuated systems, total degrees of freedom can be partitioned into active and passive degrees of freedom with an inherent nonlinear interaction among these degrees (Spong, 1997; Olfati-Saber, 2002). Property of under actuation has been displayed by several systems like underwater vehicles, twin rotor system, mobile robots, cart-pole systems, etc. and the issue of controller design for this class has been addressed by several researchers (Olfati-Saber, 2002; Quiser et al., 2007; Chyau and Feng, 2010; Wang et al., 2004; Hung et al., 2007; Marton et al., 2008; Qian et al., 2009, 2011, 2012, 2013; Qian and Yi, 2010; Chiang and Yah, 2014; Hwang et al., 2014; Aloui et al., 2011; Liu and Yamaura, 2011; Lai et al., 2015; Nguyen and Dankowicz, 2015; Kulkarni and Kumar, 2015, 2016; Choudhari and Kar, 2017). Underactuated systems are often associated with complex nonlinear dynamics which are difficult to model mathematically. An inaccurate modelling of these dynamics may undermine the system performance, control schemes developed using adaptive framework have been proved highly effective for such systems (Quiser et al., 2007; Chyau and Feng, 2010; Wang et al., 2004; Hung et al., 2007; Qian et al., 2009, 2011, 2012, 2013; Qian and Yi, 2010; Chiang and Yah, 2014; Hwang et al., 2014; Aloui et al., 2011; Liu and Yamaura, 2011; Lai et al., 2015; Nguyen and Dankowicz, 2015; Kulkarni and Kumar, 2015, 2016; Choudhari and Kar, 2017). Research findings on controller design for underactuated systems include the controller design for systems with completely known dynamics (Olfati-Saber, 2002) as well as for the systems with uncertainties (Quiser et al., 2007; Chyau and Feng, 2010; Wang et al., 2004; Hung et al., 2007; Qian et al., 2009, 2011, 2012, 2013; Qian and Yi, 2010; Chiang and Yah, 2014; Hwang et al., 2014; Aloui et al., 2011; Liu and Yamaura, 2011; Lai et al., 2015; Nguyen and Dankowicz, 2015; Kulkarni and Kumar, 2015, 2016; Choudhari and Kar, 2017).

Controller strategies for underactuated systems mainly rely on transformation or hierarchical approaches of controller design. First approach emphasise on the development of some nonlinear coordinates transformation so as to transform the original system model to into some cascade like structure which streamlines the controller design, however the cascade like structure which results from the coordinate transformation often exhibits a nonlinear relationship between the original and transformed coordinates and so results in complicated controller design (Olfati-Saber, 2002; Quiser et al., 2007; Chyau and Feng, 2010; Choudhari and Kar, 2017). Second technique utilises the hierarchical methodology to deduce the control law. This technique develops the error surfaces following some hierarchical methodology. Approach is initiated with the development of lowest order error surface either a subsystem error surface is designated as the lowest order error surface and higher order error surfaces are developed by considering the subsystem error surfaces either recursively or designating all the subsystem error surfaces as the lowest order error surfaces and a second level error surface is constructed by suitably appropriately aggregating all the subsystem error surfaces. Control law is derived so as to ensure the convergence of highest order error surface which thereby ensures the boundedness of all the lower order error surfaces. This approach does not require any nonlinear coordinate transformation and utilises the original model of the underactuated systems (Hung et al., 2007; Marton et al., 2008; Qian et al., 2009, 2011, 2012, 2013; Qian and Yi, 2010; Chiang and Yah, 2014; Hwang et al., 2014; Aloui et al., 2011; Liu and Yamaura, 2011; Lai et al., 2015; Nguyen and Dankowicz, 2015; Kulkarni and Kumar, 2015, 2016).

Backstepping is a Lyapunov function-based classical recursive approach for the development of feedback control laws. In backstepping methodology, overall system is decomposed into several smaller subsystems, thereafter the virtual control terms are designed by the recursive consideration of these subsystems and this process finally results in the designing of actual control term. Control terms are derived in agreement with some recursively constructed Lyapunov function and therefore the control term developed by applying backstepping procedure ensures the stabilisation of overall system (Khalil, 2002; Slotin and Li, 1991). Classical backstepping approach is a model-based approach and its effective implementation requires completely and accurately known system dynamics. However, the adaptive backstepping approaches are cited in the literature for the effective control of systems with uncertain dynamics (Kulkarni and Kumar, 2015; Hsu et al., 2006). These adaptive versions incorporate some adaptive approximation tool like neural network or wavelet network for the approximation of uncertain dynamics (Hsu et al., 2006; Astrom and Wittenmark, 1995). Application of backstepping methodology to underactuated systems in an approximate form has also been cited in Marton et al. (2008) and Kulkarni and Kumar (2015).

Wavelet network is a nonparametric system identification tool which can approximate any nonlinear function with arbitrary accuracy. Wavelet network can be viewed as a nonlinear regression structure where regression functions are scaled and shifted versions of some wavelet and associated scaling functions (Hsu et al., 2006). Normally, the wavelet function used for the construction of regression functions is some compactly supported, orthonormal basis of multiresolution analysis. Multiresolution allows the network construction by following a systematic methodology, starting with some coarser level resolution and gradually including the finer resolutions following the tradeoff between accuracy and computational complexity. Othonormality of the basis functions ensures a unique representation of any function with no redundancy (Mallat, 1989). Wavelet network with orthonormal wavelet basis satisfying the norms of multiresolution analysis provides an explicit function representation at different resolutions. Compact support of wavelet basis allows efficient learning and rapid convergence of training algorithms (Zhang et al., 1995; Xu and Tan, 2007). Research findings on the application of wavelet network for system identification and control are cited in Hsu et al. (2006), Xu and Tan (2007), Billings and Wei (2005), Zekri et al. (2008), Lee and Li (2012) and Xue et al. (2013).

Practical actuators are associated with the problem of amplitude limitation and cannot reproduce the control effort beyond certain limits (Gao and Selmic, 2004). Limited application of control effort degenerate the system performance or even makes the closed loop system unstable. In order to ensure the desired system performance, the saturation effects of actuator are required to be considered. Several research findings which detail the controller designing aspects with a consideration of actuator saturation are cited in Gao and Selmic (2004), Zhou et al. (2006), He and Jagannathan (2007) and Yuan et al. (2014). These techniques tackle the actuator saturation in anti-windup paradigm and either emphasise on the development of some auxiliary dynamics or on the application of some adaptive tool like neural network for the effective compensation of the nonlinear effects of saturation (Gao and Selmic, 2004; Zhou et al., 2006; He and Jagannathan, 2007; Yuan et al., 2014).

This paper presents a backstepping-based adaptive control scheme for a class of multi-input multi-output (MIMO) underactuated systems (Aloui et al., 2011) with system uncertainties and prior consideration of input constraints. In the proposed controller scheme, one control term is designated to stabilise $(n - p + 1)$ configuration variables of n dimensional configuration space whereas the remaining control terms are assigned as dedicated control terms to rest of the $(p - 1)$ configuration variables. First control term is derived by applying a two level hierarchical backstepping control design where as remaining control terms are based on classical adaptive backstepping methodology. Control scheme thus allows the application of approximate backstepping control to $(n - p + 1)$ configuration variables and classical backstepping approach to rest of the configuration variables. This approach can be considered as an optimal control approach under the restrictions imposed by the systems under consideration. This scheme ensures effective regulatory or tracking performance of the configuration variables to which dedicated control terms are assigned and at the same time ensures the stability of rest of the variables. Wavelet networks are constructed for effective approximation of system uncertainties and the nonlinear effects induced by the actuator saturation. Thus, the control scheme is semi generic in the sense that it relaxes the requirement of modelling the complex nonlinear dynamics. Also, a wavelet-based saturation compensator is used which effectively mitigates the nonlinear effects of actuator saturation. Adaptive saturation compensator not only restores the desired system performance within a short span of time but also time relaxes the constraint of measuring the clipped part of control effort (Gao and Selmic, 2004). Adaptation laws are developed for the tuning of network parameters and a robust term is incorporated to attenuate the approximation error of the wavelet network (Hsu et al., 2006; Yuan et al., 2014; Astrom and Wittenmark, 1995). Controller scheme so developed ensures the convergence of system error dynamics and boundedness of closed loop signals in presence of system uncertainties and actuator saturation.

This paper is organised as follows: preliminaries and system model are given in Section 2, whereas Section 3 describes the designing of wavelet-based adaptive controller scheme for underactuated systems with partially known system dynamics and subjected to actuator saturation. Results of the simulation performed for a MIMO underactuated system are illustrated in Section 4, whereas Section 5 concludes the paper.

2 Problem formulation and preliminaries

2.1 Actuator saturation

Practical actuators are associated with the problem of saturation and whenever the input tries to exceed the stipulated limit the actuator output does not replicate the input and gets stuck to saturation bound in either direction, depending on the sign of input. The input-output relationship of an actuator with actuator saturation can be defined as (Zhou et al., 2006)

$$u = \begin{cases} v; |v| < u_{\max} \\ u_{\max}\,\mathrm{sgn}(v); |v| \geq u_{\max} \end{cases} \tag{1}$$

where $v \in \Re$ and $u \in \Re$ are actuator input and output respectively while $u_{\max} \in \Re_+$ represents the saturation bound. As indicated by equation (1), whenever the actuator undergoes saturation certain portion of the input is clipped off and is given as

$$\Delta u = u - v \tag{2}$$

Clipping of certain portion of the input can be observed as an undesired nonlinear dynamics which is invoked by actuator saturation (Gao and Selmic, 2004). This nonlinear dynamics can degrade the system performance and may even lead to uncertainty and is therefore required to be tackled during the controller design. In this work, this nonlinear dynamics is effectively approximated and mitigated by using a wavelet network.

2.2 System formulation

Consider an uncertain MIMO underactuated system described by the following dynamics

$$\Sigma_1 = \begin{cases} \dot{x}_{2i-1} = x_{2i} \\ \dot{x}_{2i} = f_i(X) + \sum_{q=1}^{p} g_{iq}\left(\underline{X}_1\right) u_q; i = 1, 2, \ldots, n \\ y_i = x_{2i-1} \end{cases} \tag{3}$$

System described by equation (3) is composed of n interconnected subsystems in Brunovsky canonical form (Spong, 1997; Olfati-Saber, 2002) with $X = [x_1, x_2, \cdots, x_{2n}]^T \in \Re^{2n}$ as the state variables of the system, $U = [u_1, u_2, \cdots, u_p]^T \in \Re^p$ represents the control vector applied to the system through the actuators, due to the underactuation property $n > p$, $y_i(t) \in \Re$ is the output of i^{th} subsystem $f_i(X): \Re^{2n} \to \Re$ and $g_{iq}\left(\underline{X}_1\right): \Re^{2n} \to \Re$ are system nonlinearities abbreviated as f_i and g_{iq} respectively while $\underline{X}_1 = \left[x_1, x_3, \cdots, x_{2n-1}\right]^T \in \Re^n$. Nonlinearities f_i are considered as smooth uncertainties while the nonlinear function g_{iq} is either strictly positive or negative and $g_{iq} \in L_\infty$, $\forall x \in S_x$, $t \geq 0$. Here, $S_x \subset \Re^{2n}$ is some compact set of allowable state trajectories.

Objective is to design a control scheme for equation (3) with a prior consideration of uncertain dynamics and input constraints. Control law must ensure the stabilisation of all the dimensions of the configuration space and boundedness of tracking errors for all the degrees of freedom corresponding to given desired trajectories.

To facilitate the controller design, following assumptions have been considered regarding the nature of desired trajectories and input constraints.

Assumption 1: Desired trajectory vector $y_d \in \Re^n$ is selected such that $y_{id}, \dot{y}_{id}, \ddot{y}_{id} \in L_\infty, i = 1, 2, \cdots, n.$

Assumption 2: Input constraint imposed by actuator nonlinearity is in accord with the description of saturation nonlinearity (1) and (2) given in Section 2.1. Thus, for equation (3), the input-output relationship of actuator i can be described as

$$u_i = \begin{cases} v_i; |v_i| < u_{i\max} \\ u_{i\max}\,\mathrm{sgn}(v_i); |v_i| \geq u_{i\max} \end{cases} \tag{4}$$

$$u_i = v_i + \Delta u_i \tag{5}$$

where v_i and u_i are input and output of the actuator, respectively. Term v_i can be considered as the unconstrained input for equation (3), $u_{i\max}$ is saturation bound and Δu_i represents the portion of input concealed by the actuator whenever it undergoes saturation.

Under the constraints of actuator saturation (4, 5), system dynamics (3) can be expressed as

$$\Sigma_2 = \begin{cases} \dot{x}_{2i-1} = x_{2i} \\ \dot{x}_{2i} = f_i(X) + \sum_{q=1}^{p} g_{iq}\left(\underline{X}_1\right)\Delta u_q + \sum_{q=1}^{p} g_{iq}\left(\underline{X}_1\right) v_q \\ y_i = x_{2i-1}; i = 1, 2, \ldots, n - p + 1 \end{cases} \tag{6}$$

In equation (6), nonlinear term $g_{iq}(X)\Delta u_q$ represents the nonlinear effects induced by the saturation of actuator q in i^{th} subsystem and the elements of vector $v = [v_1, v_2, \cdots, v_p]^T$ represents the unconstrained control efforts .Further, the system dynamics can be expressed as

$$\Sigma_3 = \begin{cases} \dot{x}_{2i-1} = x_{2i} \\ \dot{x}_{2i} = q_i(X) + \sum_{q=1}^{p} g_{iq}\left(\underline{X}_1\right) v_q \\ y_i = x_{2i-1}; i = 1, 2, \ldots, n - p + 1 \end{cases} \tag{7}$$

where nonlinear vector field q_i is defined as

$$q_i = f_i(X) + \sum_{q=1}^{p} g_{iq}(X)\Delta u_q \tag{8}$$

System dynamics (7) can be viewed as a saturation free version of equation (3) and can be utilised for designing the auxiliary control term to be augmented with the standard control law so as to deal actuator saturation in anti-wind up paradigm.

Assumption 3: System nonlinearities g_{iq} are such that the matrix g defined as

$$g = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1p} \\ g_{21} & \cdots & \cdots & g_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ g_{n1} & \cdots & \cdots & g_{np} \end{bmatrix} \tag{9}$$

contains at least $p - 1$ linearly independent rows.

2.3 Wavelet network approximation

Wavelet decomposition of any function $f(x) \in L^2(\Re)$ can be expressed as (Zhang and Benveniste, 1992)

$$f(x) = \sum_{q \in \mathbb{Z}} \alpha_q \frac{1}{\sqrt{a_q}} \phi\left(\frac{x - b_q}{a_q}\right) \tag{10}$$

here $x \in \Omega_x \in \Re$ is the input argument, $\phi(.)$ is scaling function associated with some wavelet basis $\psi(.)$ satisfying the norms of multiresolution analysis whereas a_q, b_q are dilates , translates which are in general restricted to a dyadic lattice as $a_q = 2^{-j}$ and $b_q = k2^{-j}$ with $k, j \in \mathbb{Z}$ and α_q is the weight parameter. According to the properties of multiresolution analysis (10) can be expressed as (Mallat, 1989; Xu and Tan, 2007)

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}} \alpha_{J_o,k} 2^{J_0/2} \phi\left(2^{J_0} x - k\right) + \sum_{j \geq J_0} \sum_{k \in \mathbb{Z}} \beta_{j,k} 2^{j/2} \psi\left(2^j x - k\right) \\ &= \sum_{k \in \mathbb{Z}} \alpha_{J_0,k} \phi_{J_0,k}(x) + \sum_{j \geq J_0} \sum_{k \in \mathbb{Z}} \beta_{j,k} \psi_{j,k}(x) \end{aligned} \tag{11}$$

here, $J_0 \in \mathbb{Z}$ is the coarsest resolution level while $\alpha J_{0,k} \in \Re$ and $\beta_{j,k} \in \Re$ denotes the weights of scaling and wavelet functions.

A wavelet network can be viewed as the practical implementation of wavelet decomposition (10) where a truncated version of equation (11) is implemented. Approximation precision acts as a governing factor in deciding the size of wavelet network. Wavelet network estimation of any nonlinear function $f(x) \in L^2(\Re)$ can be expressed as (Xu and Tan, 2007)

$$\begin{aligned} \hat{f}(x) &= \sum_{k \in K_{J0}} \alpha_{J_o,k} 2^{J_0/2} \phi\left(2^{J_0} x - k\right) + \sum_{j = J_0}^{J} \sum_{k \in K_j} \beta_{j,k} 2^{j/2} \psi\left(2^j x - k\right) \\ &= \sum_{k \in K_{J0}} \alpha_{J_0,k} \phi_{J_0,k}(x) + \sum_{j = J_0}^{J} \sum_{k \in K_j} \beta_{j,k} \psi_{j,k}(x) \end{aligned} \tag{12}$$

here, the function estimation is carried using a wavelet network with $J_0 \in \mathbb{Z}$ and $J \in \mathbb{Z}$ as the coarsest and finest resolution levels respectively whereas the translates at a particular resolution level are confined to a finite set and the set $K_j \subset \mathbb{Z}^{d_j} \left(j = J_0, J_0 + 1, \cdots, J\right)$

represents translates for a particular resolution level, $d_j \in \mathbb{Z}$ represents the number of translates used at a particular resolution j.

In vector form, equation (12) can be expressed as

$$\hat{f}(x) = \alpha_{J_0}^T \phi_{J_0}(x) + \sum_{j=J_0}^{J} \beta_j^T \psi_j(x) \tag{13}$$

here, the weight vector are $\alpha_{J_0} = \left[\alpha_{J_0,k}\right]_{k \in K_{J_0}}^T$ and $\beta_j = \left[\beta_{j,k}\right]_{k \in K_j}^T$ while the scaling function and wavelet function vectors are $\phi_{J_0} = \left[\phi_{J_0,k}\right]_{k \in K_{J_0}}^T$ and $\psi_j = \left[\psi_{j,k}\right]_{k \in K_j}^T$.

According to the approximation theory of wavelet networks, there exist some integer J_N, such that with $J \geq J_N$, any nonlinear function $f(x) \in L^2(\Re)$ over a compact set $\Omega_x \subset \Re$, can be approximated to an arbitrary accuracy as (Xu and Tan, 2007)

$$f(x) = \alpha_{J_0}^{*T} \phi_{J_0}(x) + \sum_{j=J_0}^{J} \beta_j^{*T} \psi_j(x) + \varepsilon(x); \forall x \in \Omega_x \tag{14}$$

where $\alpha_{J_0}^*$ and β_j^* are optimal weight vectors and $\varepsilon(x)$ is the approximation error. For optimal weight vectors, it is assumed to be bounded by $|\varepsilon(x)| \leq \varepsilon^*$ with constant $\varepsilon^* > 0$ for $\forall x \in \Omega_x$. Optimal weight vectors are unknown and are required to be estimated.

Considering $\hat{\alpha}_{J_0}$ and $\hat{\beta}_j$ as the estimates of $\alpha_{J_0}^*$ and β_j^* then the wavelet network estimate of $f(x)$ can be defined as

$$\hat{f}(x) = \hat{\alpha}_{J_0}^T \phi_{J_0}(x) + \sum_{j=J_0}^{J} \hat{\beta}_j^T \psi_j(x) \tag{15}$$

An estimation error can be defined as

$$\tilde{f}(x) = f(x) - \hat{f}(x) = \tilde{\alpha}_{J_0}^T \phi_{J_0}(x) + \sum_{j=J_0}^{J} \tilde{\beta}_j^T \psi_j(x) + \varepsilon(x) \tag{16}$$

where $\tilde{\alpha}_{J_0}$ and $\tilde{\beta}_j$ are weight estimation errors, defined as

$$\tilde{\alpha}_{J_0} = \alpha_{J_0}^* - \hat{\alpha}_{J_0} \text{ and } \tilde{\beta}_j = \beta_j^* - \hat{\beta}_j$$

With appropriate weight update laws, weight estimation errors can be confined to a small residual set and by selecting the appropriate numbers of dilates and translates, estimation error $\tilde{f}(x)$ can be made arbitrarily small on the compact set such that the bound $\left\|\tilde{f}(x)\right\| \leq \tilde{f}_m$ is satisfied for all $x \in \Omega_x \subset \Re$.

For the estimation of multivariate functions of the form $f(x)$: $\Re^n \rightarrow \Re$, wavelet network can be constructed by using multidimensional wavelet basis. One commonly used approach for the construction of multidimensional wavelet basis of a given resolution level is tensor product of single dimensional wavelet basis of same resolution level. This approach results in the generation of separable wavelet basis and straight forward extends the properties of multiresolution analysis to multidimensional case. For n

dimensional case, there exist one scaling function $\phi_{J_0,K}$ which is obtained by the tensor product of single dimensional scaling functions $\phi_{J_0,k_i}(x_i);(1,2,\cdots,n)$ and 2^n-1 wavelet functions $\psi^q_{j,K}\left(q=1,2,\cdots,2^n-1\right)$ which are obtained by considering the single dimensional wavelet and scaling functions in different dimensions (Xu and Tan, 2007).

$$\begin{aligned}\phi_{J_0,K} &= \prod_{i=1}^{n}\phi_{J_0,k_i}(x_i)\\ \psi^q_{j,k} &= \prod_{i=1}^{n}\varphi_{j,k_i}(x_i);q=1,2,\cdots,2^n-1\end{aligned} \tag{17}$$

with

$$\begin{aligned}\psi^1_{j,K} &= \left(\prod_{i=1}^{n-1}\phi_{j,k_i}(x_i)\right)\psi_{j,k_n}(x_n);\\ \psi^{2^n-1}_{j,K} &= \prod_{i=1}^{n}\psi_{j,k_i}(x_i)\end{aligned}$$

where K = [k_1, k_2, ···, k_n]; $k_i \in K_j$ and $\varphi_{j,k_i}(x_i)$ is either $\phi_{j,k_i}(x_i)$ or $\psi_{j,k_i}(x_i)$. Multidimensional wavelet networks are associated with the problem of curse of dimensionality which implies an enormous increase in the size of wavelet network with dimensions of the problem. Curse of dimensionality results in extensively high computational burden (Zhang et al., 1995; Billings and Wei, 2005). To reduce the computational burden up to certain extent, in this work a multi input-multi output wavelet network is used for the estimation of uncertain dynamics and resolutions are considered keeping the constraint of computational tractability in view.

3 Controller designs

This section presents a backstepping-based adaptive control scheme for the system (3) to achieve the control objective. Overall, control scheme is developed in two steps, first step describes the development of pseudo control terms as per the backstepping methodology whereas second step describes the development of actual control inputs taking into account the constraints imposed by underactuation property along with system uncertainties and input constraints (Marton et al., 2008; Kulkarni and Kumar, 2015, 2016).

3.1 Step 1

While designing the standard control term for equation (3), it is assumed that the system is saturation free and $g_{iq}(X)\Delta u_q = 0$. To facilitate the designing of pseudo control terms system Σ_3, equation (7) can be rewritten as

$$\Sigma_4 = \begin{cases} \dot{\underline{X}}_1 = \underline{X}_2 \\ \dot{\underline{X}}_2 = q_i(X) + \sum_{q=1}^{p} g_{iq}(X) v_q \end{cases} \tag{18}$$

where $\underline{X}_2 = [x_2, x_4, \cdots, x_{2n}]^T \in \Re^n$ and for unconstrained system nonlinear vector field q_i is defined as $q_i = f_i(X)$.

Defining tracking error vector as

$$e_1 = \underline{X}_1 - y_d \tag{19}$$

$$\dot{e}_1 = \underline{X}_2 - \dot{y}_d \tag{20}$$

Considering a Lyapunov function of the form

$$V_1 = \frac{1}{2} e_1^T P e_1 \tag{21}$$

where $P = diag[P_{11}, P_{22}, \cdots, P_{n,n}]$; $P_{i,i} < 0$.

Differentiating equation (21) along system trajectories

$$\begin{aligned} \dot{V} &= e_1^T P \dot{e}_1 \\ &= e_1^T P(\underline{X}_2 - \dot{y}_d) \end{aligned} \tag{22}$$

with a pseudo term defined as

$$\underline{X}_{2d} = -K e_1 + \dot{y}_d \tag{23}$$

where $K = diag[k_1, k_2, \cdots, k_n]$; $k_i > 0$, equation (22) results in

$$\begin{aligned} \dot{V}_1 &= e_1^T P(\underline{X}_2 - \underline{X}_{2d} + \underline{X}_{2d} - \dot{y}_d) \\ &= e_1^T P(e_2 - K e_1) \\ &= e_1^T P e_2 - e_1^T P K e_1 \\ &= e_1^T P e_2 - e_1^T Q e_1 \end{aligned} \tag{24}$$

where error vector e_2 is defined as

$$e_2 = \underline{X}_2 - \underline{X}_{2d} \tag{25}$$

To streamline the controller design, error vectors e_1 and e_2 are suitably partitioned as

$$\begin{aligned} e_1 &= [e_{11}, e_{12}]; e_{11} \in \Re^{n-p+1}, e_{12} \in \Re^{p-1} \\ e_2 &= [e_{21}, e_{22}]; e_{21} \in \Re^{n-p+1}, e_{22} \in \Re^{p-1} \end{aligned} \tag{26}$$

With this partition, equation (24) results in

$$\begin{aligned} \dot{V}_1 &= e_{11}^T P_1 e_{21} + e_{12}^T P_2 e_{22} - e_{11}^T Q_1 e_{11} - e_{12}^T Q_2 e_{12} \\ &\le e_{11}^T P_1 e_{21} + e_{12}^T P_2 e_{22} - \lambda_{\min}(Q_1) \|e_{11}\|^2 - \lambda_{\min}(Q_2) \|e_{12}\|^2 \end{aligned} \tag{27}$$

where

$$P_1 = diag\left[P_{11}, P_{22}, \cdots, P_{(n-p+1)(n-p+1)}\right]$$

$$P_2 = diag\left[P_{(n-p+2)(n-p+2)}, \cdots, P_{n,n}\right]$$

$$Q_1 = diag\left[Q_{11}, Q_{22}, \cdots, Q_{(n-p+1)(n-p+1)}\right]$$

$$Q_2 = diag\left[Q_{(n-p+2)(n-p+2)}, \cdots, Q_{n,n}\right]$$

Here, $\lambda_{\min}(.)$ represents the minimum eigenvalues of respective matrices. It is apparent from equation (27) that convergence of e_2 to some residual set in the neighbourhood of origin ensures the uniform ultimate boundedness of e_1. Next step describes the development of control terms to ensure the boundedness of e_2.

3.2 Step 2

Due to the restrictions imposed by underactuation, classical backstepping methodology is not applicable to the systems of class (3). In order to deduce a feasible control law for the systems under consideration, an approximate backstepping approach is presented. This approach utilises a hierarchical framework to develop the system dynamics which makes the development of feasible control law. To streamline the controller designing, following assumption is taken.

Assumption 4: There exist a set of vectors of real valued elements such that

$$\Omega_\rho = \{\rho \mid \det(G) \neq 0\}; \Omega_\rho \subset \Re^{n-p+1} \tag{28}$$

where

$$g = \begin{bmatrix} \sum_{i=1}^{n-p+1} \rho_i g_{i1} & \sum_{i=1}^{n-p+1} \rho_i g_{i2} & \cdots & \sum_{i=1}^{n-p+1} \rho_i g_{ip} \\ g_{(n-p+2)1} & g_{(n-p+2)2} & \cdots & g_{(n-p+2)p} \\ \vdots & \vdots & \cdots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{np} \end{bmatrix}$$

Considering e_{21} and defining an integral error term of the form

$$S_1 = \rho^T e_{21} \tag{29}$$

where $\rho = [\rho_1, \rho_2, \cdots, \rho_{n-p+1}]^T$; $\rho \in \Omega_\rho$ is the vector of coupling parameters. Error term S_1(17) can be viewed as second level error surface which is obtained by suitably aggregating the elements of error vector e_{21} which can be considered as the first level error terms (Chyau and Feng, 2010; Wang et al., 2004). Convergence of S_1(29) ensures the boundedness of first level error terms and hence stabilisation of related subsystems.

Defining an error vector of the form

$$e_3 = \left[S_1, e_{22}^T\right]^T = \left[S_1, e_{2(n-p+2)}, \cdots, e_{2n}\right]^T \tag{30}$$

Defining a Lyapunov function of the form

$$V_2 = V_1 + \frac{1}{2} e_3^T R e_3 \tag{31}$$

where $R = diag[R_{11}, R_{22}, \cdots, R_{pp}]^T$; $R_{ii} < 0$.

Differentiation of Lyapunov function V_2(31) results in

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + e_3^T R \dot{e}_3 \\
&\le e_{11}^T P_1 e_{21} + e_{12}^T P_2 e_{22} - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 + e_3^T R \dot{e}_3 \\
&\le e_{11}^T P_1 e_{21} + e_{12}^T P_2 e_{22} - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 \\
&\quad + e_3^T R \left[S_1, \dot{e}_{2(n-p+2)}, \cdots, \dot{e}_{2n}\right]^T \\
&\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| + e_{12}^T P_2 e_{22} - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 \\
&\quad + e_3^T R\left[E - X_d + Gv\right]
\end{aligned} \tag{32}$$

where $\lambda_{\max}(P_1)$ is maximum eigenvalue of P_1 while

$$E = \begin{bmatrix} \sum_{i=1}^{n-p+1} \rho_i q_i, & q_{n-p+2}, & \cdots, & q_n \end{bmatrix}^T \text{ and}$$

$$X_d = \begin{bmatrix} \sum_{i=1}^{n-p+1} \rho_i \dot{x}_{2id}, & \dot{x}_{2(n-p+2)d}, & \cdots, & q\dot{x}_{2nd} \end{bmatrix}^T$$

Defining the control input vector as

$$v = G^{-1}\left[E - X_d + R^{-1} v_a\right] \tag{33}$$

where $v_a = \begin{bmatrix} v_{a1}, & v_{a2}, & \cdots, & v_{ap} \end{bmatrix}^T$ is the vector of stabilising inputs and is defined as

$$v_a = \left[-Ae_3 - Be_4\right] \tag{34}$$

where

$$A = diag\left[A_{11}, A_{22}, \cdots, A_{pp}\right]; A_{ii} < 0,$$

$$e_4 = \begin{bmatrix} \|e_{11}\| sign(S_i), & e_{12}^T \end{bmatrix}^T$$

$$\text{and } B = diag\left[\frac{1}{\eta}, P_{(n-p+2),(n-p+2)}, \cdots, P_{n,n}\right]; 0 < \eta < 1.$$

Substitution of control term (33 and 34) in (32) results in

$$\begin{aligned}
\dot{V}_2 &\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| + e_{12}^T P_2 e_{22} - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 \\
&\quad + -e_3^T A e_3 - e_3^T B e_4 \\
&\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| + e_{12}^T P_2 e_{22} - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 - A_{11} S_1^2 \\
&\quad - \lambda_{\min}(A_1)\|e_{22}\|^2 - \frac{1}{\eta}\|e_{11}\||S_1| - e_{22}^T P_2 e_{12}
\end{aligned}$$

here $A_1 = diag[A_{22}, \cdots, A_{pp}]$

$$\begin{aligned}\dot{V}_2 \le\ & \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 - A_{11}S_1^2 \\ & -\lambda_{\min}(A_1)\|e_{22}\|^2 - \frac{1}{\eta}\|e_{11}\||S_1|\end{aligned} \tag{35}$$

Thus, the system is stable as long as

$$\begin{aligned}&\left(\lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 + A_{11}S_1^2 + \lambda_{\min}(A_1)\|e_{22}\|^2 + \frac{1}{\eta}\|e_{11}\||S_1|\right) \ge \\ &\lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\|\end{aligned} \tag{36}$$

Thus, $\dot{V}_2$ is negative outside a compact set which indicates the uniform ultimate boundedness of all the closed loop signals. As the system nonlinearities $f_i(X)$ are assumed uncertain, implementation of the control law (33) is not feasible. To render the control law feasible, uncertain dynamics are approximated by using wavelet network.

Whenever the control effort undergoes saturation, some part of the control effort is clipped off (2), the effect of actuator saturation can be considered in terms of additional nonlinear dynamics as described by equation (8). Under the effect of actuator saturation nonlinear vector field q_i becomes $q_i = f_i(X) + \sum_{q=1}^{p} g_{iq}(X)\Delta u_q$ and accordingly changes the vector field E. In this work, the elements of nonlinear vector field E are approximated as it also accounts for the nonlinear dynamics inserted by actuator saturation. This approximation thereby allows the effective compensation of saturation dynamics and also relaxes the constraint of measuring Δu_q.

Under the effect of uncertain dynamics and actuator saturation, control term (33) becomes

$$v = G^{-1}\left[-\hat{E} + X_d + R^{-1}v_a - \frac{e_3}{2\zeta^2}\right] \tag{37}$$

where $\hat{E}$ represents the wavelet network approximation of E.

Update laws for the weight update of wavelet networks are

$$\begin{aligned}&\dot{\tilde{\alpha}}_{J0,1} = -\dot{\hat{\alpha}}_{J0,1} = -\kappa_1 R_{11} S_1 \phi_{J_0}(x) \\ &\dot{\tilde{\alpha}}_{J0,i} = -\dot{\hat{\alpha}}_{J0,i} = -\kappa_i R_{ii} e_{2(n-p+i)} \phi_{J_0}(x); i = 2,3,\cdots,p \\ &\left.\begin{aligned}&\dot{\tilde{\beta}}_{j,1}^q = -\dot{\hat{\beta}}_{j,1}^q = -\gamma_1 R_{11} S_1 \psi_j^q(x) \\ &\dot{\tilde{\beta}}_{j,i}^q = -\dot{\hat{\beta}}_{j,i}^q = -\gamma_i R_{ii} e_{2(n-p+i)} \psi_j^q(x)\end{aligned}\right\} \begin{aligned}&j = J_0, J_0+1,\cdots,J \\ &i = 2,3,\cdots,p \\ &q = 1,2,\cdots,2^n-1\end{aligned}\end{aligned} \tag{38}$$

here $\kappa_i > 0$, $\gamma_i > 0$ and $0 < \zeta < 1$. The term $\left(-\frac{e_3}{2\zeta^2}\right)$ is the robust control term inserted in control law (33) so as to attenuate the approximation error of wavelet network to a prescribed attenuation level (Hsu et al., 2006).

Controller scheme can be viewed in a generalised manner and any $(p - 1)$ subsystems with linearly independent row in equation (9) can be assigned dedicated control terms.

To examine the effectiveness of control law (37), consider the Lyapunov function of the form (Khalil, 2002; Astrom and Wittenmark, 1995)

$$V_3 = V_1 + \frac{1}{2}e_3^T Re_3 + \frac{1}{2}\sum_{i=1}^{p}\left(\frac{\tilde{\alpha}_{J0,i}^T \tilde{\alpha}_{J0,i}}{\kappa_i} + \sum_{j=J_0}^{J}\sum_{q=1}^{2^n-1}\frac{\tilde{\beta}_{j,i}^{qT}\tilde{\beta}_{j,i}^{q}}{\gamma_i}\right) \tag{39}$$

Differentiating equation (39) and substituting the control term (37)

$$\begin{aligned}\dot{V}_2 &\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| + e_{12}^T P_2 e_{22} - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 \\ &\quad + e_3^T R[E - X_d + Gv] + \sum_{i=1}^{p}\left(\frac{\tilde{\alpha}_{J0,i}^T \dot{\tilde{\alpha}}_{J0,i}}{\kappa_i} + \sum_{j=J_0}^{J}\sum_{q=1}^{2^n-1}\frac{\tilde{\beta}_{j,i}^{qT}\dot{\tilde{\beta}}_{j,i}^{q}}{\gamma_i}\right) \\ &\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 - A_{11}S_1^2 - \lambda_{\min}(A_1)\|e_{22}\|^2 \\ &\quad - \frac{1}{\eta}\|e_{11}\||S_1| + e_3^T R\tilde{E} - \frac{1}{2\zeta^2}e_3^T Re_3 + \sum_{i=1}^{p}\left(\frac{\tilde{\alpha}_{J0,i}^T \dot{\tilde{\alpha}}_{J0,i}}{\kappa_i} + \sum_{j=J_0}^{J}\sum_{q=1}^{2^n-1}\frac{\tilde{\beta}_{j,i}^{qT}\dot{\tilde{\beta}}_{j,i}^{q}}{\gamma_i}\right)\end{aligned} \tag{40}$$

Substitution of update laws (38) in (40) results in

$$\begin{aligned}\dot{V}_2 &\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 - A_{11}S_1^2 - \lambda_{\min}(A_1)\|e_{22}\|^2 \\ &\quad - \frac{1}{\eta}\|e_{11}\||S_1| + e_3^T R\varepsilon - \frac{1}{2\zeta^2}e_3^T Re_3 \\ &\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 - A_{11}S_1^2 - \lambda_{\min}(A_1)\|e_{22}\|^2 \\ &\quad - \frac{1}{\eta}\|e_{11}\||S_1| + \lambda_{\max}(R)\|\varepsilon\|\|e_3\| - \frac{1}{2\zeta^2}\lambda_{\min}(R)\|e_3\|^2 \\ &\le \lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| - \lambda_{\min}(Q_1)\|e_{11}\|^2 - \lambda_{\min}(Q_2)\|e_{12}\|^2 - A_{11}S_1^2 - \lambda_{\min}(A_1)\|e_{22}\|^2 \\ &\quad - \frac{1}{\eta}\|e_{11}\||S_1| + \frac{\lambda_{\max}(R)\zeta^2}{2\lambda_{\min}(R)}\|\varepsilon\|^2\end{aligned} \tag{41}$$

So the system stability is ensured as long as

$$\begin{aligned}&\left(\lambda_{\min}(Q_1)\|e_{11}\|^2 + \lambda_{\min}(Q_2)\|e_{12}\|^2 + A_{11}S_1^2 + \lambda_{\min}(A_1)\|e_{22}\|^2 + \frac{1}{\eta}\|e_{11}\||S_1|\right) \ge \\ &\left(\lambda_{\max}(P_1)\|e_{11}\|\|e_{21}\| + \frac{\lambda_{\max}(R)\zeta^2}{2\lambda_{\min}(R)}\|\varepsilon\|^2\right)\end{aligned} \tag{42}$$

Thus, the control law (37) ensures the uniform ultimate boundedness of all the closed loop signals in presence of uncertain dynamics and actuator saturation.

4 Simulation results

To demonstrate the effectiveness of the controller scheme developed in previous section [equation (39)], a simulation is carried out with an objective to stabilise a three link planner robot manipulator with two active and one passive joint under the consideration of uncertain dynamics and actuator saturation. Robotic links are connected using revolute joints in serial link mechanism with first link connected to the base. Dynamics considered for simulation is meant for a robot manipulator with first passive joint and next two active joints. Euler-Lagrange equation for the system dynamics is as under (Liu and Yamaura, 2011).

$$M\left(\underline{X}_1\right)\dot{\underline{X}}_2 + C\left(\underline{X}_1, \underline{X}_2\right) + H\left(\underline{X}_1\right) = \Gamma u \tag{43}$$

where

$$M\left(\underline{X}_1\right) = \begin{bmatrix} a_1 + a_2 + a_3 & a_2 + a_3 & a_3 \\ a_2 + a_3 & b_2 + b_3 + 2m_3 l_2 c_3 \cos x_5 & b_3 + m_3 l_2 c_3 \cos x_5 \\ a_3 & b_3 + m_3 l_2 c_3 \cos x_5 & b_3 \end{bmatrix}$$

$$C\left(\underline{X}_1, \underline{X}_2\right) = \begin{bmatrix} \left(-r_1\left(m_3 l_1 c_2 + m_3 l_1 l_2\right)\sin\left(x_3\right) - r_3 m_3 l_2 c_3 \sin\left(x_5\right) - r_3 m_3 l_1 c_3 \sin\left(x_3 + x_5\right)\right) \\ \left(x_2^2\left(\left(m_3 l_1 c_2 + m_3 l_1 l_2\right)\sin\left(x_3\right) + m_3 l_1 c_3 \sin\left(x_3 + x_5\right)\right) - r_2 m_3 l_2 c_3 \sin\left(x_5\right)\right) \\ \left(x_2^2\left(m_3 l_2 c_3 \sin\left(x_5\right) + m_3 l_1 l_3 \sin\left(x_3 + x_5\right)\right) + r_1 m_3 l_2 c_3 \sin\left(x_5\right)\right) \end{bmatrix}$$

$$H\left(\underline{X}_1\right) = \begin{bmatrix} \begin{pmatrix} g\left(m_1 c_1 + \left(m_2 + m_3\right) l_1\right)\sin\left(x_1\right) + g\left(m_2 c_2 + m_3 l_1\right)\sin\left(x_1 + x_3\right) \\ + g m_3 c_3 \sin\left(x_1 + x_3\right) \end{pmatrix} \\ g\left(m_2 c_2 + m_3 l_2\right)\sin\left(x_1 + x_3\right) + g m_3 c_3 \sin\left(x_1 + x_3\right) g m_3 c_3 \sin\left(x_1 + x_3\right) \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

here, $M\left(\underline{X}_1\right) \in \Re^{3\times 3}, C\left(\underline{X}_1, \underline{X}_2\right) \in \Re^{3\times 1}$ and $H\left(\underline{X}_1\right) \in \Re^{3\times 1}$ are inertia matrix, carioles matrix and gravity matrix respectively.

Element of the vector $\underline{X}_1 = \left[x_1, x_3, x_5\right]^T \in \Re^{3\times 1}$ denotes the link angles. Variable x_1 denotes the angular position of the first link with respect to vertical axis whereas x_2 and x_3 denotes the angular positions of next two respective links with respect to axis passing through the centre of mass of the preceding link. Vector $\underline{X}_2 = \left[x_2, x_4, x_6\right]^T \in \Re^{3\times 1}$ contains corresponding velocity variables. $u = \left[u_1 \quad u_2\right]^T \in \Re^{2\times 1}$ is the torque vector applied to the system . Other variables appearing in the system dynamics are

$$a_1 = b_1 + \left(m_3 l_1 c_2 + m_3 l_1 l_2\right)\cos x_3 + m_3 l_1 c_3 \cos\left(x_3 + x_5\right)$$

$$a_2 = b_2 + \left(m_3 l_1 c_2 + m_3 l_1 l_2\right)\cos x_3 + m_3 l_2 c_3 \cos x_5$$

$$a_3 = b_3 + m_3 l_2 c_3 \cos x_3 + m_3 l_1 c_3 \cos\left(x_3 + x_5\right)$$

$$r_1 = 2x_2 x_4 + x_4^2$$

$$r_2 = 2x_2 x_6 + 2x_4 x_6 + x_6^2$$

$$r_3 = 2x_2 x_4 + 2x_4 x_6 + 2x_2 x_6 + x_4^2 + x_6^2$$

System parameters m_i, l_i, c_i and $I_i(i = 1, 2. 3)$ are link parameters and respectively denotes the mass, length, distance from i^{th} joint to centre of mass and moment of inertia about the centre of mass for i^{th} link with

$$b_1 = I_1 + m_1 c_1^2 + m_2 l_1^2 + m_3 l_1^2$$
$$b_2 = I_2 + m_2 c_2^2 + m_3 l_2^2$$
$$b_3 = I_3 + m_3 c_3^2$$

whereas g is acceleration due to gravity (Liu and Yamaura, 2011).

System dynamics (43) can be represented in the following state space form

$$\Sigma_6 = \begin{cases} \dot{x}_{2i-1} = x_{2i} \\ \dot{x}_{2i} = f_i + g_{i1} u_1 + g_{i2} u_2 ; i = 1,2,3 \\ y_i = x_i \end{cases} \tag{44}$$

Control scheme (37) is applied to above dynamics (44) with an objective to stabilise it about $[0, 0, 0, 0, 0, 0]^T$ under the magnitude constraints imposed by system actuators and consideration of nonlinearities $f_i(i = 1, 2, 3)$ as system uncertainties. For the clear illustration of the effectiveness of control scheme simulation is carried out for two cases

4.1 Case 1

Control term u_1 is designed to assure the stabilisation of first and second link dynamics whereas u_2 is assigned as dedicated controller to third link.

After considering the nonlinear effects of saturation dynamics (3), system (44) can be rewritten as

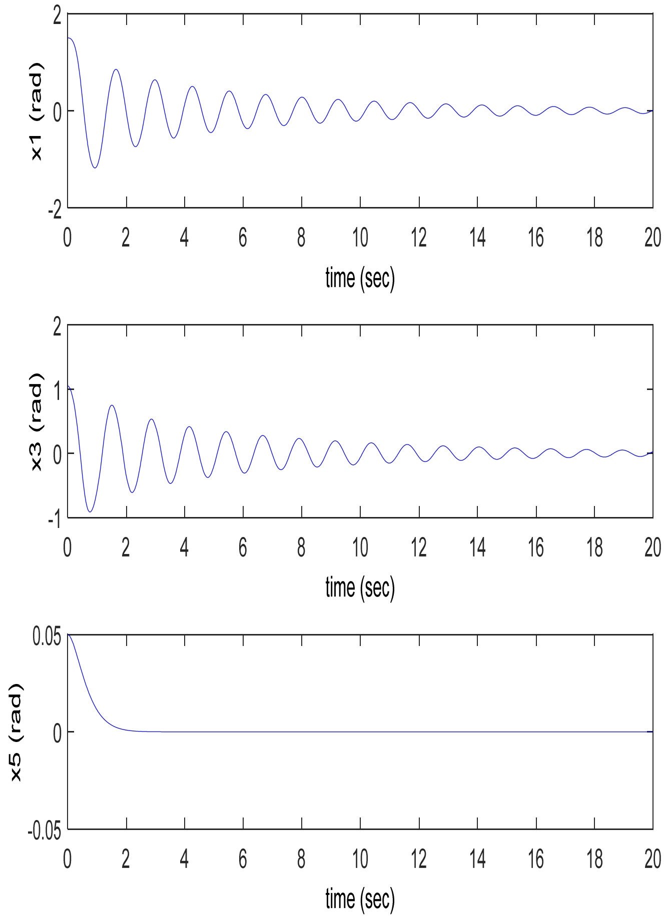
$$\Sigma_7 = \begin{cases} \dot{x}_{2i-1} = x_{2i} \\ \dot{x}_{2i} = q_i + g_{i1} v_1 + g_{i2} v_2 ; i = 1,2,3 \\ y_i = x_i \end{cases} \tag{45}$$

where

$$q_p = f_p + \sum_{i=1}^{2} g_{pi} \Delta u_i ; p = 1,2,3 \tag{46}$$

Here, the nonlinear term $g_{pi}\Delta u_i$ represents the nonlinear effect of saturation dynamics and the nonlinearity q_p thus, combines the nonlinear system dynamics as well as nonlinearity induced by the saturation. These nonlinearities are approximated by using wavelet neural network. Wavelet approximation of the nonlinear saturation dynamics also relaxes the requirement of measuring the saturation error [equation (3)].

Figure 1 Trajectories of link angles with unconstrained actuators (see online version for colours)



A wavelet network with $n = 6$ and arguments $(x_1, x_2, x_3, x_4, x_5, x_6)$ is constructed using Daubechies wavelet (db3) for the approximation of system nonlinearities. The coarsest and finest resolution levels for the wavelet network are taken as $J_0 = 1$ and $J = 3$ respectively. Number of translates at coarsest resolution level for (x_1, x_3, x_5) are taken as $K_{11} = 3$ and those for (x_2, x_4, x_6) are taken as $K_{12} = 5$. Translates are made double when resolution is increased by 1. Online adjustment of weight parameters is carried out using adaptation laws (38) with initial conditions set to zero for all the wavelet parameters.

Figure 2 Control efforts with unconstrained actuators (see online version for colours)

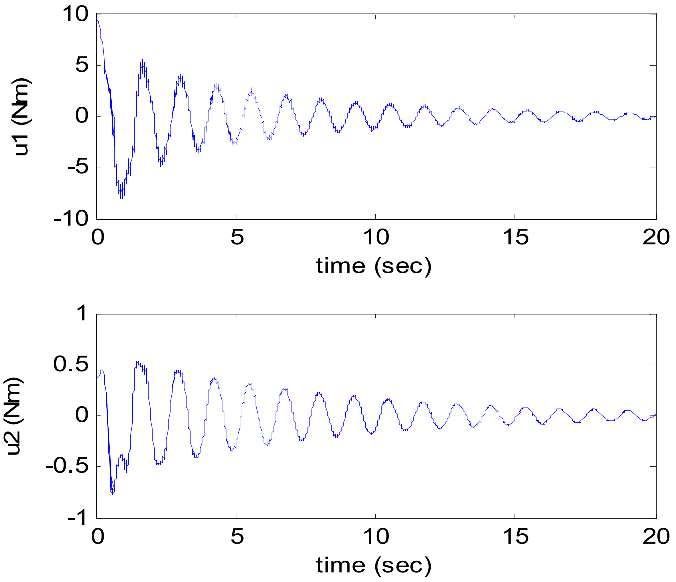


Figure 3 Trajectories of state variables under actuator saturation (see online version for colours)

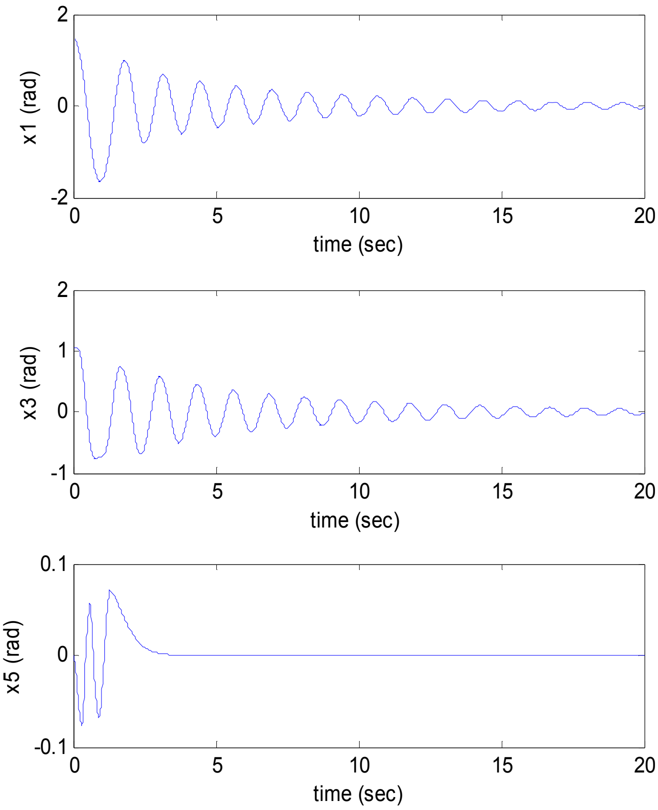
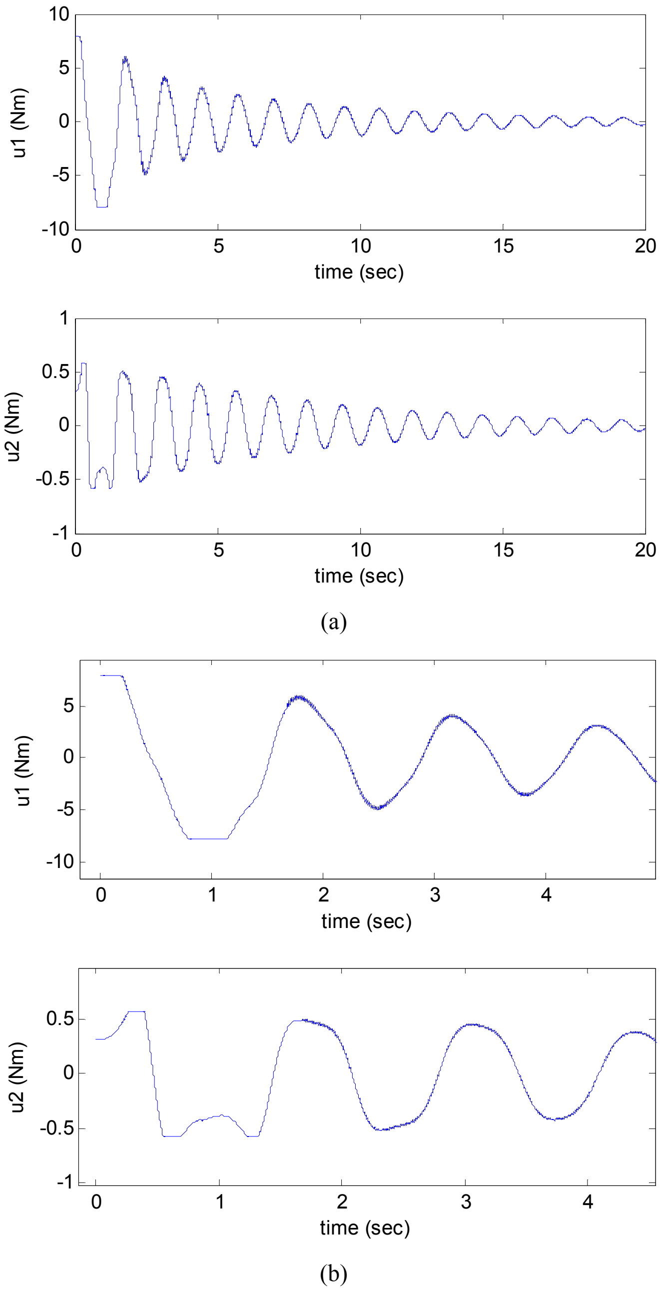


Figure 4 (a) Control efforts under actuator saturation (b) Expanded view (see online version for colours)



To highlight the controller performance under the condition of saturation, the simulation is carried out in two phases. During the first phase, actuator saturation is neglected and the system is assumed unconstrained. Simulation is carried out with following system parameters (Liu and Yamaura, 2011), initial conditions and controller settings:

$$l_i = 0.150\text{m}; l_2 = 0.150\text{m}; l_3 = 0.200\text{m}$$
$$m_3 = 0.216\text{kg}; c_3 = 0.952\text{m};$$
$$x(0) = [1.5, 0, 1.05, 0, 0, 0]^T$$
$$P = diag[1,1,1]\ K = diag[0.75, 3.75, 3.2]$$
$$m_2 c_2 = 0.1429\text{kgm}; m_1 c_1 + m_2 l_1 = 0.983\text{kgm}$$
$$b_1 = 0.1389\text{kgm}^2; b_2 = 0.0299\text{kgm}^2; b_3 = 0.0028\text{kgm}^3$$
$$\rho = [-0.5, 0.5]^T\ R = diag[1,1]$$
$$A = diag[3,2]$$
$$\eta = 0.5; \zeta = 0.1;$$
$$\kappa_i = 0.5; \gamma_i = 0.5; (i = 1, 2)$$

Simulation results are shown in Figures 1and 2. With the control scheme (37), convergence of link angles to the close neighbourhood of origin can be observed.

Figure 5 Trajectories of link angles with unconstrained actuators (see online version for colours)

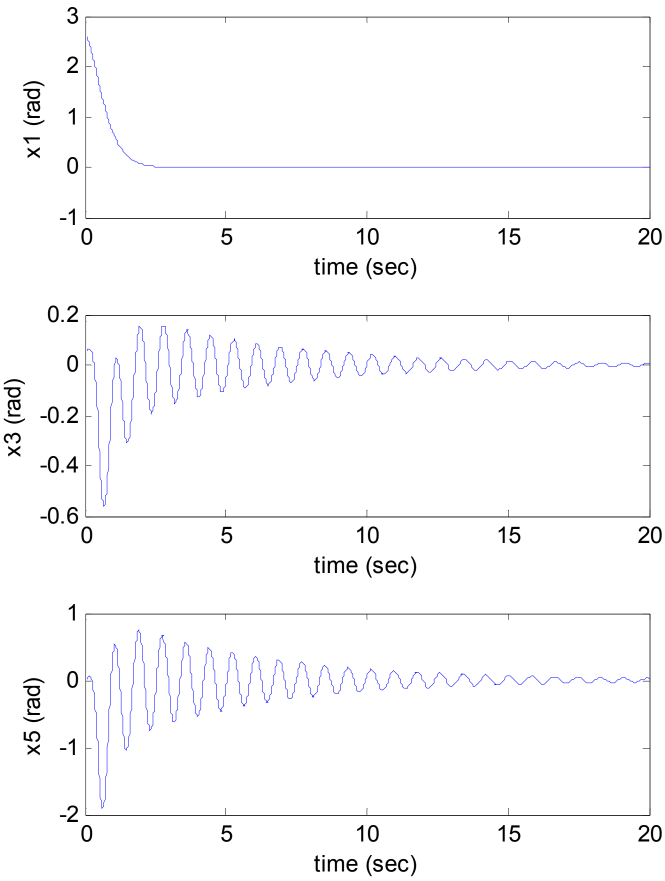


Figure 6 Control efforts with unconstrained actuators (see online version for colours)

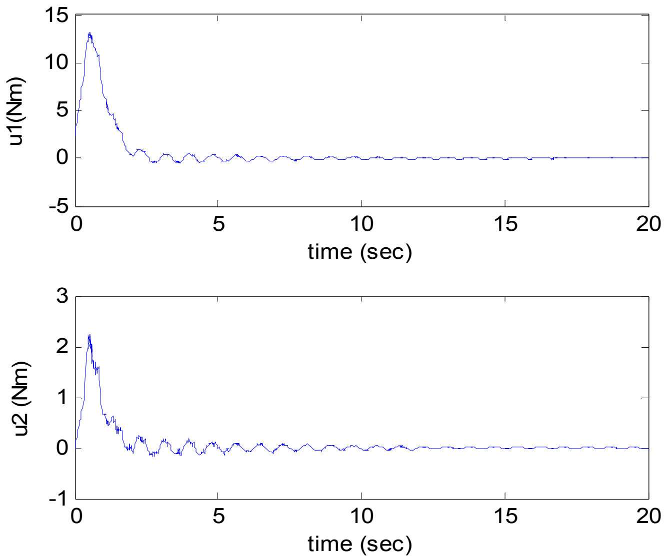


Figure 7 Trajectories of state variables under actuator saturation (see online version for colours)

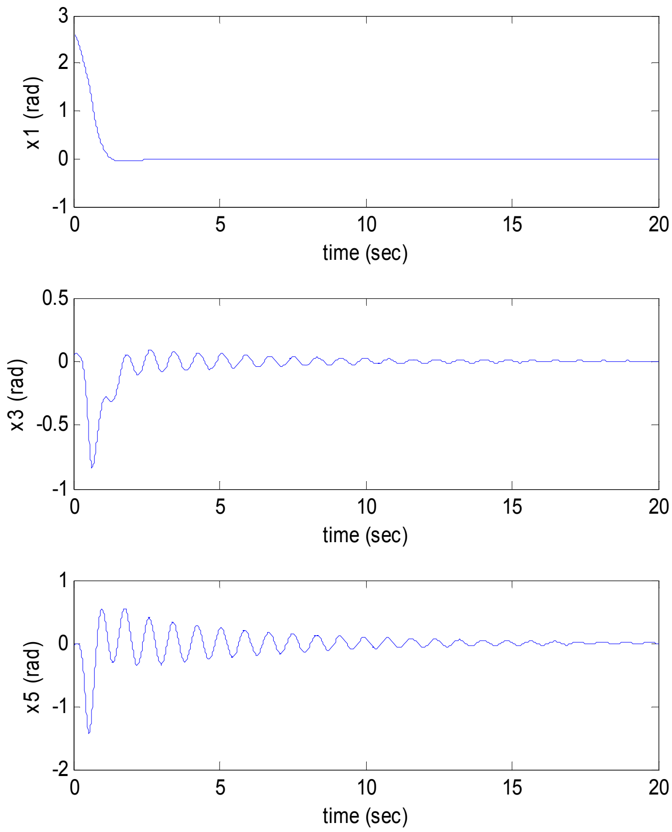
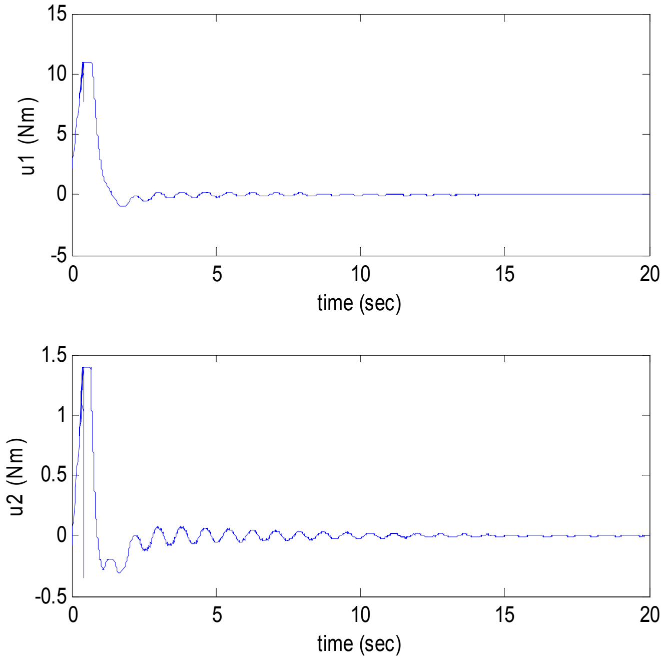
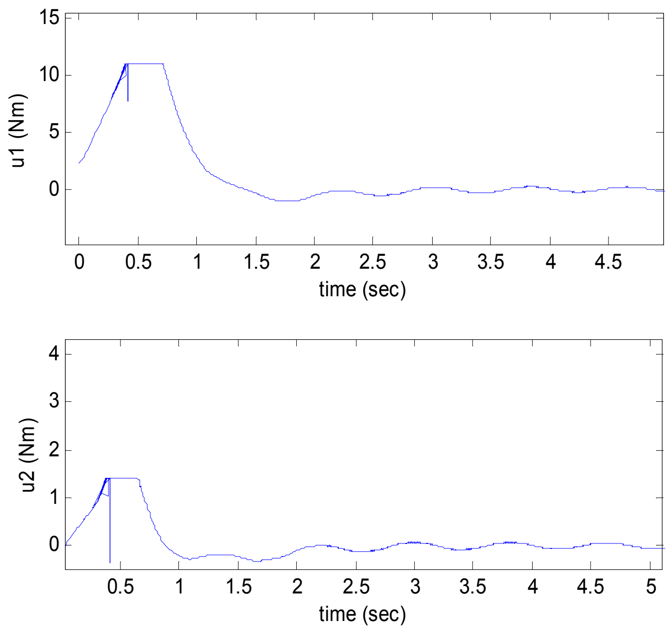


Figure 8 (a) Control efforts under actuator saturation (b) Extended view (see online version for colours)

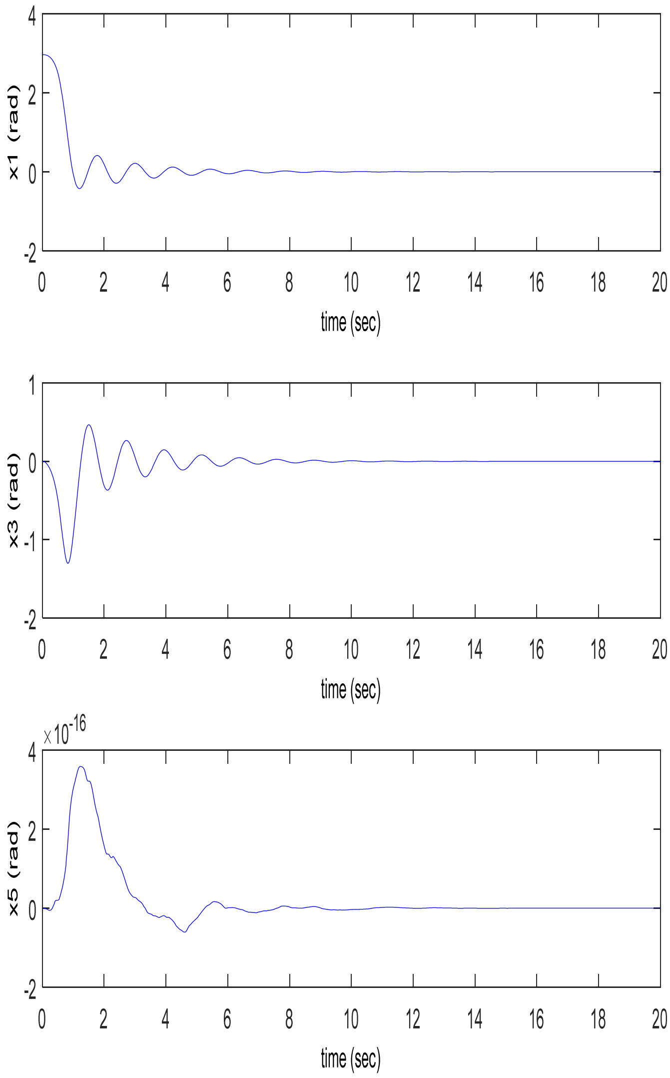


(a)



(b)

Figure 9 Trajectories followed by link angles (see online version for colours)



During the second phase of simulation, simulation is carried out for the same initial conditions and controller parameter settings with the consideration of actuator saturation with saturation limits $u_{1\max}$ = 8.5 Nm and $u_{2\max}$ = 0.6 Nm. Results of the simulation performed are shown in Figures 3 and 4. As reflected by the figures, system performance is almost similar to the unconstrained response obtained during the first phase. Control efforts initially undergo saturation thereby causing a slight detuning of the system response and invoking the nonlinear effects of saturation. However, wavelet network, due to its potential to approximate the nonlinearities accurately and rapidly, reshape the control efforts and drive the actuators out of saturation within a short span of time. Control efforts, thereafter retune the system response rapidly to its original unconstrained form.

4.2 Case 2

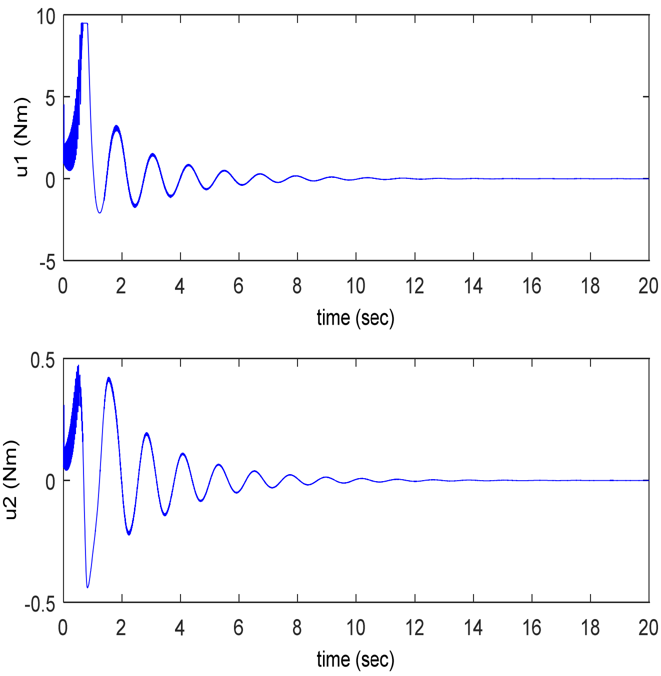
This time control term u_2 is designed to assure the stabilisation of second and third link dynamics and u_1 is assigned as dedicated controller to first link. Simulation is carried for the following settings:

$$x(0) = [2.62, 0, 0.05, 0, -0.05, 0]^T$$
$$P = diag[1,1,1]\ K = diag[1.75, 3.1, 6.5]$$
$$A = diag[3, 2]$$
$$\eta = 0.5; \zeta = 0.1;$$
$$\kappa_i = 0.5; \gamma_i = 0.5; (i = 1, 2)$$
$$u_{1\max} = 12\text{Nm}; u_{2\max} = 1.8\text{Nm}$$

Simulation results are shown in Figures 5, 6, 7 and 8. With the control scheme (37), convergence of link angles to the close neighbourhood of origin can be observed

To illustrate the effectiveness of the proposed scheme, a comparison study is carried out; the closed loop response of the system under consideration is simulated with the initial settings $x(0) = [17\pi/18, 0, 0, 0, \pi/60, 0]^T$ taken in Lai et al. (2015). Simulation is carried out considering the system uncertainties and actuator saturation. System response and control efforts are shown in Figures 9 and 10 respectively. As reflected by the state trajectories, despite of system uncertainties and actuator saturation, link angles rapidly converge to equilibrium position and settle down in 10 seconds.

Figure 10 Control effort (see online version for colours)



5 Conclusions

This paper presents a semi generic adaptive control scheme, developed by integrating the backstepping methodology with hierarchical framework to ensure the stabilisation of a class of uncertain multi- input multi- output underactuated systems with actuator saturation. Under the constraints imposed by the property of underactuation, a feasible control scheme is derived by integrating backstepping and hierarchical controller design scheme. In the controller scheme presented one control term is derived by applying hierarchical framework to appropriate number of subsystems whereas the remaining controllers are assigned as dedicated controllers to rest of the subsystems. Backstepping methodology is used to derive the feedback control laws. Control scheme so derived ensures the uniform ultimate boundedness of all the closed loop signals. Wavelet neural networks are used for the approximation of uncertain nonlinear dynamics as well as the nonlinear dynamics invoked by the actuator saturation. Convergence analysis of the error dynamics is carried out in the Lyapunov sense. Effectiveness of the controller scheme is illustrated through the simulation

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