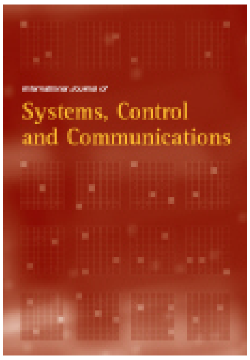

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Approximation of fractional-order MIMO system – a unified approach

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Abstract: In this paper, a method is presented to approximate a class of fractional-order multiple-input multiple-output (MIMO) system in discrete-delta domain. The modelling approach is unconventional in the sense that it does not encompass the traditional generating functions for approximating the fractional-order MIMO system. Instead, the novel delta operator has been endorsed for accomplishing the approximation as it facilitates the unified representation of both the continuous-time system and discrete-time system together. The transfer function matrix (TFM) of the continuous-time fractional-order MIMO system is derived in the first stage. The TFM is approximated in the next stage using delta operator and the continued fraction expansion (CFE) method. The approximation technique has been illustrated by taking suitable example.

Keywords: fractional-order system; FOS; multiple-input multiple-output; MIMO; delta operator; continued fraction expansion; CFE; transfer function matrix; TFM.

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1 Introduction

Fractional calculus is a subject which offers a generalised framework to express both the non-integer order and the integer-order differentiation and integration together. The modelling of many physical systems is better accomplished using fractional-order models expressed through fractional-order differential equation, fractional-order transfer function or fractional-order state space representation (Monje et al., 2010). At present, the research on fractional-order system (FOS) is prevalent in many disciplines of engineering and science. The application fields of FOS embrace wide range like electrochemical process (Sabatier et al., 2006), thermal processes (Gabano and Poinot, 2011), biological systems (Biswas et al., 2017), viscoelasticity (Mashayekhi et al., 2018), robotics (Kumar et al., 2020), electronics (AbdelAty et al., 2019), etc. In control theory, CRONE controller

(Oustaloup, 1991), fractional-order PID controller (Podlubny, 1999) have evidenced their superior performances over the traditional controllers through exploiting the benefits of additional degrees-of-freedom. Accordingly, the design of fractional-order controller has been widely addressed in many recent literatures (Mecheri et al., 2021; Saidi et al., 2018; Kumar and Narayan, 2017; Katal and Narayan, 2017). The FOS is infinite dimensional and therefore, it is not feasible to employ them directly for practical implementations like controller design, realisation, etc. In reality, the FOS is approximated to an equivalent integer order model within a specific frequency range. The efficacy of the approximation is measured in terms of closeness of the frequency response and time response characteristics between the original FOS and its corresponding approximated version. Two types of approximation methods are prevalent in literature – one is the continuous-time approximation methods which are developed in s domain and another is the discrete-time approximation methods which are developed in z domain. These rational approximation methods have been used to approximate the single-input single-output (SISO) systems (Bouafoura et al., 2011; Krajewski and Viaro, 2014) and also the multiple-input multiple-output (MIMO) systems (Rachid et al., 2011; Latawiec et al., 2017). The proposed work belongs to the class of discrete-time approximation method to model the fractional-order MIMO system in delta domain instead of conventional z-domain. A remarkable study on fractional-order modelling and control of MIMO process is detailed in the contemporary literature (Li, 2015). In Khanra et al. (2013), the reduced order approximation of the fractional-order MIMO system has been obtained through matching the suitable numbers of approximate generalised time moments and the approximate generalised Markov parameters. In Ivanova et al. (2016), the time domain simulation of fractional-order MIMO system has been performed employing Oustaloup's approximation and Trigeassou's variant. In Mishra and Chandra (2013), the particle swarm optimisation method has been applied to model the fractional-order MIMO system through minimising the objective function based on the integral-square-error.

There are numerous approaches to obtain the discrete model of the fractional order system. The indirect approach finds the approximate model of the FOS in s-domain and then it discretises the system (Martinez-Patiño et al., 2022). Direct approach is more popular as it approximates the fractional power term straightaway in z-domain using a generating function and expansion method (De Keyser et al., 2018). There is another approach which uses asymptotic sampling-zeros to find the approximate discrete model of FOS (Yucra et al., 2013). Several discrete-time operators have been employed to model the fractional-order MIMO systems. For example, the Grunwald-Letnikov operator has been used for the said purpose as seen in Rachid et al. (2010). The recognised Tustin operator has also been used for approximating the fractional-order MIMO system (Juchem et al., 2019). The use of Al-Alaoui operator has been mentioned in the contemporary literature for approximation of the fractional-order MIMO system (Rydel, 2019). All these operators are designated as generating functions of z domain which have been employed to model the fractional-order MIMO system. Generally, the discrete-time systems are modelled using shift operator or its allied frequency variable z. This modelling approach does not correspond to the original continuous-time system at low sampling time limit or fast sampling rate. On the other hand, the delta operator theory articulates the concept of unification of discrete-time system and the continuous-time system at low sampling time limit (Middleton and Goodwin, 1990; Maione, 2011; Swarnakar et al., 2019). However, this stimulating property of the delta operator has not been casted yet to model the fractional-order MIMO system. Therefore, the presented work capitalises this opportunity for approximating a class of fractional-order MIMO system using a discrete-time approximation technique in delta domain. The work has been done in two stages. In the first stage, the transfer function matrix (TFM) is attained from the fractional-order state space model. The TFM includes the SISO transfer functions corresponding to the relevant input-output pair. In the next stage, all the fractional power terms in the TFM are identified and they are approximated in the delta domain through continued fraction expansion (CFE) method. The CFE method is well-known in the literature for expanding the generating function in rational form (Song et al., 2014). The other expansion method like power series expansion method (PSE) has not been preferred in this work as it results the higher order approximation compared to CFE method (De Keyser et al., 2018). All the SISO transfer functions inside the TFM are approximated in delta domain to get the overall approximation of the fractional-order MIMO system. The frequency responses and the time responses of the original MIMO system and the approximated MIMO system have been compared at separate sampling instants.

The paper is organised in four sections. Section 1 deals with the introductory part. Section 2 describes the fundamentals of the delta operator theory and elaborates the procedure to approximate a class of fractional-order MIMO system in delta domain from the fractional-order state space model and its corresponding TFM. Section 3 discusses about the simulation results. The influence of changing the sampling time on the resultant approximation has been analysed in line with the delta operator theory. Section 4 accomplishes the work through relevant concluding remarks.

2 Approximation methodology using delta operator

In time domain, the delta operator is represented as below:

$$\delta = \frac{q-1}{\Delta} \tag{1}$$

where q denotes the forward shift operator and Δ implies the sampling time. The delta operator merges to the continuous-time derivate operator as shown in the equation (2).

$$\lim_{\Delta \to 0} \delta h(t) = \lim_{\Delta \to 0} \frac{h(t+\Delta)-h(t)}{\Delta} = \frac{d}{dt} h(t) \tag{2}$$

The complex domain description of the delta operator is quite analogous to the equation (1) as given below:

$$\gamma = \frac{z-1}{\Delta} \quad (z = e^{s\Delta}) \tag{3}$$

From equation (3), the delta operator is finally obtained as shown below:

$$s = \frac{1}{\Delta} \ln(1+\Delta\gamma) \tag{4}$$

From equation (3), following is obtained:

$$\lim_{\Delta \to 0} \gamma = \lim_{\Delta \to 0} \frac{e^{s\Delta} - 1}{\Delta} = s \tag{5}$$

So, the Laplacian frequency variable s merges to the delta-domain frequency variable γ at the low sampling time as seen from the equation (5). Equations (2) and (5) indicate the unification property of the delta operator. This exciting property of the delta operator has been exploited to approximate a class of fractional-order MIMO system. The non-commensurate state space model of the fractional-order MIMO system is represented as below:

$$\begin{aligned} D^{(\rho)}x &= Mx + Nu \\ y &= Lx + Eu \end{aligned} \tag{6}$$

where $\rho \in R^{+}$, $D^{(\rho)}x = \left[\frac{d^{\rho_1}}{dt^{\rho_1}} x_1, \frac{d^{\rho_2}}{dt^{\rho_2}} x_2 .... \frac{d^{\rho_f}}{dt^{\rho_f}} x_f \right]^T$, $x \in R^{f\times 1}$, $u \in R^{k\times 1}$, $y \in R^{j\times 1}$, $M \in R^{f\times f}$, $N \in R^{f\times k}$, $L \in R^{j\times f}$, $E \in R^{j\times k}$. The number of outputs and inputs are j and k, respectively. The system will have the commensurate form if $\rho_1 = \rho_2 = \rho_3 \text{ .... } = \rho_f$ and in that case, $D^{(\rho)}x = \frac{d^{\rho}}{dt^{\rho}} [x_1, x_2 .... x_f]^T$. The TFM of the aforementioned state space model results the following expression:

$$H(s) = L\left[s^{(\rho)}I - M\right]^{-1} N + E \tag{7}$$

where $s^{(\rho)}I = diag(s^{\rho_1}, s^{\rho_2} ... s^{\rho_f})$.

After simplifying the equation (7), the TFM of the state space model will be as follows:

$$H(s) = \begin{bmatrix} h_{11}(s) & \cdots & h_{1k}(s) \\ \vdots & \ddots & \vdots \\ h_{j1}(s) & \cdots & h_{jk}(s) \end{bmatrix} \tag{8}$$

Each element of $H(s)$ refers to a fractional-order transfer function. Let, a single fractional power term existing in a fractional-order transfer function is represented as $s^{\rho_i} = s^{[\rho_i]} \times s^{\rho_i - [\rho_i]}$, where $s^{[\rho_i]}$ represents the integer part of s^{ρ_i} and $s^{\rho_i - [\rho_i]}$ represents the fractional part of s^{ρ_i}. For example, $s^{2.7} = s^2 \times s^{2.7-0.7}$. Hence, the integer part of $s^{2.7}$ is s^2 and the fractional part is $s^{0.7}$. For purely fractional power term, the part $s^{[\rho_i]} = 1$. For example, a purely fractional power term $s^{0.6}$ can be expressed as $s^{\cdot 6} = s^0 \times s^{0.6}$. Now, equation (4) is modified employing trapezoidal rule as shown below:

$$1 + \Delta\gamma = \left(e^{as}\right) \times \left(e^{-as}\right)^{-1} \approx \frac{1 + as}{1 - as} \quad (a = 0.5\Delta) \tag{9}$$

Rearranging equation (9), we obtain the following relationship:

$$s \approx \frac{n_1}{d_1} \tag{10}$$

where $n_1 = \gamma$ and $d_1 = 1 + 0.5\Delta\gamma$.

Equation (10) is used as a generating function in this work which is a modified form of equation (4). Equation (10) can be used directly to replace integer part, i.e., $s^{[\rho_i]}$ in delta domain. Using equation (10), the fractional power term containing both the integer-part and fractional-part is represented as below:

$$s^{\rho_i} = \left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{[\rho_i]} \times \left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{\rho_i-[\rho_i]} \tag{11}$$

Now, the term $\left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{[\rho_i]}$ represents the integer part which is obtained directly from equation (10). However, $\left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{\rho_i-[\rho_i]}$ is an irrational part which requires integer order approximation. This has been done using CFE method and accordingly equation (11) is represented as below:

$$s^{\rho_i} \approx \left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{[\rho_i]} \times \left\{CFE\left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{\rho_i-[\rho_i]}\right\} \tag{12}$$

The mathematical representation of CFE approximation is given as below (Khovanskii, 1963; Maione and Lazarević, 2019):

$$(b+1)^p = \left[1+\frac{pb}{1}{}_{+}\frac{(1-p)b}{2}{}_{+}\frac{(1+p)b}{3}{}_{+}\frac{(2-p)b}{2}{}_{+}\frac{(2+p)b}{5}{}_{+}\frac{(3-p)b}{2}{}_{+}\frac{(3+p)b}{7}\cdots\right] \tag{13}$$

It is assumed that third-order CFE delivers the moderate approximation. Higher order CFE has not been used to restrict the order of the overall approximate model. Substituting $b=\left(\frac{\gamma}{1+0.5\Delta\gamma}-1\right)$ and $p = \rho_i - [\rho_i]$, the third order CFE approximation results the following expression:

$$\left\{CFE\left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{\rho_i-[\rho_i]}\right\} = \frac{\sum_{c=0}^{3}\alpha_c\gamma^{3-c}}{\sum_{c=0}^{3}\beta_c\gamma^{3-c}} \tag{14}$$

The coefficients α_c (c = 0, 1, 2, 3) and β_c (c = 0, 1, 2, 3) are shown in Table 1.

Using equations (10), (14) and Table 1, the continuous-time TFM represented by equation (8) is approximated as below:

$$H^{\delta}(\gamma) = \begin{bmatrix} h_{11}^{\delta}(\gamma) & \cdots & h_{1k}^{\delta}(\gamma) \\ \vdots & \ddots & \vdots \\ h_{j1}^{\delta}(\gamma) & \cdots & h_{jk}^{\delta}(\gamma) \end{bmatrix} \tag{15}$$

The frequency response and time response of $H(s)$ and $H^{\delta}(\gamma)$ are compared to evaluate the effectiveness of the approximation.

Table 1 Coefficients of third order approximation of $\left\{CFE\left(\frac{\gamma}{1+0.5\Delta\gamma}\right)^{p}\right\}$ where $p = \rho_i - [\rho_i]$

$\alpha_0 = A_0 + 0.5A_1\Delta + 0.25A_2\Delta^2 + 0.125A_3\Delta^3$
$\alpha_1 = A_1 + A_2\Delta + 0.75A_3\Delta^2$
$\alpha_3 = A_2 + 1.5A_3\Delta$
$\alpha_3 = A_3$
$\beta_0 = A_3 + 0.5A_2\Delta + 0.25A_1\Delta^2 + 0.125A_0\Delta^3$
$\beta_1 = A_2 + A_1\Delta + 0.75A_0\Delta^2$
$\beta_2 = A_1 + 1.5A_0\Delta$
$\beta_3 = A_0$
$A_0 = 6 + p(p^2 + 6p + 11)$
$A_1 = 54 - p(3p^2 + 6p - 27)$
$A_2 = 54 + p(3p^2 - 6p - 27)$
$A_3 = 6 - p(p^2 - 6p + 11)$

3 Simulation and results

The following two-input, two-output fractional-order MIMO system is considered (Khanra et al., 2013):

$$\begin{pmatrix} D^{1.65}x_1 \\ D^{1.28}x_2 \\ D^{0.87}x_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -20 & -10 \end{bmatrix} x + \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \\ 5 & 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} u \quad (16)$$

The TFM corresponding to above state space model is given as below:

$$H(s) = \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \quad (17)$$

where

$$h_{11}(s) = \frac{5s^{1.65} + 20}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}$$

$$h_{12}(s) = \frac{0.8s^{2.15} + 3s^{1.65} + 8s^{1.28} + 27.2}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}$$

$$h_{21}(s) = \frac{10s^{2.93} + 5s^{1.65} + 5}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}$$

$$h_{22}(s) = \frac{0.5s^{3.8} + 11s^{2.93} + 0.2s^{2.15} + 13s^{1.65} + 0.4s^{1.28} + 8.2}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}$$

The TFM shown in equation (17) contains five fractional power terms. The fractional power terms are $s^{1.65}$, $s^{3.8}$, $s^{2.93}$, $s^{2.15}$ and $s^{1.28}$. Using equation (14) and Table 1, the fractional part of each fractional power term is approximated for Δ = 0.1 sec and Δ = 0.01 sec, respectively.

Subsequently, using equation (10) and the approximants of each fractional term, the TFM of equation (17) is approximated to obtain $H^{\delta}(\gamma)$ as shown in equation (18).

$$H^{\delta}(\gamma) = \begin{bmatrix} h_{11}^{\delta}(\gamma) & h_{12}^{\delta}(\gamma) \\ h_{21}^{\delta}(\gamma) & h_{22}^{\delta}(\gamma) \end{bmatrix} \tag{18}$$

where $h_{11}^{\delta}(\gamma), h_{12}^{\delta}(\gamma), h_{21}^{\delta}(\gamma)$ and $h_{22}^{\delta}(\gamma)$ are the corresponding rational approximants of $h_{11}(s)$, $h_{12}(s)$, $h_{21}(s)$ and $h_{22}(s)$ in delta domain. Now, the approximation of $H(s)$, i.e., $H^{\delta}(\gamma)$ has been computed taking two different sampling instants at Δ = 0.1 sec and Δ = 0.01 sec, respectively as shown in equations (19) and (20).

$$H^{\delta_1}(\gamma) = H^{\delta}(\gamma)\Big|_{\Delta=0.1} = \begin{bmatrix} h_{11}^{\delta_1}(\gamma) & h_{12}^{\delta_1}(\gamma) \\ h_{21}^{\delta_1}(\gamma) & h_{22}^{\delta_1}(\gamma) \end{bmatrix} \tag{19}$$

$$H^{\delta_2}(\gamma) = H^{\delta}(\gamma)\Big|_{\Delta=0.01} = \begin{bmatrix} h_{11}^{\delta_2}(\gamma) & h_{12}^{\delta_2}(\gamma) \\ h_{21}^{\delta_2}(\gamma) & h_{22}^{\delta_2}(\gamma) \end{bmatrix} \tag{20}$$

The four SISO transfer functions of $H^{\delta_1}(\gamma)$ are $h_{11}^{\delta_1}(\gamma), h_{12}^{\delta_1}(\gamma), h_{21}^{\delta_1}(\gamma)$ and $h_{22}^{\delta_1}(\gamma)$. Similarly, the corresponding transfer functions for $H^{\delta_2}(\gamma)$ are denoted as $h_{11}^{\delta_2}(\gamma), h_{12}^{\delta_2}(\gamma), h_{21}^{\delta_2}(\gamma)$ and $h_{22}^{\delta_2}(\gamma)$. The elements of $H^{\delta_1}(\gamma)$ and $H^{\delta_2}(\gamma)$ are computed using equation (10) and Table 1 as shown from equations (21) to equations (28).

$$h_{11}^{\delta_1}(\gamma) = \frac{\begin{matrix} 3{,}729\gamma^{12} + 3.05\times10^{5}\gamma^{11} + 9.977\times10^{6}\gamma^{10} + 1.666\times10^{8}\gamma^{9} + 1.52\times10^{9}\gamma^{8} \\ +7.85\times10^{9}\gamma^{7} + 2.531\times10^{10}\gamma^{6} + 5.412\times10^{10}\gamma^{5} + 7.606\times10^{10}\gamma^{4} \\ +6.49\times10^{10}\gamma^{3} + 2.955\times10^{10}\gamma^{2} + 6.552\times10^{9}\gamma + 5.565\times10^{8} \end{matrix}}{\begin{matrix} 8.3\times10^{5}\gamma^{12} + 3.737\times10^{7}\gamma^{11} + 6.215\times10^{8}\gamma^{10} + 4.952\times10^{9}\gamma^{9} \\ +2.122\times10^{10}\gamma^{8} + 5.403\times10^{10}\gamma^{7} + 8.647\times10^{10}\gamma^{6} + 8.812\times10^{10}\gamma^{5} \\ +5.709\times10^{10}\gamma^{4} + 2.475\times10^{10}\gamma^{3} + 7.337\times10^{9}\gamma^{2} + 1.347\times10^{9}\gamma + 1.113\times10^{8} \end{matrix}} \tag{21}$$

$$h_{12}^{\delta_1}(\gamma)=\frac{\begin{aligned}&1.861\times10^{7}\gamma^{18}+1.777\times10^{9}\gamma^{17}+7.274\times10^{10}\gamma^{16}+1.678\times10^{12}\gamma^{15}\\&+2.417\times10^{13}\gamma^{14}+2.293\times10^{14}\gamma^{13}+1.479\times10^{15}\gamma^{12}+6.659\times10^{15}\gamma^{11}\\&+2.139\times10^{16}\gamma^{10}+4.978\times10^{16}\gamma^{9}+8.451\times10^{16}\gamma^{8}+1.041\times10^{17}\gamma^{7}\\&+9.167\times10^{16}\gamma^{6}+5.617\times10^{16}\gamma^{5}+2.32\times10^{16}\gamma^{4}+6.222\times10^{15}\gamma^{3}\\&+1.026\times10^{15}\gamma^{2}+9.381\times10^{13}\gamma+3.611\times10^{12}\end{aligned}}{\begin{aligned}&2.202\times10^{9}\gamma^{18}+1.361\times10^{11}\gamma^{17}+3.509\times10^{12}\gamma^{16}+5.001\times10^{13}\gamma^{15}\\&+4.4\times10^{14}\gamma^{14}+2.536\times10^{15}\gamma^{13}+9.957\times10^{15}\gamma^{12}+2.744\times10^{16}\gamma^{11}\\&+5.423\times10^{16}\gamma^{10}+7.785\times10^{16}\gamma^{9}+8.162\times10^{16}\gamma^{8}+6.276\times10^{6}\gamma^{7}\\&+3.563\times10^{16}\gamma^{6}+1.505\times10^{16}\gamma^{5}+4.714\times10^{15}\gamma^{4}+1.059\times10^{15}\gamma^{3}\\&+1.587\times10^{14}\gamma^{2}+1.392\times10^{13}\gamma+5.31\times10^{11}\end{aligned}} \tag{22}$$

$$h_{21}^{\delta_1}(\gamma)=\frac{\begin{aligned}&3.573\times10^{5}\gamma^{12}+1.922\times10^{7}\gamma^{11}+3.803\times10^{8}\gamma^{10}+3.438\times10^{9}\gamma^{9}\\&+1.486\times10^{10}\gamma^{8}+3.417\times10^{10}\gamma^{7}+4.574\times10^{10}\gamma^{6}+4.09\times10^{10}\gamma^{5}\\&+3.023\times10^{10}\gamma^{4}+1.882\times10^{10}\gamma^{3}+7.672\times10^{9}\gamma^{2}+1.645\times10^{9}\gamma\\&+1.391\times10^{8}\end{aligned}}{\begin{aligned}&8.3\times10^{5}\gamma^{12}+3.737\times10^{7}\gamma^{11}+6.215\times10^{8}\gamma^{10}+4.952\times10^{9}\gamma^{9}\\&+2.122\times10^{10}\gamma^{8}+5.403\times10^{10}\gamma^{7}+8.647\times10^{10}\gamma^{6}+8.812\times10^{10}\gamma^{5}\\&+5.709\times10^{10}\gamma^{4}+2.475\times10^{10}\gamma^{3}+7.337\times10^{9}\gamma^{2}+1.347\times10^{9}\gamma\\&+1.113\times10^{8}\end{aligned}} \tag{23}$$

$$h_{22}^{\delta_1}(\gamma)=\frac{\begin{aligned}&1.672\times10^{9}\gamma^{18}+1.084\times10^{11}\gamma^{17}+2.932\times10^{12}\gamma^{16}+4.361\times10^{13}\gamma^{15}\\&+3.965\times10^{14}\gamma^{14}+2.327\times10^{15}\gamma^{13}+9.135\times10^{15}\gamma^{12}+2.48\times10^{16}\gamma^{11}\\&+4.797\times10^{16}\gamma^{10}+6.817\times10^{16}\gamma^{9}+7.34\times10^{16}\gamma^{8}+6.153\times10^{16}\gamma^{7}\\&+4.059\times10^{16}\gamma^{6}+2.06\times10^{16}\gamma^{5}+7.639\times10^{15}\gamma^{4}+1.94\times10^{15}\gamma^{3}\\&+3.123\times10^{14}\gamma^{2}+2.83\times10^{13}\gamma+1.089\times10^{12}\end{aligned}}{\begin{aligned}&2.202\times10^{9}\gamma^{18}+1.361\times10^{11}\gamma^{17}+3.509\times10^{12}\gamma^{16}+5.001\times10^{13}\gamma^{15}\\&+4.4\times10^{14}\gamma^{14}+2.536\times10^{15}\gamma^{13}+9.957\times10^{15}\gamma^{12}+2.744\times10^{16}\gamma^{11}\\&+5.423\times10^{16}\gamma^{10}+7.785\times10^{16}\gamma^{9}+8.162\times10^{16}\gamma^{8}+6.276\times10^{16}\gamma^{7}\\&+3.563\times10^{16}\gamma^{6}+1.505\times10^{16}\gamma^{5}+4.714\times10^{15}\gamma^{4}\\&+1.059\times10^{15}\gamma^{3}+1.587\times10^{14}\gamma^{2}+1.392\times10^{13}\gamma+5.31\times10^{11}\end{aligned}} \tag{24}$$

$$h_{11}^{\delta_2}(\gamma)=\frac{\begin{aligned}&1.634\gamma^{12}+887.7\gamma^{11}+1.675\times10^{5}\gamma^{10}+1.285\times10^{7}\gamma^{9}\\&+3.702\times10^{8}\gamma^{8}+2.962\times10^{9}\gamma^{7}+1.21\times10^{10}\gamma^{6}+3.115\times10^{10}\gamma^{5}\\&+5.237\times10^{10}\gamma^{4}+5.232\times10^{10}\gamma^{3}+2.639\times10^{10}\gamma^{2}\\&+6.252\times10^{9}\gamma+5.565\times10^{8}\end{aligned}}{\begin{aligned}&3.357\times10^{4}\gamma^{12}+4.796\times10^{6}\gamma^{11}+1.666\times10^{8}\gamma^{10}+2.08\times10^{9}\gamma^{9}\\&+1.143\times10^{10}\gamma^{8}+3.416\times10^{10}\gamma^{7}+6.176\times10^{10}\gamma^{6}+6.929\times10^{10}\gamma^{5}\\&+4.771\times10^{10}\gamma^{4}+2.159\times10^{10}\gamma^{3}+6.685\times10^{9}\gamma^{2}+1.287\times10^{9}\gamma\\&+1.113\times10^{8}\end{aligned}} \tag{25}$$

$$h_{12}^{\delta_2}(\gamma)=\frac{\begin{array}{l}9354\gamma^{18}+3.929\times10^{6}\gamma^{17}+5.779\times10^{8}\gamma^{16}+3.96\times10^{10}\gamma^{15}\\+1.392\times10^{12}\gamma^{14}+2.611\times10^{13}\gamma^{13}+2.777\times10^{14}\gamma^{12}+1.793\times10^{15}\gamma^{11}\\+7.535\times10^{15}\gamma^{10}+2.167\times10^{16}\gamma^{9}+4.384\times10^{16}\gamma^{8}+6.278\times10^{16}\gamma^{7}\\+6.284\times10^{16}\gamma^{6}+4.283\times10^{16}\gamma^{5}+1.924\times10^{16}\gamma^{4}\\+5.509\times10^{15}\gamma^{3}+9.559\times10^{14}\gamma^{2}+9.0888\times10^{13}\gamma+3.611\times10^{12}\end{array}}{\begin{array}{l}3.661\times10^{7}\gamma^{18}+6.093\times10^{9}\gamma^{17}+3.112\times10^{11}\gamma^{16}+7.444\times10^{12}\gamma^{15}\\+9.82\times10^{13}\gamma^{14}+7.746\times10^{14}\gamma^{13}+3.858\times10^{15}\gamma^{12}+1.278\times10^{16}\gamma^{11}\\+2.927\times10^{16}\gamma^{10}+4.744\times10^{16}\gamma^{9}+5.495\times10^{16}\gamma^{8}+4.575\times10^{16}\gamma^{7}\\+2.765\times10^{16}\gamma^{6}+1.23\times10^{16}\gamma^{5}+4.037\times10^{15}\gamma^{4}+9.483\times10^{14}\gamma^{3}\\+1.482\times10^{14}\gamma^{2}+1.349\times10^{13}\gamma+5.31\times10^{11}\end{array}} \tag{26}$$

$$h_{21}^{\delta_2}(\gamma)=\frac{\begin{array}{l}4047\gamma^{12}+1.117\times10^{6}\gamma^{11}+6.828\times10^{7}\gamma^{10}+1.377\times10^{9}\gamma^{9}\\+8.439\times10^{9}\gamma^{8}+2.342\times10^{10}\gamma^{7}+3.443\times10^{10}\gamma^{6}+3.131\times10^{10}\gamma^{5}\\+2.328\times10^{10}\gamma^{4}+1.555\times10^{10}\gamma^{3}+6.876\times10^{9}\gamma^{2}+1.57\times10^{9}\gamma\\+1.391\times10^{8}\end{array}}{\begin{array}{l}3.357\times10^{4}\gamma^{12}+4.796\times10^{6}\gamma^{11}+1.666\times10^{8}\gamma^{10}+2.08\times10^{9}\gamma^{9}\\+1.143\times10^{10}\gamma^{8}+3.416\times10^{10}\gamma^{7}+6.176\times10^{10}\gamma^{6}+6.929\times10^{10}\gamma^{5}\\+4.771\times10^{10}\gamma^{4}+2.159\times10^{10}\gamma^{3}+6.685\times10^{9}\gamma^{2}+1.287\times10^{9}\gamma\\+1.113\times10^{8}\end{array}} \tag{27}$$

$$h_{22}^{\delta_2}(\gamma)=\frac{\begin{array}{l}2.095\times10^{7}\gamma^{18}+3.841\times10^{9}\gamma^{17}+2.181\times10^{11}\gamma^{16}+5.806\times10^{12}\gamma^{15}\\+8.377\times10^{13}\gamma^{14}+7.017\times10^{14}\gamma^{13}+3.581\times10^{15}\gamma^{12}+1.177\times10^{16}\gamma^{11}\\+2.62\times10^{16}\gamma^{10}+4.124\times10^{16}\gamma^{9}+4.786\times10^{16}\gamma^{8}+4.263\times10^{16}\gamma^{7}\\+2.986\times10^{16}\gamma^{6}+1.618\times10^{16}\gamma^{5}+6.404\times10^{15}\gamma^{4}+1.722\times10^{15}\gamma^{3}\\+2.909\times10^{14}\gamma^{2}+2.742\times10^{13}\gamma+1.089\times10^{12}\end{array}}{\begin{array}{l}3.661\times10^{7}\gamma^{18}+6.093\times10^{9}\gamma^{17}+3.112\times10^{11}\gamma^{16}+7.444\times10^{12}\gamma^{15}\\+9.82\times10^{13}\gamma^{14}+7.746\times10^{14}\gamma^{13}+3.858\times10^{15}\gamma^{12}+1.278\times10^{16}\gamma^{11}\\+2.927\times10^{16}\gamma^{10}+4.744\times10^{16}\gamma^{9}+5.495\times10^{16}\gamma^{8}+4.575\times10^{16}\gamma^{7}\\+2.765\times10^{16}\gamma^{6}+1.23\times10^{16}\gamma^{5}+4.037\times10^{15}\gamma^{4}+9.483\times10^{14}\gamma^{3}\\+1.482\times10^{14}\gamma^{2}+1.349\times10^{13}\gamma+5.31\times10^{11}\end{array}} \tag{28}$$

The frequency responses of $h_{\mu\eta}(s)$, $h_{\mu\eta}^{\delta_1}(\gamma)$ and $h_{\mu\eta}^{\delta_2}(\gamma)$ are plotted taking four combinations of μ and η viz. ($\mu = 1$, $\eta = 1$), ($\mu = 1$, $\eta = 2$), ($\mu = 2$, $\eta = 1$) and ($\mu = 2$, $\eta = 2$) as shown in Figures 1, 2, 3 and 4, respectively. In general, all the approximations fit well in the entire frequency range (0.01~100 rad/sec) except some inconsistencies noticed in the high frequency range. The frequency response graph of $h_{\mu\eta}^{\delta_2}(\gamma)$ has shown better match with the corresponding $h_{\mu\eta}(s)$ as compared to $h_{\mu\eta}^{\delta_1}(\gamma)$. The mean absolute errors (MAE) are listed in Tables 2 and 3 after magnitude and phase approximation of $H(s)$ The mathematical expressions of MAE are given separately for computation of magnitude error and phase error as below:

$$MAE_{mag(dB)} = \frac{\sum_{v=1}^{Q} 20\log_{10}\left|H_{ideal}(\omega_v) - H^{\delta}(\omega_v)\right|}{Q} \tag{29}$$

$$MAE_{pha(\text{degree})} = \frac{\sum_{v=1}^{Q} \left|\angle H_{ideal}(\omega_v) - \angle H^{\delta}(\omega_v)\right|}{Q} \tag{30}$$

where Q denotes the total number of frequency-points or ω_v, $H_{ideal}(\omega_v)$ and $\angle H_{ideal}(\omega_v)$ represent the magnitude response and phase response of the original fractional-order MIMO system at $\omega = \omega_v(1 \le v \le Q)$; $H^{\delta}(\omega_v)$ and $\angle H^{\delta}(\omega_v)$ represent the magnitude response and phase response of the approximated MIMO system at $\omega = \omega_v(1 \le v \le Q)$. In this work, the MAE has been computed taking $Q = 200$ in the band 10^{-2} rad/sec to 10^2 rad/sec. Two other sampling instants $\Delta = 0.5$ sec and $\Delta = 0.04$ sec have also been taken to observe the effect of reducing Δ on MAE. The data of Tables 2 and 3 indicate that the errors in magnitude approximation are less as compared to phase approximation. Therefore, the overall approximation errors are reduced when Δ is decreased gradually.

Table 2 MAE resulted after magnitude approximation of $H(s)$ in delta-domain for different sampling times

Sampling time (sec)	*Approximation error for $h_{11}(s)$ (dB)*	*Approximation error for $h_{12}(s)$ (dB)*	*Approximation error for $h_{21}(s)$ (dB)*	*Approximation error for $h_{22}(s)$ (dB)*
$\Delta = 0.5$	8.715	8.587	2.522	1.2488
$\Delta = 0.1$	2.44	2.48	0.95	0.588
$\Delta = 0.04$	2.211	1.658	0.675	0.111
$\Delta = 0.01$	0.38	0.437	0.172	0.096

Figure 1 Comparison of the frequency responses among $h_{11}(s)$, $h_{11}^{\delta_1}(\gamma)$ and $h_{11}^{\delta_2}(\gamma)$

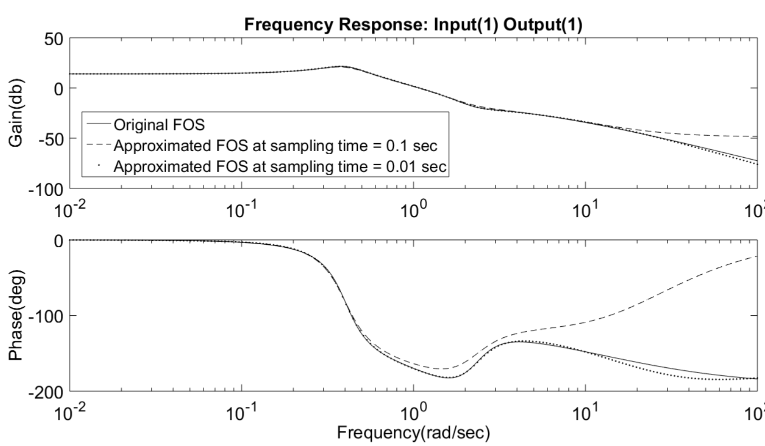


Table 3 MAE resulted after phase approximation of $H(s)$ in delta-domain for different sampling times

Sampling time (sec)	*Approximation error for $h_{11}(s)$ (degree)*	*Approximation error for $h_{12}(s)$ (degree)*	*Approximation error for $h_{21}(s)$ (degree)*	*Approximation error for $h_{22}(s)$ (degree)*
$\Delta = 0.5$	56.77	53.556	19.327	194.5
$\Delta = 0.1$	30.7	28.23	11.14	4.608
$\Delta = 0.04$	1.92	6.076	5.365	1.384
$\Delta = 0.01$	1.74	2.47	2.173	0.6732

Figure 2 Comparison of the frequency responses among $h_{12}(s)$, $h_{12}^{\delta_1}(\gamma)$ and $h_{12}^{\delta_2}(\gamma)$

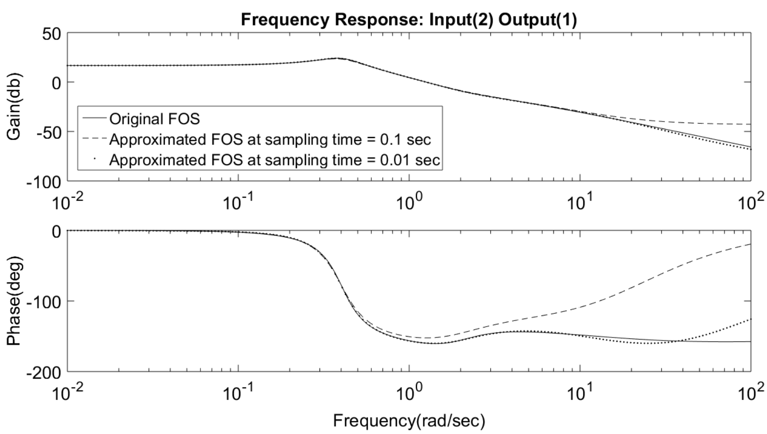


Figure 3 Comparison of the frequency responses among $h_{21}(s)$, $h_{21}^{\delta_1}(\gamma)$ and $h_{21}^{\delta_2}(\gamma)$

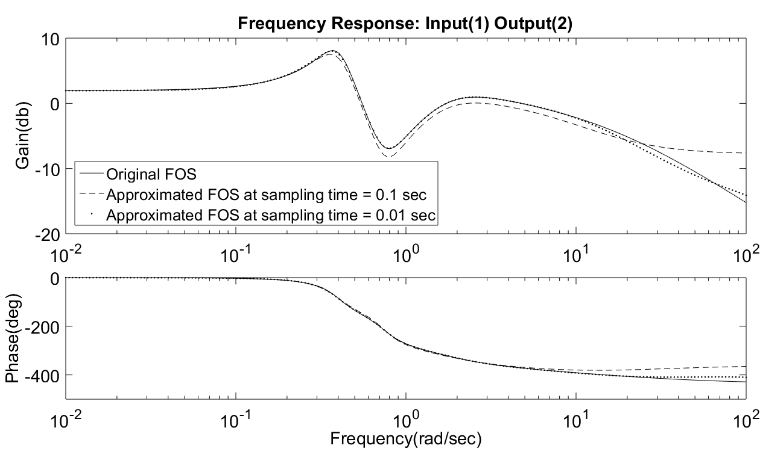
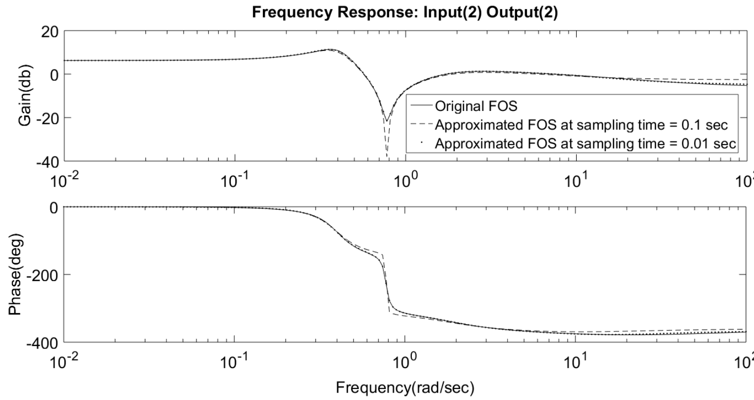
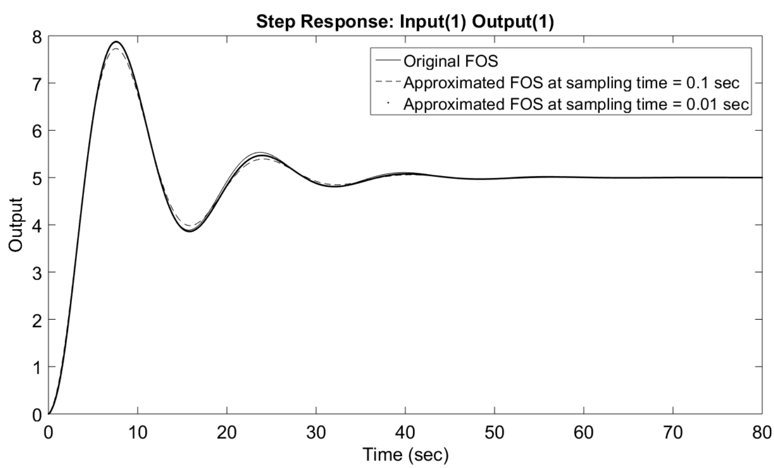


Figure 4 Comparison of the frequency responses among $h_{22}(s)$, $h_{22}^{\delta_1}(\gamma)$ and $h_{22}^{\delta_2}(\gamma)$



The step response graphs of $h_{\mu\eta}(s)$, $h_{\mu\eta}^{\delta_1}(\gamma)$ and $h_{\mu\eta}^{\delta_2}(\gamma)$ are shown in Figures 5, 6, 7 and 8, respectively. Both the approximants, i.e., $h_{\mu\eta}^{\delta_1}(\gamma)$ and $h_{\mu\eta}^{\delta_2}(\gamma)$ match convincingly with the corresponding underlying FOS $h_{\mu\eta}(s)$. However, the step response graphs of $h_{\mu\eta}^{\delta_2}(\gamma)$ seem to be more accurate than the corresponding step response graphs of $h_{\mu\eta}^{\delta_1}(\gamma)$ in terms of closeness to the ideal step response characteristics.

Figure 5 Comparison of the step responses among $h_{11}(s)$, $h_{11}^{\delta_1}(\gamma)$ and $h_{11}^{\delta_2}(\gamma)$



Therefore, the simulation results shown in Figures 1~8 deliver the same impression in the frequency domain and also in the time domain, i.e., the discrete-time FOS inclines to the original FOS as the sampling time tends to zero. These simulation outcomes compliment the fundamental insight of the delta operator theory towards unifying both the continuous-time and discrete-time fractional-order MIMO system at low sampling time limit.

Figure 6 Comparison of the step responses among $h_{12}(s)$, $h_{12}^{\delta_1}(\gamma)$ and $h_{12}^{\delta_2}(\gamma)$

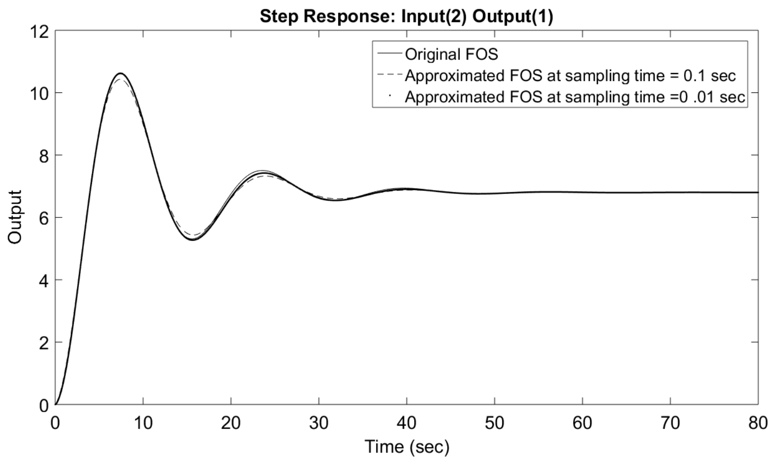
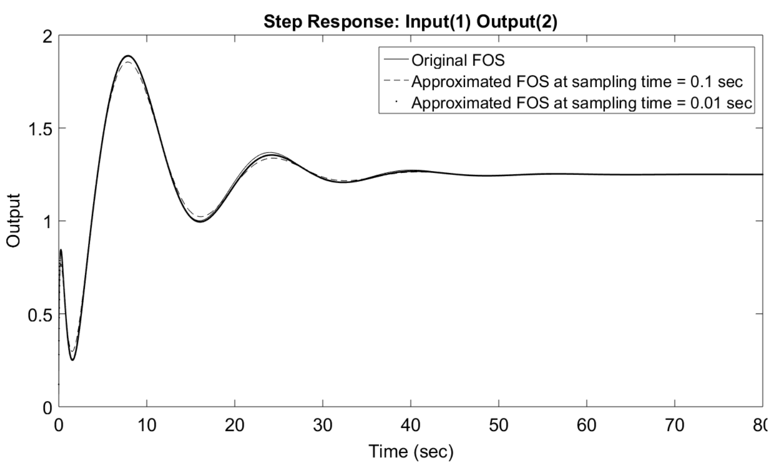


Figure 7 Comparison of the step responses among $h_{21}(s)$, $h_{21}^{\delta_1}(\gamma)$ and $h_{21}^{\delta_2}(\gamma)$



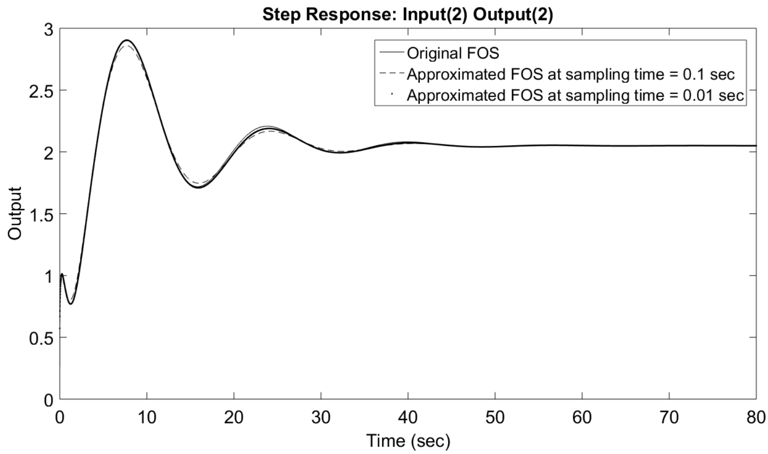
The outcomes of the proposed method have been compared with one of the established methods of z-domain. The Tustin approximation is given by the following equation:

$$s = \frac{2}{\Delta}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{31}$$

The fractional-power term s^p $(0 < p < 1)$ is approximated to a third order transfer function of z-domain using Tustin method in combination CFE as given below (Chen et al., 2004; Calderon and Sarango, 2022):

$$s^p = \left(\frac{2}{\Delta}\right)^p CFE\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^p \approx \frac{-\left(p^3-4p\right)+\left(6p^2-9\right)z-15pz^2+15pz^3}{\left(p^3-4p\right)+\left(6p^2-9\right)z+15pz^2+15pz^3} \tag{32}$$

Figure 8 Comparison of the step responses among $h_{22}(s)$, $h_{22}^{\delta_1}(\gamma)$ and $h_{22}^{\delta_2}(\gamma)$



All the elements of the TFM represented by equation (17) are approximated in z-domain using equations (31) and (32). Taking $\Delta = 0.01$ sec, the approximated MIMO system of z-domain is represented as below:

$$H_T(z) = \begin{bmatrix} \hbar_{11}(z) & \hbar_{12}(z) \\ \hbar_{21}(z) & \hbar_{22}(z) \end{bmatrix} \tag{33}$$

where

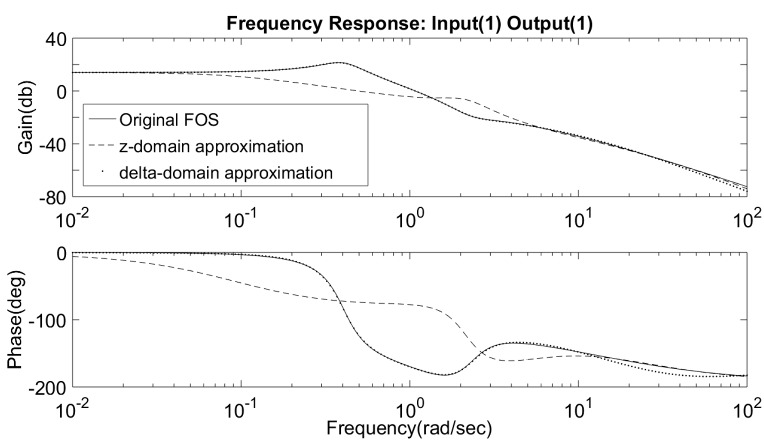
$$\hbar_{11}(z) = \frac{\begin{array}{l} 105.7z^{12} + 220.2z^{11} - 139.7z^{10} - 535.3z^{9} - 73.98z^{8} \\ +445.3z^{7} + 171.6z^{6} - 149z^{5} - 74.2z^{4} + 20.24z^{3} \\ +11.67z^{2} - 0.8286z - 0.571 \end{array}}{\begin{array}{l} 2.058\times10^{6}z^{12} - 4.243\times10^{6}z^{11} - 2.555\times10^{6}z^{10} \\ +9.991\times10^{6}z^{9} - 1.663\times10^{6}z^{8} - 7.899\times10^{6}z^{7} \\ +3.315\times10^{6}z^{6} + 2.427\times10^{6}z^{5} - 1.334\times10^{6}z^{4} \\ -2.933\times10^{5}z^{3} + 1.951\times10^{5}z^{2} + 1.082\times10^{4}z - 9{,}102 \end{array}}$$

$$\hbar_{12}(z) = \frac{\begin{array}{l} 7.344\times10^{4}z^{18} + 1.346\times10^{5}z^{17} - 2.105\times10^{5}z^{16} - 4.882\times10^{5}z^{15} \\ +1.94\times10^{5}z^{14} + 7.203\times10^{5}z^{13} - 1.77\times10^{4}z^{12} - 5.579\times10^{5}z^{11} \\ -8.29\times10^{4}z^{10} + 2.438\times10^{5}z^{9} + 5.934\times10^{4}z^{8} - 5.963\times10^{4}z^{7} \\ -1.82\times10^{4}z^{6} + 7368z^{5} + 2710z^{4} - 306.6z^{3} - 162.8z^{2} - 8.359z + 0.537 \end{array}}{\begin{array}{l} 4.631\times10^{8}z^{18} - 7.556\times10^{8}z^{17} - 1.503\times10^{9}z^{16} + 2.899\times10^{9}z^{15} \\ +1.728\times10^{9}z^{14} - 4.511\times10^{9}z^{13} - 6.599\times10^{8}z^{12} + 3.654\times10^{9}z^{11} \\ -2.234\times10^{8}z^{10} - 1.653\times10^{9}z^{9} + 2.682\times10^{8}z^{8} + 4.196\times10^{8}z^{7} \\ -8.212\times10^{7}z^{6} - 5.846\times10^{7}z^{5} + 9.964\times10^{6}z^{4} + 4.209\times10^{6}z^{3} \\ -3.548\times10^{5}z^{2} - 1.278\times10^{5}z - 5{,}963 \end{array}}$$

$$\hbar_{21}(z)=\frac{\begin{array}{l}1.864\times10^5z^{12}-8.922\times10^4z^{11}-6.295\times10^5z^{10}\\+2.786\times10^5z^9+8.032\times10^5z^8-3.193\times10^5z^7\\-4.796\times10^5z^6+1.613\times10^5z^5+1.373\times10^5z^4\\-3.383\times10^4z^3-1.874\times10^4z^2+2407z+1,006\end{array}}{\begin{array}{l}2.058\times10^6z^{12}-4.243\times10^6z^{11}-2.555\times10^6z^{10}\\+9.991\times10^6z^9-1.663\times10^6z^8-7.899\times10^6z^7\\+3.315\times10^6z^6+2.427\times10^6z^5-1.334\times10^6z^4\\-2.933\times10^5z^3+1.951\times10^5z^2+1.082\times10^4z-9,102\end{array}}$$

$$\hbar_{22}(z)=\frac{\begin{array}{l}2.567\times10^8z^{18}-3.79\times10^8z^{17}-8.698\times10^8z^{16}+1.455\times10^9z^{15}\\+1.096\times10^9z^{14}-2.265\times10^9z^{13}-5.755\times10^8z^{12}+1.836\times10^9z^{11}\\+4.054\times10^7z^{10}-8.3\times10^8z^9+7.813\times10^7z^8+2.098\times10^8z^7\\-2.923\times10^7z^6-2.876\times10^7z^5+3.671\times10^6z^4+1.977\times10^6z^3\\-1.208\times10^5z^2-5.403\times10^4z-2,586\end{array}}{\begin{array}{l}4.631\times10^8z^{18}-7.556\times10^8z^{17}-1.503\times10^9z^{16}+2.899\times10^9z^{15}\\+1.728\times10^9z^{14}-4.511\times10^9z^{13}-6.599\times10^8z^{12}+3.654\times10^9z^{11}\\-2.234\times10^8z^{10}-1.653\times10^9z^9+2.682\times10^8z^8+4.196\times10^8z^7\\-8.212\times10^7z^6-5.846\times10^7z^5+9.964\times10^6z^4+4.209\times10^6z^3\\-3.548\times10^5z^2-1.278\times10^5z-5,963\end{array}}$$

Figure 9 Comparison of the frequency responses among $h_{11}(s)$ and its approximations in two discrete domains at $\Delta = 0.01$ sec



The comparison of the frequency responses between two discrete-domains are shown in Figures 9~12 for $\Delta = 0.01$ sec. It is clearly reflected that the delta-domain approximation is closer to the ideal characteristics while the *z*-domain approximation deviates more from the ideal graph. A comparative study on approximation errors is shown in Table 4. It is observed that the magnitude approximation in *z*-domain offers moderate result. However, the phase errors are showing larger values in *z*-domain. On the other hand, the delta-domain approximation exhibits lesser magnitude and phase errors and hence, it is offering better approximation of fractional-order MIMO system.

Table 4 Comparison of MAE resulted after magnitude and phase approximation of $H(s)$ in z-domain and delta-domain for $\Delta = 0.01$ sec

MAE	*Approximation error for $h_{11}(s)$*		*Approximation error for $h_{12}(s)$*		*Approximation error for $h_{21}(s)$*		*Approximation error for $h_{22}(s)$*	
	z domain	*Delta domain*	*z domain*	*Delta domain*	*z domain*	*Delta domain*	*z domain*	*Delta domain*
Magnitude (dB)	4.417	0.38	4.055	0.437	3.51	0.172	2.49	0.096
Phase (degree)	28.01	1.74	25.41	2.47	207.2	2.173	202.2	0.6732

Figure 10 Comparison of the frequency responses among $h_{12}(s)$ and its approximations in two discrete domains at $\Delta = 0.01$ sec

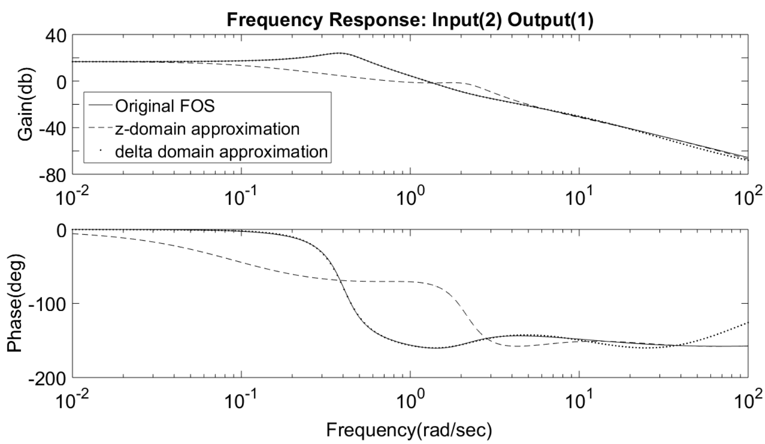


Figure 11 Comparison of the frequency responses among $h_{21}(s)$ and its approximations in two discrete domains at $\Delta = 0.01$ sec

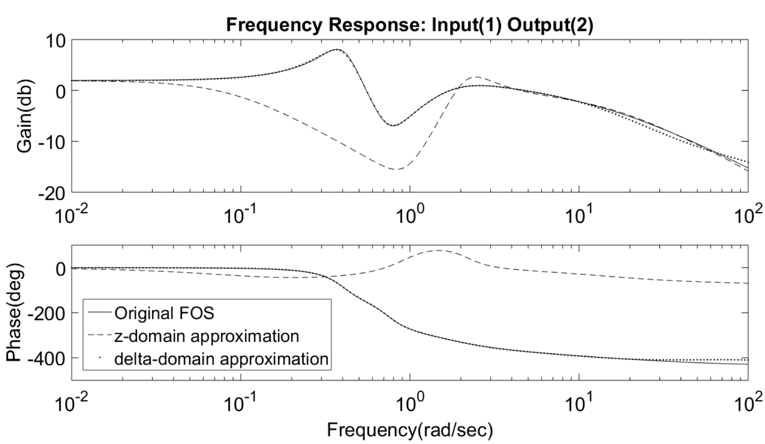
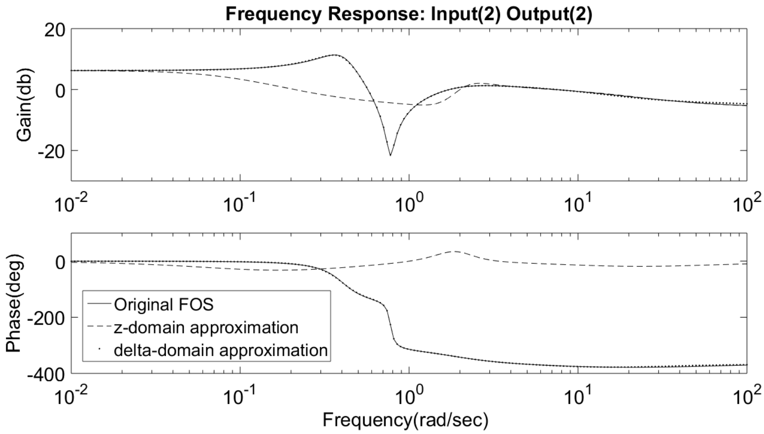


Figure 12 Comparison of the frequency responses among $h_{22}(s)$ and its approximations in two discrete domains at $\Delta = 0.01$ sec



4 Conclusions

In this paper, a methodology has been presented to approximate a class of fractional-order MIMO system in discrete-time domain. An unconventional approach has been deliberated instead of the traditional approaches through casting the delta operator and involving the same as a generating function. The frequency responses and the step responses validate that the approximated fractional-order MIMO system exhibits a great similarity to the original FOS as the sampling time approaches to a lower limit. This evidently validates the central theme of the delta operator theory towards unifying the discrete-time fractional-order MIMO system and its continuous-time counterpart together at the low sampling time. The proposed approach has been compared with one of the well-known *z*-domain methods to establish its superiority. However, the approximated model of the MIMO system results higher order which leaves a future scope of research towards simplifying the realisation of the overall MIMO system in the delta-domain through further model order reduction.

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