International Journal of Operational Research

ISSN online: 1745-7653 - ISSN print: 1745-7645
https://www.inderscience.com/ijor

Analysis of MAP/PH(1), PH(2), PH(3)/1 queueing system with two modes of heterogeneous service, standby server, vacation, impatient behaviour of customers, additional service, start-up time, breakdown and phase type repairs

G. Ayyappan, K. Thilagavathy

DOI: 10.1504/IJOR.2021.10042582

Article History:

Received:	09 February 2021
Accepted:	29 April 2021
Published online:	12 January 2024

Analysis of MAP/PH(1), PH(2), PH(3)/1 queueing system with two modes of heterogeneous service, standby server, vacation, impatient behaviour of customers, additional service, start-up time, breakdown and phase type repairs

G. Ayyappan and K. Thilagavathy*

Department of Mathematics,
Pondicherry Engineering College,
Pillaichavady, Puducherry, India
Email: ayyappanpec@hotmail.com
Email: thilagakarthik95@gmail.com
*Corresponding author

Abstract: In this article, we consider a single server queue in which customers arrive according to the Markovian arrival process (MAP) and their corresponding two modes of service based on phase-type (PH) distribution. The main server may affect by breakdown while offering service whether it is any one of the modes of service or additional service immediately go for the repair process. At that moment, the service process switchover to the standby server until the main server rejuvenated from the phase-type repair. When vacation completion epoch, the main server will do the start-up process. Using the matrix-analytic method, we investigated the total number of customers in the system under steady-state probability vector. We examined the stability condition, busy period and characteristics of some performance measures of the system are discussed. Numerical results are tabulated and graphical representations are provided for a clear view of our model.

Keywords: PH distribution; Markovian arrival process; MAP; standby server; impatient behaviour; additional service.

Reference to this paper should be made as follows: Ayyappan, G. and Thilagavathy, K. (2024) 'Analysis of MAP/PH(1), PH(2), PH(3)/1 queueing system with two modes of heterogeneous service, standby server, vacation, impatient behaviour of customers, additional service, start-up time, breakdown and phase type repairs', *Int. J. Operational Research*, Vol. 49, No. 1, pp.85–130.

Biographical notes: G. Ayyappan received his MSc in Mathematics from the Bharathidasan University in 1990, and his MPhil and PhD in Mathematics from the Annamalai University in India in 1995. He is currently working as a Professor in the Department of Mathematics, Pondicherry Engineering College, Puducherry, India. His research interests include queueing theory and reliability. He is a member of the Indian Society of Technical Education in Probability and Statistics. He has 25 years of teaching experience. He has published more than 150 research articles.

K. Thilagavathy received her Master of Science in Mathematics from the Kanchi Mamunivar Center for Post Graduate Studies, Puducherry. Currently, she is working on her PhD research program under the guidance of Dr. G. Ayyappan in the Department of Mathematics, Pondicherry Engineering College. Her area of research interests include queueing models with correlated arrivals and priority queues.

1 Introduction

The versatile Markovian point process has been first introduced and examined by Neuts (1979). He has discussed the PH-distributions, point process and some applications including the Poisson process, PH renewal process, Markov-modulated Poisson process and the Markov-arrival process. Chakravarthy (2010) has described the point process and marked point process which are the first building blocks in the development of MAP and the PH distributions are the generalisations of exponential distributions. He has also described the Poisson process and PH-renewal process are the second building blocks in the development of MAP concepts and then he incorporated the distinct types of special cases of MAP. Ayyappan and Karpagam (2019) have studied the non-Markovian batch arrival and bulk service queueing model with standby server, single vacation and some performance measures including the expected waiting time. Chakravarthy and Agarwal (2003) incorporated the machine repair problem with the concepts of unreliable server, service and repair process based on phase-type distribution. They have analysed their model using matrix analytic methods.

Jain et al. (2004) have incorporated the machine repair system deliberates the concepts of N or more units has failed then the removable repairman turns on for start the repair work afterward if there is no failed unit for repair, the repairman will turn off. The arriving failed unit may renege due to impatient behaviour during the repairman is busy. Ke (2003) has analysed the $M/G/1$ queue with vacation under N-policy, unpredictable breakdowns and do start-up process for serving the waiting customers until the system becomes empty. Whenever the server affected by breakdown immediately goes for the repair process and also analysed the cost optimisation. Kumar and Sharma (2019) have analysed the multi-server Markovian queueing model with reneging and retention of reneging customers. Kumar and Arumuganathan (2008) have incorporated a single server retrial queueing system with two phases of heterogeneous service, batch arrival of customers follows the Poisson process and discussed the Bernoulli schedule vacation. Sudhesh and Azhagappan (2019) described the heterogeneous multi-server queue with balking and reneging of customers. Kannadasan and Sathiyamoorthi (2021) described a single server queue with working vacation serving at a slower rate during the start-up period. Dudin and Nishimura (1999) have studied the batch Markovian arrival process for arriving customers with controllable two service modes follow the general distribution.

Wang et al. (2007) have described a single removable and unreliable server with unpredictable breakdowns and also discussed when the server affect by the breakdown which is emergently recovered and it has need start-up process of preparatory time to start the service again. They have developed a cost analysis for the optimisation of their model. Ke (2001) has incorporated a single server non-Markovian queueing model with two types of vacations in which server can go type one vacation which the system is empty likewise when vacation completion epoch if the number of customers in the system less than Q again the server will take type one vacation or else number of customers in the system is Q $\leq$ N, the server goes for type two vacation. However, if the number of customers in the system reaches N or more and the vacation completion epoch, the server turned on and do some preparatory work on the basis of the start-up process. Furthermore, he has analysed the cost model to attaining the threshold optimisation. Kalita and Choudhury (2021) have analysed the non-Markovian single server queueing system in which the server takes the maximum number of random vacation up to finds a minimum of one message waiting in a queue when the vacation completion epoch and the server would be dormant in the system after completing the maximum number of random vacations. Singh et al. (2020) have incorporated the bulk arrival retrial queue with the unreliable server, optional additional service and arrival of negative customers.

Bagyam and Udayachandrika (2011) have studied non-Markovian retrial queueing model with two types of service. Ayyappan and Udayageetha (2020) have analysed the retrial queue with priority services, start-up and closedown time, vacation, impatient customers and working breakdown which is rendering service at a slower rate even after affected by the breakdown while offering normal service to the customers. Nobel and Tijms (1999) have analysed a single server queue station with batch arrival of customers follows the Poisson process and dealt with the arriving customer has an option to choose any one of the modes of service in which it deliberates high speed and regular speed service. They also added the switch over times for changing one mode to another mode of service. Choudhury and Kalita (2017) have investigated the non-Markovian queue with a breakdown, delayed repair and two types of service from the server, the customer who has an option to get the same service once again if the customer is not satisfied with the essential service. Baruah et al. (2012) have analysed the batch arrival queueing system with two types of heterogeneous service deliberates the arriving customers could choose any one types of service, re-service and impatient behaviour of customers. Although several aspects of queueing models with the concept of two types of service in the literature, no works have been done with the concepts of two modes of heterogeneous service, additional service, standby server subject to unpredictable breakdowns and phase-type repair process. Therefore, in this paper we investigate the $MAP/PH(1), PH(2), PH(3)/1$ queueing system using matrix-analytic methods with start-up time, vacation, two modes of heterogeneous service, standby server, breakdown, both the service and repair follows the phase-type distribution and arrival considered under MAP is the suitable way of approaching the correlated and non-correlated arrival concepts.

The overall goal and intentions of our paper are given as follows:

- We formulate some scenarios faced by customers in the banking sector as well as in internet banking into mathematical form.
- We derive the stability condition of the system for the condition check of our model to remain balanced. To analyse the busy period of the system and the waiting time analysis for the customers.

- To describe the Total cost of the system for optimisation of our proposed model. To examine the consequence of the various parameters on the characteristics of the system performance measures whether they increase or decrease according to the situations.

The motivation for this article comes from the banking sector. In banking systems, the arriving customers usually demand either deposit or withdrawal of their amounts (two modes of heterogeneous service) then they received either any one of the modes of heterogenous services then after some people may demand print the details of the transactions in their passbook (additional service) or else they will leave out of the system. While the customer receiving service from any one of the counters, the computer or the printer may be struck by the breakdown then that service receiving customer would take care of by any other staff member (standby server). During the vacation period the waiting customer who may leave out of the system (reneging) due to impatient behaviour. Similarly, consider the internet banking sector, the customers who use this source either will pay the bills, fees to their corresponding educational sector, etc. otherwise they will deposit their amounts in the investment avenues like equity, bonds, mutual funds, exchange-traded fund (ETF) and a cryptocurrency (Bitcoin, Litecoin, Ethereum, Altcoin and Ripple), etc. While using internet banking with the help of any one of the broadband Wi-fi connection which may not come properly in some situations, in those moments people will connect any other Wi-fi or smart-phone internet connection (standby server) during the repair period of that broadband Wi-fi connection. When the transaction completion epoch they would make transaction pdf print or download it (additional service) on their laptop, personal computer or smart-phones if their transaction has done by using NEFT platform in internet banking.

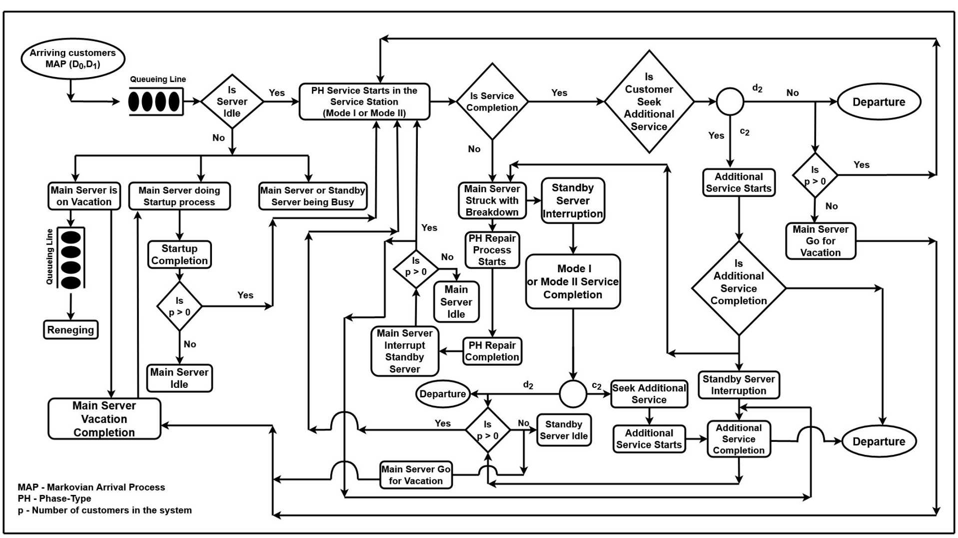
1.1 Structure of our manuscript

The remaining structure of the overall manuscript is structured as follows. We elaborate on the detailed description of our mathematical model in Section 2. In Section 3, we generated a matrix formulation of our model. In Section 4, we discussed the stability condition, the steady-state probability vector of our model and the computation of the R matrix. In Section 5, we have analysed the busy period of the system. The system performance measures have been discussed in Section 6 and we described the cost analysis for our model in Section 7. In section 8 we analysed the waiting time distribution and in Section 9, presented some tabulated numerical and illustrated graphical representations through exemplifications. The conclusion of our model has given in Section 10.

2 The mathematical model description

We consider a classical queueing model in which customers arrive based on MAP with representation (D_0, D_1) of the square matrix of order is m such that D_0 represents there are no arriving customers in the system, D_1 represents the arrival of customers in the system. The main server (MS) offering the two modes of heterogeneous service in which arriving customers have an option they can choose either mode I service with probability c_1 or mode II service with probability d_1 and the additional service also offer when service completion epoch of any one of the modes of service of which all these services follow the phase-type distribution. The mode I and mode II phase-type service of the main service with representation (α, T) and (γ, U) of order n_1 and n_2 with $T^0 + Te = 0$ and $U^0 + Ue = 0$ such that $T^0 = -Te$ and $U^0 = -Ue$. While the main server offering mode I or mode II service to the customers, if the main server struck with breakdown immediately standby server will interrupt and carry over the service process and it follows the PH distribution with representation $(\alpha_1, \theta_1 T)$ and $(\gamma_1, \theta_2 U)$ of order n_1 and n_2 where $0<\theta_1, \theta_2<1$. The vectors α, γ and α_1, γ_1 are denoting for the purpose of differentiating the main server and the standby server and we are taking α, γ and α_1, γ_1 are the same such that $\alpha = \alpha_1$ and $\gamma = \gamma_1$. After receiving service from the main server the customer who may wish to get an additional service immediately they will get additional service with probability c_2 otherwise, leave the system with probability d_2 such that $c_2 + d_2 = 1$ and the additional service follows PH distribution with representation (β, R) and the same scenario will happen when the standby server in the system. While offering additional service to the customers if the main server struck with breakdown then immediately standby server (SS) would interrupt and carry over the additional service and it follows a phase-type distribution with representation $(\beta_1, \theta_3 R)$ of order is n_3 with $R^0 + Re = 0$ such that $R^0 = -Re$ and $0<\theta_3<1$. The vector β and β_1 are denoting for the purpose of differentiating the main server and the standby server and we are taking β and β_1 are the same such that $\beta = \beta_1$. Whenever the main server struck with a breakdown of whether the main server is offering either any one of the two modes of service or additional service to the customers immediately standby server would interrupt the main server and carry over the respective service, then the main server goes for the repair process under phase-type distribution with representation (δ, S) of order l with $S^0 + Se = 0$ such that $S^0 = -Se$ and the breakdown time follows an exponential distribution with parameter τ.

Figure 1 Schematic representation of our proposed model



When the main server rejuvenation is over from the phase-type repair process then come back to the service station and then who will interrupt the standby server and carry over the service process whether it is any one of the modes of service or additional service. Meanwhile, when the main server rejuvenated from the repair at that moment if the standby server is in an idle state then the main server also will be in the idle state until the customer arrives at the system. After completing service to the customers the main server can go for a vacation if there is no customer in the system and the vacation times follow an exponential distribution with parameter η. However, when the vacation completion whether the customer in the system or there is no one in the system the main server will make the start-up process for the purpose of giving service to the customers in which whether the customer in the system the main server will start the service or else the main server being idle up to the customer's arrival and the start-up times follows an exponential distribution with parameter σ. During the vacation period of the main server, the customers who are waiting in the queueing line who may lose patience and renege from the system with parameter ζ and it is exponentially distributed (see Figure 1).

3 The matrix generation – QBD process

In this section, we describe the notation of our model as follows for the purpose of generating the QBD process.

3.1 Notations for matrix generation

- $\otimes$ – Kronecker product of two different dimension matrices by using this symbol.
- $\oplus$ – Kronecker sum of two different dimension matrices by using this symbol.
- I_m – it denotes an m-dimensional Identity matrix.
- e – column vector of suitable dimension each of its entry is 1.
- $e_1 = e_{3m+lm}$.
- $e_2 = e_{2m+n_1m+n_2m+n_3m+ln_1m+ln_2m+ln_3m}$.
- $e_3 = e_{2+n_1+n_2+n_3+ln_1+ln_2+ln_3}$.
- $e_1(1)$ – column vector of dimension $\{(3 + l)m \times 1\}$ with first $\{m\}$ entries as 1 and the rest of the entries are zero.
- $e_1(2)$ – column vector of dimension $\{(3 + l)m \times 1\}$ with $\{m + 1\}$ to $\{2m\}$ entries as 1 and the rest of the entries are zero.
- $e_1(3)$ – column vector of dimension $\{(3 + l)m \times 1\}$ with $\{2m + 1\}$ to $\{3m\}$ entries as 1 and the rest of the entries are zero.
- $e_1(4)$ – column vector of dimension $\{(3 + l)m \times 1\}$ with $\{3m + 1\}$ to $\{(3m + lm)\}$ entries as 1 and the rest of the entries are zero.
- $e_2(1)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with first $\{m\}$ entries as 1 and the rest of the entries are zero.

- $e_2(2)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with $\{m + 1\}$ to $\{2m\}$ entries as 1 and the rest of the entries are zero.
- $e_2(3)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with $\{2m + 1\}$ to $\{2m + n_1m\}$ entries as 1 and the rest of the entries are zero.
- $e_2(4)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with $\{2m + n_1m + 1\}$ to $\{2m + n_1m + n_2m\}$ entries as 1 and the rest of the entries are zero.
- $e_2(5)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with $\{2m + n_1m + n_2m + 1\}$ to $\{2m + n_1m + n_2m + n_3m\}$ entries as 1 and the rest of the entries are zero.
- $e_2(6)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with $\{2m + n_1m + n_2m + n_3m + 1\}$ to $\{2m + n_1m + n_2m + n_3m + ln_1m\}$ entries as 1 and the rest of the entries are zero.
- $e_2(7)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with $\{2m + n_1m + n_2m + n_3m + ln_1m + 1\}$ to $\{2m + n_1m + n_2m + n_3m + ln_1m + ln_2m\}$ entries as 1 and the rest of the entries are zero.
- $e_2(8)$ – column vector of dimension $\{(2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)m \times 1\}$ with $\{2m + n_1m + n_2m + n_3m + ln_1m + ln_2m + 1\}$ to $\{2m + n_1m + n_2m + n_3m + ln_1m + ln_2m + ln_3m\}$ entries as 1 and the rest of the entries are zero.
- Let us denote λ be the fundamental arrival rate and it is defined as $\lambda = \pi_1 D_1 e_m$, where π_1 is the probability vector of the generator matrix $D = D_0 + D_1$, governs transitions of the MAP. Let the π_1 such that $\pi_1 D = 0$, $\pi_1 e = 1$.
- The mode I service rate of the main server is denoted as δ_1 and the mode I service rate of the standby server is denoted as $\theta_1\delta_1$, where $\delta_1 = [\alpha(-T)^{-1}e_{n_1}]^{-1}$.
- The mode II service rate of the main server is denoted as δ_2 and the mode II service rate of the standby server is denoted as $\theta_2\delta_2$, where $\delta_2 = [\gamma(-U)^{-1}e_{n_2}]^{-1}$.
- Additional service rate of the main server is denoted as δ_3 and the additional service rate of the standby server is denoted as $\theta_3\delta_3$, where $\delta_3 = [\beta(-R)^{-1}e_{n_3}]^{-1}$.
- The repair rate of the main server is denoted as Ψ, where $\Psi = [\delta(-S)^{-1}e_l]^{-1}$.
- $N(t)$ indicates the number of customers in the system at time t.
- V(t) indicates the status of the server at time t, where

$$V(t) = \begin{cases} 0, & \text{if the main server is on vacation} \\ 1, & \text{if the main server doing the start-up process} \\ 2, & \text{if the main server is the idle state} \\ 3, & \text{if the standby server is the idle state} \\ & \quad \text{while the main server is under PH repair process} \\ 4, & \text{if the main server offering mode I service} \\ 5, & \text{if the main server offering mode II service} \\ 6, & \text{if the main server offering additional service} \\ 7, & \text{if the standby server offering mode I service} \\ & \quad \text{while the main server is under PH repair process} \\ 8, & \text{if the standby server offering mode II service} \\ & \quad \text{while the main server is under PH repair process} \\ 9, & \text{if the standby server offering additional service} \\ & \quad \text{while the main server is under PH repair process} \end{cases}$$

- $I(t)$ indicates the repair process considered by phases.
- $J_1(t)$ indicates the mode I service considered by phases.
- $J_2(t)$ indicates the mode II service considered by phases.
- $J_3(t)$ indicates the additional service considered by phases.
- $M(t)$ indicates the arrival process considered by phases.

Let $\{(N(t), V(t), I(t), J_1(t), J_2(t), J_3(t), M(t)) : t \geq 0\}$ is the continuous time Markov chain with state-level independent quasi-birth-and-death process whose state space is as follows:

$$\Omega = l(0) \cup l(p).$$

where

$$\begin{aligned} l(0) = &\{(0, q_1, s) : q_1 = 0, 1, 2; 1 \leq s \leq m\} \\ &\cup \{(0, 3, q_2, s) : 1 \leq q_2 \leq l; 1 \leq s \leq m\}. \end{aligned}$$

for $p \geq 1$,

$$\begin{aligned} l(p) = &\{(p, q_1, s) : q_1 = 0, 1; 1 \leq s \leq m\} \\ &\cup \{(p, 4, r_1, s) : 1 \leq r_1 \leq n_1; 1 \leq s \leq m\} \\ &\cup \{(p, 5, r_2, s) : 1 \leq r_2 \leq n_2; 1 \leq s \leq m\} \\ &\cup \{(p, 6, r_3, s) : 1 \leq r_3 \leq n_3; 1 \leq s \leq m\} \\ &\cup \{(p, 7, q_2, r_1, s) : 1 \leq q_2 \leq l; 1 \leq r_1 \leq n_1; 1 \leq s \leq m\} \\ &\cup \{(p, 8, q_2, r_2, s) : 1 \leq q_2 \leq l; 1 \leq r_2 \leq n_2; 1 \leq s \leq m\} \\ &\cup \{(p, 9, q_2, r_3, s) : 1 \leq q_2 \leq l; 1 \leq r_3 \leq n_3; 1 \leq s \leq m\}. \end{aligned}$$

The infinitesimal matrix generation of the QBD process is given by,

$$Q = \begin{bmatrix} B_{00} & B_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\ B_{10} & A_1 & A_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\ \mathbf{0} & A_2 & A_1 & A_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & A_2 & A_1 & A_0 & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_2 & A_1 & A_0 & \mathbf{0} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \end{bmatrix}$$

$$B_{00} = \begin{bmatrix} D_0 - \eta I_m & \eta I_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_0 - \sigma I_m & \sigma I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S^0 \otimes I_m & S \oplus D_0 \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_1\alpha \otimes D_1 & d_1\gamma \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1\alpha_1 \otimes D_1 & I_l \otimes d_1\gamma_1 \otimes D_1 & \mathbf{0} \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} \zeta I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ d_2 T^0 \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ d_2 U^0 \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R^0 \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes d_2\theta_1 T^0 \otimes I_m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes d_2\theta_2 U^0 \otimes I_m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes \theta_3 R^0 \otimes I_m \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1^{22} & A_1^{23} & A_1^{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_1^{33} & \mathbf{0} & A_1^{35} & A_1^{36} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_1^{44} & A_1^{45} & \mathbf{0} & A_1^{47} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_1^{55} & \mathbf{0} & \mathbf{0} & A_{58} \\ \mathbf{0} & \mathbf{0} & A_{63} & \mathbf{0} & \mathbf{0} & A_{66} & \mathbf{0} & A_{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_1^{74} & \mathbf{0} & \mathbf{0} & A_1^{77} & A_1^{78} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{85} & \mathbf{0} & \mathbf{0} & A_{88} \end{bmatrix},$$

where

$$A_1^{11} = D_0 - (\eta + \zeta) I_m; \quad A_1^{12} = \eta I_m; \quad A_1^{22} = D_0 - \sigma I_m; \quad A_1^{23} = c_1\alpha \otimes \sigma I_m;$$
$$A_1^{24} = d_1\gamma \otimes \sigma I_m; \quad A_1^{33} = T \oplus D_0 - \tau I_{n_1 m}; \quad A_1^{35} = c_2 T^0 \otimes \beta \otimes I_m;$$
$$A_1^{36} = \tau\delta \otimes I_{n_1} \otimes I_m; \quad A_1^{44} = U \oplus D_0 - \tau I_{n_2 m}; \quad A_1^{45} = c_2 U^0 \otimes \beta \otimes I_m;$$
$$A_1^{47} = \tau\delta \otimes I_{n_2} \otimes I_m; \quad A_1^{55} = R \oplus D_0 - \tau I_{n_3 m}; \quad A_1^{58} = \tau\delta \otimes I_{n_3} \otimes I_m;$$
$$A_1^{63} = S^0 \otimes I_{n_1} \otimes I_m; \quad A_1^{66} = S \oplus \theta_1 T \oplus D_0; \quad A_1^{68} = I_l \otimes c_2\theta_1 T^0 \otimes \beta_1 \otimes I_m;$$
$$A_1^{74} = S^0 \otimes I_{n_2} \otimes I_m; \quad A_1^{77} = S \oplus \theta_2 U \oplus D_0; \quad A_1^{78} = I_l \otimes c_2\theta_2 U^0 \otimes \beta_1 \otimes I_m;$$
$$A_1^{85} = S^0 \otimes I_{n_3} \otimes I_m; \quad A_1^{88} = S \oplus \theta_3 R \oplus D_0.$$

$$A_0 = \begin{bmatrix} D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_1} \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{n_2} \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{n_3} \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes I_{n_1} \otimes D_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes I_{n_2} \otimes I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes I_{n_3} \otimes I_m \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \zeta I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_1 d_2 T^0 \alpha \otimes I_m & d_1 d_2 T^0 \otimes \gamma \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_1 d_2 U^0 \otimes \alpha \otimes I_m & d_1 d_2 U^0 \gamma \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_1 R^0 \otimes \alpha \otimes I_m & d_1 R^0 \otimes \gamma \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1 d_2 \theta_1 T^0 \alpha_1 \otimes I_m & I_l \otimes d_1 d_2 \theta_1 T^0 \otimes \gamma_1 \otimes I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1 d_2 \theta_2 U^0 \otimes \alpha_1 \otimes I_m & I_l \otimes d_1 d_2 \theta_2 U^0 \gamma_1 \otimes I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1 \theta_3 R^0 \otimes \alpha_1 \otimes I_m & I_l \otimes d_1 \theta_3 R^0 \otimes \gamma_1 \otimes I_m & \mathbf{0} \end{bmatrix}.$$

4 Stability condition

We analyse our model under some conditions that the system is stable.

4.1 Analysis of stability condition

Let us define the matrix A as $A = A_0 + A_1 + A_2$. It clearly shows that the arrangement of the square matrix A of order is $(2m + n_1 m + n_2 m + n_3 m + l n_1 m + l n_2 m + l n_3 m)$ and this matrix is irreducible infinitesimal generator matrix.

Let $\boldsymbol{\xi}$ be the steady-state probability vector of A satisfying $\boldsymbol{\xi}$A = 0 and $\boldsymbol{\xi}$e = 1. The vector $\boldsymbol{\xi}$ is partitioned by $\boldsymbol{\xi}$ = (ξ_0, ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, ξ_7), where ξ_0 and ξ_1 are of dimension m, ξ_2 is of dimension $n_1 m$, ξ_3 is of dimension $n_2 m$, ξ_4 is of dimension $n_3 m$, ξ_5 is of dimension $l n_1 m$, ξ_6 is of dimension $l n_2 m$ and ξ_7 is of dimension $l n_3 m$. The Markov process has the quasi-birth-and-death structure, there exits stability of our model should satisfy the condition $\boldsymbol{\xi} A_0$e < $\boldsymbol{\xi} A_2$e, which is the necessary and sufficient condition of a QBD process. The vector $\boldsymbol{\xi}$ is calculated by solving the following equations:

$$\begin{aligned}
&\xi_0[D - \eta I_m] = 0, \\
&\xi_0[\eta I_m] + \xi_1[D - \sigma I_m] = 0, \\
&\xi_1[c_1 \alpha \otimes \sigma I_m] + \xi_2[(T + c_1 d_2 T^0 \alpha) \oplus D - \tau I_{n_1 m}] + \xi_3[c_1 d_2 U^0 \otimes \alpha \otimes I_m] \\
&+\xi_4[c_1 R^0 \otimes \alpha \otimes I_m] + \xi_5[S^0 \otimes I_{n_1} \otimes I_m] = 0, \\
&\xi_1[d_1 \gamma \otimes \sigma I_m] + \xi_2[d_1 d_2 T^0 \otimes \gamma \otimes I_m] + \xi_3[(U + d_1 d_2 U^0 \gamma) \oplus D - \tau I_{n_2 m}] \\
&+\xi_4[d_1 R^0 \otimes \gamma \otimes I_m] + \xi_6[S^0 \otimes I_{n_2} \otimes I_m] = 0, \\
&\xi_2[c_2 T^0 \otimes \beta \otimes I_m] + \xi_3[c_2 U^0 \otimes \beta \otimes I_m] + \xi_4[R \oplus D - \tau I_{n_3 m}] \\
&+\xi_7[S^0 \otimes I_{n_3} \otimes I_m] = 0, \\
&\xi_2[\tau \delta \otimes I_{n_1} \otimes I_m] + \xi_5[S \oplus (\theta_1 T + c_1 d_2 \theta_1 T^0 \alpha_1) \oplus D] \\
&+\xi_6[I_l \otimes c_1 d_2 \theta_2 U^0 \otimes \alpha_1 \otimes I_m] + \xi_7[I_l \otimes c_1 \theta_3 R^0 \otimes \alpha_1 \otimes I_m] = 0, \\
&\xi_3[\tau \delta \otimes I_{n_2} \otimes I_m] + \xi_5[I_l \otimes d_1 d_2 \theta_1 T^0 \otimes \gamma_1 \otimes I_m] \\
&+\xi_6[S \oplus (\theta_2 U + d_1 d_2 \theta_2 U^0 \gamma_1) \oplus D] + \xi_7[I_l \otimes d_1 \theta_3 R^0 \otimes \gamma_1 \otimes I_m] = 0,
\end{aligned}$$

$$\xi_4[\tau\delta \otimes I_{n_3} \otimes I_m] + \xi_5[I_l \otimes c_2\theta_1 T^0 \otimes \beta_1 \otimes I_m]$$
$$+\xi_6[I_l \otimes c_2\theta_2 U^0 \otimes \beta_1 \otimes I_m] + \xi_7[S \oplus \theta_3 R \oplus D] = 0.$$

subject to normalising condition

$$(\xi_0 + \xi_1)e_m + \xi_2 e_{n_1 m} + \xi_3 e_{n_2 m} + \xi_4 e_{n_3 m} + \xi_5 e_{l n_1 m} + \xi_6 e_{l n_2 m} + \xi_7 e_{l n_3 m} = 1.$$

After some of the algebraical manipulation, the stability condition $\boldsymbol{\xi} A_0 \mathrm{e} < \boldsymbol{\xi} A_2 \mathrm{e}$ which is turns to be

$$\{(\xi_0 + \xi_1)[D_1 e_m] + \xi_2[e_{n_1} \otimes D_1 e_m] + \xi_3[e_{n_2} \otimes D_1 e_m] + \xi_4[e_{n_3} \otimes D_1 e_m]$$
$$+\xi_5[e_l \otimes e_{n_1} \otimes D_1 e_m] + \xi_6[e_l \otimes e_{n_2} \otimes D_1 e_m] + \xi_7[e_l \otimes e_{n_3} \otimes D_1 e_m]\}$$
$$< \{\xi_0[\zeta I_m] + \xi_2[d_2 T^0 \otimes e_m] + \xi_3[d_2 U^0 \otimes e_m] + \xi_4[R^0 \otimes I_m]$$
$$+\xi_5[e_l \otimes d_2\theta_1 T^0 \otimes e_m] + \xi_6[e_l \otimes d_2\theta_2 U^0 \otimes e_m]$$
$$+\xi_7[e_l \otimes \theta_3 R^0 \otimes e_m]\}.$$

4.2 *Analysis of steady-state probability vector*

Let us take the variable $\mathbf{x}$ be the steady-state probability vector of Q and it is partitioned as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$. Mention that $\mathbf{x}_0$ is of dimension $(3m + lm)$, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...$ are of dimension $(2m + n_1 m + n_2 m + n_3 m + l n_1 m + l n_2 m + l n_3 m)$. Then, the vector $\mathbf{x}$ satisfies the condition $\mathbf{x}Q = 0$ and $\mathbf{x}e = 1$.

However, when the stability condition has been satisfied and the subvectors of $\mathbf{x}$ except for $\mathbf{x}_0$ and $\mathbf{x}_1$ commensurate to the different level states are given by the equation

$$\mathbf{x}_j = \mathbf{x}_1 R^{j-1}, j \geqslant 2$$

where the rate matrix R denotes the minimal non-negative solution of the matrix quadratic equation as $R^2 A_2 + R A_1 + A_0 = 0$. Since our system is stable and the square matrices A_0, A_1, A_2 whose row sums are equal to zero, then the rate matrix R is a square matrix of order $(2m + n_1 m + n_2 m + n_3 m + l n_1 m + l n_2 m + l n_3 m)$, it is obtained from the above quadratic equation and also satisfies the relation $R A_2 e = A_0 e$.

The sub vectors $\mathbf{x}_0$ and $\mathbf{x}_1$ has been acquired by solving the following equations

$$\mathbf{x}_0 B_{00} + \mathbf{x}_1 B_{10} = 0.$$
$$\mathbf{x}_0 B_{01} + \mathbf{x}_1 (A_1 + R A_2) = 0.$$

subject to the normalising condition is

$$\mathbf{x}_0 e_1 + \mathbf{x}_1 (I - R)^{-1} e_2 = 1.$$

Thus, the R matrix could be calculated mathematically using essential steps in the logarithmic reduction algorithm.

For later use, we could make partition as

$$\mathbf{x}_0 = (u_{00}, u_{01}, u_{02}, u_{03}) \text{ and } \mathbf{x}_i = (v_{i0}, v_{i1}, v_{i4}, v_{i5}, v_{i6}, v_{i7}, v_{i8}, v_{i9}) \quad i \geq 1$$

such that their dimensions are specified in Table 1.

Table 1 Vector notation vs. dimension

Vector notation	*Dimension*
$u_{0k}(k = 0, 1, 2)$	m
u_{03}	lm
$v_{ik}(k = 0, 1)$	m
v_{i4}	$n_1 m$
v_{i5}	$n_2 m$
v_{i6}	$n_3 m$
v_{i7}	$ln_1 m$
v_{i8}	$ln_2 m$
v_{i9}	$ln_3 m$

The interpretation of the vectors in the steady-state is as follows:

u_{00} the main server is on vacation with there is no customer in the system and the arrival is considered in any one of m phases

u_{01} the main server is doing the start-up process with there is no customer in the system and the arrival is considered in any one of m phases

u_{02} the main server is being idle with there is no customer in the system and the arrival is considered in any one of m phases

u_{03} the standby server is being idle while the main server is under phase-type repair process with there is no customer in the system and the arrival is considered in any one of m phases

v_{i0} the system has precisely i ($i \geq 1$) customers with the main server is on vacation and the arrival is considered in any one of m phases

v_{i1} the system has precisely i ($i \geq 1$) customers with the main server is doing start-up process and the arrival is considered in any one of m phases

v_{i4} the system has precisely i ($i \geq 1$) customers in which the main server is offering mode I service with the arrival and service process are considered in various phases

v_{i5} the system has precisely i ($i \geq 1$) customers in which the main server is offering mode II service with the arrival and service process are considered in various phases

v_{i6} the system has precisely i ($i \geq 1$) customers in which the main server is offering additional service with the arrival and service process are considered in various phases

v_{i7} the system has precisely i ($i \geq 1$) customers in which the standby server is offering mode I service during the main server is under phase-type repair process with the arrival and service process are considered in various phases

v_{i8} the system has precisely i ($i \geq 1$) customers in which the standby server is offering mode II service during the main server is under phase-type repair process with the arrival and service process are considered in various phases

v_{i9} the system has precisely i ($i \geq 1$) customers in which the standby server is offering Additional service during the main server is under phase-type repair process with the arrival and service process are considered in various phases.

The following equations which are incurred from the quasi-birth-and-death process of the infinitesimal matrix Q,

$$
\begin{aligned}
&u_{00}[D_0 - \eta I_m] + v_{10}[\zeta I_m] + v_{14}[d_2 T^0 \otimes I_m] + v_{15}[d_2 U^0 \otimes I_m] \\
&+v_{16}[R^0 \otimes I_m] = 0. \\
&u_{00}[\eta I_m] + u_{01}[D_0 - \sigma I_m] = 0. \\
&u_{01}[\sigma I_m] + u_{02}[D_0] + u_{03}[S^0 \otimes I_m] = 0. \\
&u_{03}[S \oplus D_0] + v_{17}[I_l \otimes d_2\theta_1 T^0 \otimes I_m] + v_{18}[I_l \otimes d_2\theta_2 U^0 \otimes I_m] \\
&+v_{19}[I_l \otimes \theta_3 R^0 \otimes I_m] = 0. \\
&u_{00}[D_1] + v_{10}[D_0 - (\eta + \zeta) I_m] + v_{20}[\zeta I_m] = 0. \\
&u_{01}[D_1] + v_{10}[\eta I_m] + v_{11}[D_0 - \sigma I_m] = 0. \\
&u_{02}[c_1\alpha \otimes D_1] + v_{11}[c_1\alpha \otimes \sigma I_m] + v_{14}[T \oplus D_0 - \tau I_{n_1 m}] \\
&+v_{17}[S^0 \otimes I_{n_1} \otimes I_m] + v_{24}[c_1 d_2 T^0 \alpha \otimes I_m] + v_{25}[c_1 d_2 U^0 \otimes \alpha \otimes I_m] \\
&+v_{26}[c_1 R^0 \otimes \alpha \otimes I_m] = 0. \\
&u_{02}[d_1\gamma \otimes D_1] + v_{11}[d_1\gamma \otimes \sigma I_m] + v_{15}[U \oplus D_0 - \tau I_{n_2 m}] \\
&+v_{18}[S^0 \otimes I_{n_2} \otimes I_m] + v_{24}[d_1 d_2 T^0 \otimes \gamma \otimes I_m] + v_{25}[d_1 d_2 U^0 \gamma \otimes I_m] \\
&+v_{26}[d_1 R^0 \otimes \gamma \otimes I_m] = 0. \\
&v_{14}[c_2 T^0 \otimes \beta \otimes I_m] + v_{15}[c_2 U^0 \otimes \beta \otimes I_m] + v_{16}[R \oplus D_0 - \tau I_{n_3 m}] \\
&+v_{19}[S^0 \otimes I_{n_3} \otimes I_m] = 0. \\
&u_{03}[I_l \otimes c_1\alpha_1 \otimes D_1] + v_{14}[\tau\delta \otimes I_{n_1} \otimes I_m] + v_{17}[S \oplus \theta_1 T \oplus D_0] \\
&+v_{27}[I_l \otimes c_1 d_2 \theta_1 T^0 \alpha_1 \otimes I_m] + v_{28}[I_l \otimes c_1 d_2 \theta_2 U^0 \otimes \alpha_1 \otimes I_m] \\
&+v_{29}[I_l \otimes c_1 \theta_3 R^0 \otimes \alpha_1 \otimes I_m] = 0. \\
&u_{03}[I_l \otimes d_1\gamma_1 \otimes D_1] + v_{15}[\tau\delta \otimes I_{n_2} \otimes I_m] + v_{18}[S \oplus \theta_2 U \oplus D_0] \\
&+v_{27}[I_l \otimes d_1 d_2 \theta_1 T^0 \otimes \gamma_1 \otimes I_m] + v_{28}[I_l \otimes d_1 d_2 \theta_2 U^0 \gamma_1 \otimes I_m] \\
&+v_{29}[I_l \otimes d_1 \theta_3 R^0 \otimes \gamma_1 \otimes I_m] = 0. \\
&v_{\{(i-1)0\}}[D_1] + v_{\{i0\}}[D_0 - (\eta + \zeta) I_m] + v_{\{(i+1)0\}}[\zeta I_m] = 0, \quad i \geq 2. \\
&v_{\{(i-1)1\}}[D_1] + v_{\{i0\}}[\eta I_m] + v_{\{i1\}}[D_0 - \sigma I_m] = 0, \quad i \geq 2. \\
&v_{\{(i-1)4\}}[I_{n_1} \otimes D_1] + v_{\{i1\}}[c_1\alpha \otimes \sigma I_m] + v_{\{i4\}}[T \oplus D_0 - \tau I_{n_1 m}] \\
&+v_{\{i7\}}[S^0 \otimes I_{n_1} \otimes I_m] + v_{\{(i+1)4\}}[c_1 d_2 T^0 \alpha \otimes I_m] \\
&+v_{\{(i+1)5\}}[c_1 d_2 U^0 \otimes \alpha \otimes I_m] + v_{\{(i+1)6\}}[c_1 R^0 \otimes \alpha \otimes I_m] = 0, \quad i \geq 2. \\
&v_{\{(i-1)5\}}[I_{n_2} \otimes D_1] + v_{\{i1\}}[d_1\gamma \otimes \sigma I_m] + v_{\{i5\}}[U \oplus D_0 - \tau I_{n_2 m}] \\
&+v_{\{(i8)\}}[S^0 \otimes I_{n_2} \otimes I_m] + v_{\{(i+1)4\}}[d_1 d_2 T^0 \otimes \gamma \otimes I_m] \\
&+v_{\{(i+1)5\}}[d_1 d_2 U^0 \gamma \otimes I_m] + v_{\{(i+1)6\}}[d_1 R^0 \otimes \gamma I_m] = 0, \quad i \geq 2. \\
&v_{\{(i-1)6\}}[I_{n_3} \otimes D_1] + v_{\{i4\}}[c_2 T^0 \otimes \beta \otimes I_m] + v_{\{i5\}}[c_2 U^0 \otimes \beta \otimes I_m] \\
&+v_{\{i6\}}[R \oplus D_0 - \tau I_{n_3 m}] + v_{\{i9\}}[S^0 \otimes I_{n_3} \otimes I_m] = 0, \quad i \geq 2.
\end{aligned}
$$

$$v_{\{(i-1)7\}}[I_l \otimes I_{n_1} \otimes D_1] + v_{\{i4\}}[\tau\delta \otimes I_{n_1} \otimes I_m] + v_{\{i7\}}[S \oplus \theta_1 T \oplus D_0]$$
$$+v_{\{(i+1)7\}}[I_l \otimes c_1 d_2 \theta_1 T^0 \alpha_1 \otimes I_m] + v_{\{(i+1)8\}}[I_l \otimes c_1 d_2 \theta_2 U^0 \otimes \alpha_1 \otimes I_m]$$
$$+v_{\{(i+1)9\}}[I_l \otimes c_1 \theta_3 R^0 \otimes \alpha_1 \otimes I_m] = 0, \quad i \geq 2.$$
$$v_{\{(i-1)8\}}[I_l \otimes I_{n_2} \otimes D_1] + v_{\{i5\}}[\tau\delta \otimes I_{n_2} \otimes I_m] + v_{\{i8\}}[S \oplus \theta_2 U \oplus D_0]$$
$$+v_{\{(i+1)7\}}[I_l \otimes d_1 d_2 \theta_1 T^0 \otimes \gamma_1 \otimes I_m] + v_{\{(i+1)8\}}[I_l \otimes d_1 d_2 \theta_2 U^0 \gamma_1 \otimes I_m]$$
$$+v_{\{(i+1)9\}}[I_l \otimes d_1 \theta_3 R^0 \otimes \gamma_1 \otimes I_m] = 0, \quad i \geq 2.$$
$$v_{\{(i-1)9\}}[I_l \otimes I_{n_3} \otimes D_1] + v_{\{i6\}}[\tau\delta \otimes I_{n_3 \otimes I_m}]$$
$$+v_{\{i7\}}[I_l \otimes c_2 \theta_1 T^0 \otimes \beta_1 \otimes I_m] + v_{\{i8\}}[I_l \otimes c_2 \theta_2 U^0 \otimes \beta_1 \otimes I_m]$$
$$+v_{\{i9\}}[S \oplus \theta_3 R \oplus D_0] = 0, \quad i \geq 2.$$

subject to normalising condition

$$[u_{00} + u_{01} + u_{02}]e_m + [u_{03}]e_{lm} + \sum_{i=1}^{\infty}[v_{i0} + v_{i1}]e_m + \sum_{i=1}^{\infty}[v_{i4}]e_{n_1 m}$$
$$+ \sum_{i=1}^{\infty}[v_{i5}]e_{n_2 m} + \sum_{i=1}^{\infty}[v_{i6}]e_{n_3 m} + \sum_{i=1}^{\infty}[v_{i7}]e_{l n_1 m} + \sum_{i=1}^{\infty}[v_{i8}]e_{l n_2 m}$$
$$+ \sum_{i=1}^{\infty}[v_{i9}]e_{l n_3 m} = 1.$$

4.3 Computation of R matrix

There are so many algorithms for finding the rate matrix R. However, here we have given two algorithms.

4.3.1 Iterative algorithm

One can easily evaluate the rate matrix using the recursive procedure as follows:

- *Step 0*

  $R(0) = 0.$
- *Step 1*

  $R(n+1) = A_0(A_1)^{-1} + R^2(n)A_2(-A_1)^{-1}, n = 0, 1, 2, 3, ...$

  Continue Step 1 until $||R(n+1) - R(n)||_\infty < \epsilon$.

4.3.2 Logarithmic reduction algorithm

Logarithmic reduction algorithm is developed by Latouche and Ramaswami (1993) which have fast convergence and here we list steps involved in the logarithm reduction algorithm as follows:

- *Step 0*

  $H \leftarrow (-A_1)^{-1}A_0$, $L \leftarrow (-A_1)^{-1}A_2$, $G = L$, and $T = H$.

- *Step 1*

  $U = HL + LH$

  $M = H^2$

  $H \leftarrow (I - U)^{-1}M$

  $M \leftarrow L^2$

  $L \leftarrow (I - U)^{-1}M$

  $G \leftarrow G + TL$

  $T \leftarrow TH$

  Continue Step 1 until $||e - Ge||_\infty < \epsilon$.

- *Step 2*

  $R = -A_0(A_1 + A_0G)^{-1}$.

5 Analysis of the busy period

- A busy period is nothing but the interval between the customers arrives into the empty system and afterward the first interval once again the system becomes empty. So, it is the first passage from level 1 to 0. The busy cycle describes the first return time to level 0 with at least one visit to a state at any other level.
- Prior to examining the busy period, we introduce an overview of the fundamental period. Under consideration of the QBD process, it is the first passage time from level j to level $j-1, j \geqslant 2$.
- The cases $j = 0, 1$ corresponding to the boundary states have to be discussed individually. Note that for each and every level j, $j \geqslant 1$ there corresponds $(2m + n_1m + n_2m + n_3m + ln_1m + ln_2m + ln_3m)$ states. Thus by the state (j, k) of level j, we mention that the k^{th} state of level j when the states are arranged in alphabetical order.
- Let us denote $G_{kk'}(u, \mathrm{x})$ be the conditional probability that it starts in the state (j, k) at time $t = 0$, then the QBD process visits the level $j - 1$ but not later than time x, we can make changes to u transitions to the left and also entering the state (j, k').

Let us introduce the concept of the joint transform

$$\widetilde{G}_{kk'}(z,s) = \sum_{u=1}^{\infty} z^u \int_0^\infty e^{-s\mathrm{x}} dG_{kk'}(u,\mathrm{x}) \quad ; |\ z\ | \leq 1, Re(s) \geq 0.$$

and the matrix is denoted as $\widetilde{G}(z,s) = \widetilde{G}_{kk'}(z,s)$ then the above-defined matrix $\widetilde{G}(z,s)$ satisfies the equation

$$\widetilde{G}(z,s) = z(SI - A_1)^{-1}A_2 + (SI - A_1)^{-1}A_0\widetilde{G}^2(z,s).$$

The matrix of $G = G_{kk'} = \widetilde{G}(1,0)$ would be taken for the first passage times, exclude for the boundary states. If we already know the matrix R then we could find the matrix G using the result

$$G = -(A_1 + RA_2)^{-1}A_2.$$

Otherwise, we may use the concept of a logarithmic reduction algorithm method to find the values of the G matrix.

Notations of boundary level states for busy period

- $G_{kk'}^{(1,0)}(u,\mathrm{x})$ indicates the conditional probability is discussed for the first passage time from level 1 to level 0 at time $t = 0$.
- $G_{kk'}^{(0,0)}(u,\mathrm{x})$ indicates the conditional probability is discussed for the return time to level 0.
- $\mathbb{F}_{1j}$ indicates the average first passage time from the level j to level $j-1$, given that the process is in the state (j,k) at time $t = 0$.
- $\vec{\mathbb{F}}_1$ indicates the column vector with entries $\mathbb{F}_{1j}$.
- $\mathbb{F}_{2j}$ indicates the average number of customers to be served during the first passage time from level j to level $j-1$, given that the first passage time begins in the state (j,k).
- $\vec{\mathbb{F}}_2$ indicates the column vector with entries $\mathbb{F}_{2j}$.
- $\vec{\mathbb{F}}_1^{(1,0)}$ indicates the average first passage time from level 1 to level 0.
- $\vec{\mathbb{F}}_2^{(1,0)}$ indicates the average number of service completed during the first passage time from level 1 to level 0.
- $\vec{\mathbb{F}}_1^{(0,0)}$ indicates the first return time to level 0.
- $\vec{\mathbb{F}}_2^{(0,0)}$ indicates the average number of service completion in between first return time to level 0.

The following equations which are given $\widetilde{G}^{(1,0)}(z,s)$ and $\widetilde{G}^{(0,0)}(z,s)$ are for the boundary levels 1 and 0 respectively.

$$\widetilde{G}^{(1,0)}(z,s) = z(SI - A_1)^{-1}B_{10} + (SI - A_1)^{-1}A_0\widetilde{G}(z,s)\widetilde{G}^{(1,0)}(z,s)$$
$$\widetilde{G}^{(0,0)}(z,s) = (SI - B_{00})^{-1}B_{01}\widetilde{G}^{(1,0)}(z,s)$$

Thus, the following instances are calculated using the matrices as G, $\widetilde{G}^{(0,0)}(1,0)$ and $\widetilde{G}^{(1,0)}(1,0)$ are stochastic in nature.

$$\vec{\mathbb{F}}_1 = -\left.\frac{\partial}{\partial s}\widetilde{G}(z,s)\right|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}e_2 \tag{1}$$

$$\vec{\mathbb{F}}_2 = \frac{\partial}{\partial z}\widetilde{G}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}A_2e_2 \tag{2}$$

$$\vec{\mathbb{F}}_1^{(1,0)} = -\frac{\partial}{\partial s}\widetilde{G}^{(1,0)}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0G]^{-1}(A_0\vec{\mathbb{F}}_1 + e_2) \tag{3}$$

$$\vec{\mathbb{F}}_2^{(1,0)} = \frac{\partial}{\partial z}\widetilde{G}^{(1,0)}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0G]^{-1}(A_0\vec{\mathbb{F}}_2 + B_{10}e_1) \tag{4}$$

$$\vec{\mathbb{F}}_1^{(0,0)} = -\frac{\partial}{\partial s}\widetilde{G}^{0,0}(z,s)\bigg|_{z=1,s=0} e = -B00^{-1}[B_{01}\vec{\mathbb{F}}_1^{(1,0)} + e_1] \tag{5}$$

$$\vec{\mathbb{F}}_2^{(0,0)} = \frac{\partial}{\partial z}\widetilde{G}^{(0,0)}(z,s)\bigg|_{z=1,s=0} e = -B_{00}^{-1}[B_{01}\vec{\mathbb{F}}_2^{(1,0)}]. \tag{6}$$

6 Measures of system performance

We investigate the qualitative behaviour of our model under a steady state. In this section, we itemised a few performances of the characteristics of system measures along with their expressions for computation as follows,

- The probability that the main server is on vacation

$$P_V = \sum_{s=1}^{m}\mathbf{x}_{00s} + \sum_{p=1}^{\infty}\sum_{s=1}^{m}\mathbf{x}_{p0s}e = \mathbf{x}_0\, e_1(1) + \mathbf{x}_1(I-R)^{-1}\, e_2(1)$$

- The probability that the main server is doing the start-up process

$$P_S = \sum_{s=1}^{m}\mathbf{x}_{01s} + \sum_{p=1}^{\infty}\sum_{s=1}^{m}\mathbf{x}_{p1s} = \mathbf{x}_0\, e_1(2) + \mathbf{x}_1(I-R)^{-1}\, e_2(2)$$

- The probability that the main server is being idle

$$P_{MI} = \sum_{s=1}^{m}\mathbf{x}_{02s} = \mathbf{x}_0\, e_1(3)$$

- The probability that the standby server is being idle

$$P_{SI} = \sum_{q_2=1}^{l}\sum_{s=1}^{m}\mathbf{x}_{03q_2s} = \mathbf{x}_0\, e_1(4)$$

- The probability that the main server offering mode I service

$$P_{MBI} = \sum_{p=1}^{\infty}\sum_{r_1=1}^{n_1}\sum_{s=1}^{m}\mathbf{x}_{p4r_1s} = \mathbf{x}_1(I-R)^{-1}\, e_2(3)$$

- The probability that the main server offering mode II service

$$P_{MBII} = \sum_{p=1}^{\infty} \sum_{r_2=1}^{n_2} \sum_{s=1}^{m} \mathbf{x}_{p5r_2s} = \mathbf{x}_1 (I-R)^{-1}\, e_2(4)$$

- The probability that the main server offering additional service

$$P_{MBA} = \sum_{p=1}^{\infty} \sum_{r_3=1}^{n_3} \sum_{s=1}^{m} \mathbf{x}_{p6r_3s} = \mathbf{x}_1 (I-R)^{-1}\, e_2(5)$$

- The probability that the standby server offering mode I service

$$P_{SBI} = \sum_{p=1}^{\infty} \sum_{q_2=1}^{l} \sum_{r_1=1}^{n_1} \sum_{s=1}^{m} \mathbf{x}_{p7r_1s} = \mathbf{x}_1 (I-R)^{-1}\, e_2(6)$$

- The probability that the standby server offering mode II service

$$P_{SBII} = \sum_{p=1}^{\infty} \sum_{q_2=1}^{l} \sum_{r_2=1}^{n_2} \sum_{s=1}^{m} \mathbf{x}_{p8r_2s} = \mathbf{x}_1 (I-R)^{-1}\, e_2(7)$$

- The probability that the standby server offering additional service

$$P_{SBA} = \sum_{p=1}^{\infty} \sum_{q_2=1}^{l} \sum_{r_3=1}^{n_3} \sum_{s=1}^{m} \mathbf{x}_{p9r_3s} = \mathbf{x}_1 (I-R)^{-1}\, e_2(8)$$

- The probability that the main server is being busy

$$P_{MB} = P_{MBI} + P_{MBII} + P_{MBA}$$

- The probability that the standby server is being busy

$$P_{SB} = P_{SBI} + P_{SBII} + P_{SBA}$$

- Expected number of customers in the system

$$E_{system} = \sum_{p=1}^{\infty} p\, \mathbf{x}_p\, e = \mathbf{x}_1 (I-R)^{-2}\, e_2$$

- Average number of customers in the queue

$$\begin{aligned} E_{queue} &= \mathbf{x}_1 (I-R)^{-2}\, [e_2(1) + e_2(2)] + \mathbf{x}_1 R (I-R)^{-2}\, [e_2(3) + e_2(4) + e_2(5)] \\ &\quad + \mathbf{x}_1 R (I-R)^{-2}\, [e_2(6) + e_2(7) + e_2(8)] \end{aligned}$$

- The rate at which impatient behaviour of reneging customers

$$\mathscr{B} = \zeta \Big[\sum_{p=1}^{\infty} \sum_{s=1}^{m} \mathbf{x}_{p0s} \Big] = \zeta [\mathbf{x}_1 (I-R)^{-1}\, e_2(1)]$$

7 Cost analysis

Now we are imposing a cost associated with few characteristics of performance measures of the system for our model under study, we construct a cost function TC is defined by:

$$\begin{aligned}\text{TC} = {} & C_H E_{system} + C_V P_V + C_S P_S + C_{MI} P_{MI} + C_{SI} P_{SI} + C_{MBI} P_{MBI} \\ & + C_{MBII} P_{MBII} + C_{MBA} P_{MBA} + C_{SBI} P_{SBI} + C_{SBII} P_{SBII} \\ & + C_{SBA} P_{SBA} + \sigma C_1 + \tau C_2 + \delta_1 C_3 + \delta_2 C_4 + \delta_3 C_5 + \theta_1 \delta_1 C_6 + \theta_2 \delta_2 C_7 \\ & + \theta_3 \delta_3 C_8 + \Psi C_9 + \mathscr{B} C_{10}\end{aligned}$$

where

TC	total cost of the system per unit time
C_H	holding customers per unit time for each customer in the system
C_V	cost incurred during the main server is on vacation
C_S	cost incurred due to the main server doing start-up process
C_{MI}	cost incurred due to the main server being idle
C_{SI}	cost incurred due to standby server being idle while the main server under phase-type repair
C_{MBI}	cost incurred during the main server being busy with mode I service
C_{MBII}	cost incurred during the main server being busy with mode II service
C_{MBA}	cost incurred during the main server being busy with additional service
C_{SBI}	cost incurred during standby server being busy with mode I service while the main server under phase-type repair
C_{SBII}	cost incurred during standby server being busy with mode II service while the main server under phase-type repair
C_{SBA}	cost incurred during standby server being busy with additional service while the main server under phase-type repair
C_1	cost incurred for start-up process of the main server
C_2	cost incurred for the main server struck with a breakdown during a busy period
C_3	cost incurred by the main server for providing mode I service
C_4	cost incurred by the main server for providing mode II service
C_5	cost incurred by the main server for providing additional service
C_6	cost incurred by the standby server for providing mode I service
C_7	cost incurred by the standby server for providing mode II service
C_8	cost incurred by the standby server for providing additional service

C_9 cost incurred for the main server rejuvenated from repair

C_{10} cost incurred due to impatient behaviour of reneging customers.

8 Analysis of waiting time distribution

In this section, we perform an analysis of the distribution of the waiting period of a customer who arrives in the queueing line using the first passage time analysis. Let $W(t)$ indicates the distribution function of the waiting time is to consider the incoming(tagged) customer to the queueing line. If the server is idle when any customer arrives, then there is no delay in getting the service from the server, or else if the server is busy or on vacation, they have to wait in the queueing line for the aim of getting service from the server.

Let us introduce the absorption time in a Markov chain with state space is given by

$$\widetilde{\Omega} = (*) \cup \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, ...\}.$$

On entering into the absorbing state $(*)$, which corresponds to the tagged customer will begin to receive service without waiting and the absorbing state is defined as follows:

$$(*) = \{(0,2),\ (0,3)\}$$

The level state 0 is as follows,

$$\overline{0} = \{(0,0),\ (0,1)\}$$

for $p \geq 1$, the level state for p is given by

$$\begin{aligned}
\overline{p} = \{(p, q_1) :\ q_1 = 0, 1\} \cup \{(p, 4, r_1) : 1 \leq r_1 \leq n_1\} \\
\cup\{(p, 5, r_2) : 1 \leq r_2 \leq n_2\} \\
\cup\{(p, 6, r_3) : 1 \leq r_3 \leq n_3\} \\
\cup\{(p, 7, q_2, r_1) : 1 \leq q_2 \leq l; 1 \leq r_1 \leq n_1\} \\
\cup\{(p, 8, q_2, r_2) : 1 \leq q_2 \leq l; 1 \leq r_2 \leq n_2\} \\
\cup\{(p, 9, q_2, r_3) : 1 \leq q_2 \leq l; 1 \leq r_3 \leq n_3\}
\end{aligned}$$

The transition matrix $\widetilde{Q}$ of the absorbing Markov chain is given by

$$\widetilde{Q} = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\
E_0 & L_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\
E_1 & L_2 & L_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & L_3 & L_1 & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & L_3 & L_1 & \mathbf{0} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots
\end{bmatrix}$$

where its entries of $\widetilde{Q}$ are as follows:

$$E_0 = \begin{bmatrix} \eta \\ \sigma \end{bmatrix}, \quad L_0 = \begin{bmatrix} -\eta & \mathbf{0} \\ \mathbf{0} & -\sigma \end{bmatrix},$$

$$E_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ e_l \otimes d_2\theta_1 T^0 \\ e_l \otimes d_2\theta_2 T^0 \\ e_l \otimes \theta_3 R^0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} \zeta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ d_2 T^0 & \mathbf{0} \\ d_2 U^0 & \mathbf{0} \\ R^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$L_1 = \begin{bmatrix} -(\zeta+\eta) & \eta & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\sigma & c_1\sigma\alpha & d_1\sigma\gamma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T-\tau I_{n_1} & \mathbf{0} & c_2 T^0 \otimes \beta & \tau\delta \otimes I_{n_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & U-\tau I_{n_2} & c_2 U^0 \otimes \beta & \mathbf{0} & \tau\delta \otimes I_{n_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & R-\tau I_{n_3} & \mathbf{0} & \mathbf{0} & \tau\delta \otimes I_{n_3} \\ \mathbf{0} & \mathbf{0} & S^0 \otimes I_{n_1} & \mathbf{0} & \mathbf{0} & S \oplus \theta_1 T & \mathbf{0} & I_l \otimes c_2\theta_1 T^0 \otimes \beta_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & S^0 \otimes I_{n_2} & \mathbf{0} & \mathbf{0} & S \oplus \theta_2 U & I_l \otimes c_2\theta_2 U^0 \otimes \beta_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S^0 \otimes I_{n_3} & \mathbf{0} & \mathbf{0} & S \oplus \theta_3 R \end{bmatrix},$$

$$L_3 = \begin{bmatrix} \zeta & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_1 d_2 T^0 \alpha & d_1 d_2 T^0 \otimes \gamma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_1 d_2 U^0 \otimes \alpha & d_1 d_2 U^0 \gamma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c_1 R^0 \otimes \alpha & d_1 R^0 \otimes \gamma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1 d_2 \theta_1 T^0 \alpha_1 & I_l \otimes d_1 d_2 \theta_1 T^0 \otimes \gamma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1 d_2 \theta_2 U^0 \otimes \alpha_1 & I_l \otimes d_1 d_2 \theta_2 U^0 \gamma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1 \theta_3 R^0 \otimes \alpha_1 & I_l \otimes d_1 \theta_3 R^0 \otimes \gamma_1 & \mathbf{0} \end{bmatrix}.$$

Let us define $\mathbf{z}(0) = (\mathbf{z}_0(0),\ \mathbf{z}_1(0),\ \mathbf{z}_2(0),\ \mathbf{z}_3(0), ...)$ which is the conditional probability distribution of the state of system defined on the arrival of the tagged customers and the vector $\mathbf{z}_0(0)$ can be further partitioned as follows:

$$\mathbf{z}_0(0) = (\mathbf{z}_{00}, \mathbf{z}_{01})$$

However, the arrival process follows the Markovian property and it is observed that the arrival stationary probability distribution of the system size is as follows:

$$\mathbf{z}_{0k} = u_{0k} \left[\frac{D_1 e_m}{\lambda} \right], \qquad k = 0, 1$$

for $i \geq 1$,

$$\mathbf{z}_i(0) = \mathbf{x}_i \left[I_{2+n_1+n_2+n_3+ln_1+ln_2+ln_3} \otimes \frac{D_1 e_m}{\lambda} \right]$$

where λ indicates the fundamental arrival rate of the MAP. Now, let us define $\mathbf{z}(t) = (\mathbf{z}_*(t), \mathbf{z}_0(t),\ \mathbf{z}_1(t),\ \mathbf{z}_2(t),\ \mathbf{z}_3(t), ...)$, where $\mathbf{z}_i(t)$ is a row vector of order $\{1 \times (2 + n_1 + n_2 + n_3 + ln_1 + ln_2 + ln_3)\}$, where $i \geq 1$ and $\mathbf{z}_0(t)$ is a (1×2) vector. The components of $\mathbf{z}_i(t)$ are the probabilities that at time t, the continuous-time Markov chain of the respective states of level i with the generator $\widetilde{Q}$. Here, $\mathbf{z}_*(t)$ is the probability that the process is in the absorbing state at time t. Clearly, $W(t) = \mathbf{z}_*(t)$, for $t \geq 0$.

The differential equation $\mathbf{z}'(t) = \mathbf{z}(t)\widetilde{Q}$ where $t \geq 0$ becomes

$$\mathbf{z}'_*(t) = \sum_{i=0}^{1} \mathbf{z}_i(t)E_i$$
$$\mathbf{z}'_0(t) = \mathbf{z}_0(t)L_0 + \mathbf{z}_1(t)L_2$$
$$\mathbf{z}'_i(t) = \mathbf{z}_i(t)L_1 + \mathbf{z}_{i+1}(t)L_3, \quad for \quad i \geq 1$$

where $'$ denotes the derivative with respect to t.

The Laplace-Stieltjes Transform (LST) of the first passage time to level 1 is specified by the row vector $\omega(s)$ is as follows,

$$\omega(s) = \sum_{i=1}^{\infty} \mathbf{z}_i(0)[(sI - L_1)^{-1}L_3]^{i-1} \tag{7}$$

Let the LST of the absorbing time to the state $(*)$ commensurate the process begins at state level $i = 0, 1$, it would be indicated by $\phi(i, s)$. Hence, we have

$$\phi(0, s) = [sI - L_0]^{-1}E_0 \tag{8}$$
$$\phi(1, s) = [sI - L_1]^{-1}L_2\phi(0, s) + [sI - L_1]^{-1}E_1 \tag{9}$$

Thus, we observe that the LST for the distribution of waiting time $\widetilde{W}(s)$ is given by

$$\widetilde{W}(s) = \mathbf{z}_0(0)\phi(0, s) + \omega(s)\phi(1, s). \tag{10}$$

8.1 Expected waiting time

The expected waiting time is specified by

$$E(W) = -\widetilde{W}'(0) = -\mathbf{z}_0(0)\phi'(0, 0) - \omega'(0)e_3 - \omega(0)\phi'(1, 0). \tag{11}$$

Suppose if the system is in the level state $i = 0$, then the average time to enter the absorbing state $(*)$ is denoted by the first term of equation (11). Likewise, if the system is in the level state $i \geq 1$, then the average time to enter the absorbing state $(*)$ is denoted by the last two terms of equation (11).

On differentiating (8), (9) and substitute $s = 0$, we get

$$\phi'(0, 0) = (-1)[-L_0]^{-2}E_0 \tag{12}$$
$$\phi'(1, 0) = (-1)[-L_1]^{-2}L_2\phi(0, 0) + [-L_1]^{-1}L_2\phi'(0, 0) - [-L_1]^{-2}E_1 \tag{13}$$

By making use of the expression (12) together with the primary condition $\mathbf{z}(0) = (\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0), \mathbf{z}_3(0), ...)$, one can easily determine the first term of (11). Then from equation (7), we have

$$\omega(0) = \sum_{i=1}^{\infty} \mathbf{z}_i(0)K^{i-1} \tag{14}$$

where $K = [-L_1]^{-1}L_3$. Since K is a stochastic matrix, we have

$$\omega(0)e_3 = 1 - z_0(0)e. \tag{15}$$

With the help of the expressions (13) and (14) together with the primary condition $\mathbf{z}_0 = (\mathbf{z}_0(0),\ \mathbf{z}_1(0),\ \mathbf{z}_2(0), ...)$, one can evaluate the last term of (11). On differentiating (7) and take $s = 0$, we get

$$\omega'(0) = (-1)\sum_{i=1}^{\infty}\mathbf{z}_{i+1}(0)\sum_{j=0}^{i-1}K^j[-L_1]^{-1}K^{i-j} \tag{16}$$

Since K is stochastic in nature, we have

$$(-1)\omega'(0)e_3 = \sum_{i=1}^{\infty}\mathbf{z}_{i+1}(0)\sum_{j=0}^{i-1}K^j[-L_1]^{-1}e_3. \tag{17}$$

We can compute the value of $(-1)\omega'(0)e_3$. Now, let us consider the stochastic matrix K_2 such that $I - K + K_2$ is the non-singular and generalised inverse of the form $(I - K)$ and K is irreducible (see Kemeny and Snell, 1960), then the matrix K_2 maybe chosen as $K_2 = e_3k_0$, where k_0 is the stationary probability vector of K such that $k_0K = k_0$ and $k_0e_3 = 1$. Moreover, the following expression has the property that $KK_2 = K_2K = K_2$. Then we get,

$$\sum_{j=0}^{i-1}K^j(I - K + K_2) = I - K^i + iK_2, \quad for \ \ i \geq 1. \tag{18}$$

Substituting (18) in (17) and after doing some of the simplifications we will get as follows,

$$\begin{aligned}(-1)\omega'(0)e_3 = \Bigg\{&\mathbf{x}_1(I-R)^{-1}\left[I_{2+n_1+n_2+n_3+ln_1+ln_2+ln_3}\otimes\frac{D_1e_m}{\lambda}\right] - \omega(0)\\ &+ \mathbf{x}_1R(I-R)^{-2}\left[I_{2+n_1+n_2+n_3+ln_1+ln_2+ln_3}\otimes\frac{D_1e_m}{\lambda}\right]K_2\Bigg\}\\ &\times[I - K + K_2]^{-1}[-L_1]^{-1}e_3\end{aligned} \tag{19}$$

Hence, we have calculated all the terms of (11) and therefore we could easily compute the expected waiting time.

9 Numerical results

In this part, we analyse the model behaviour in the form of numerical and graphical illustrations. The following five different types of MAP representations have a different structure of variance and correlation. Consider the first three types of arrival processes, namely ERLA, EXPA and HYP-EXPA corresponds to renewal processes and therefore their correlation is 0. The arrival process of MAP-NC and MAP-PC are correlated arrivals with the correlation between two successive inter-arrival times are given by –0.4889 and 0.4889. These five arrival processes have coefficient of variation of intervals between arrivals of 0.3333, 1, 5.0388, 1.9868, and 1.9861. These five sets of arrival values are taken as input data and these values are incurred from the works of Chakravarthy (2010).

- *Arrival in Erlang (ERLA):*
$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$
- *Arrival in exponential (EXPA):*
$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \end{bmatrix}$$
- *Arrival in hyper-exponential (HYP-EXPA):*
$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$
- *Arrival in MAP-negative correlation (MAP-NC):*
$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99241 \\ 223.539 & 0 & 2.258 \end{bmatrix}$$
- *Arrival in MAP-positive correlation (MAP-PC):*
$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.99241 & 0 & 0.01002 \\ 2.258 & 0 & 223.539 \end{bmatrix}$$

Let us consider three phase-type distributions for the service process. The normalisation of these three representations has been made to obtain both the service times and repair times with representations δ_1, δ_2, δ_3 and Ψ. We will use the notations ERLX, EXPX and HYP-EXPX respectively for Erlang, exponential and hyper-exponential cases dealing with X-type distribution where X=S, R depending on whether the services or repairs are under consideration. Hence, EXPS corresponds to services that are modelled using exponential, whereas HYP-EXPR corresponds to hyper-exponential repairs. These sets of service values are taken as input data incurred from the works of Chakravarthy (2010).

- *Erlang (ERLX):*
$$\alpha = \gamma = \beta = \delta = \big(1, 0\big), \quad T = U = R = S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$
- *Exponential (EXPX):*
$$\alpha = \gamma = \beta = \delta = \big(1\big), \quad T = U = R = S = \begin{bmatrix} -1 \end{bmatrix}$$
- *Hyper-exponential (HYP-EXPX):*
$$\alpha = \gamma = \beta = \delta = \big(0.8, 0.2\big), \qquad T = U = R = S = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}$$

Illustrative Example 9.1

Consider the effect of breakdown rate of main server (τ) versus the expected number of customers in the system (E_{system}). We choose $\lambda = 1$; $\delta_1 = 3$; $\delta_2 = 3$; $\delta_3 = 2$; $\theta_1 = 0.8$; $\theta_2 = 0.8$; $\theta_3 = 0.7$; $\Psi = 8$; $\eta = 2$; $\zeta = 3$; $c_1 = 0.5$; $d_1 = 0.5$; $\sigma = 2$; $c_2 = 0.6$; $d_2 = 0.4$.

Table 2 Breakdown rate of the main server vs. E_{system} – Erlang service

τ	*ERLS*				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
0.2	1.238200	1.671396	3.575535	1.800199	87.611276
0.6	1.258173	1.709399	3.714660	1.840488	90.429772
1.0	1.277551	1.746354	3.850494	1.879648	93.162437
1.4	1.296365	1.782304	3.983086	1.917724	95.813207
1.8	1.314643	1.817289	4.112495	1.954757	98.385776
2.2	1.332409	1.851344	4.238789	1.990789	100.883614
2.6	1.349687	1.884508	4.362040	2.025857	103.309981
3.0	1.366500	1.916812	4.482325	2.060000	105.667947
3.4	1.382868	1.948291	4.599721	2.093250	107.960405
3.8	1.398808	1.978973	4.714309	2.125643	110.190083
4.2	1.414340	2.008888	4.826168	2.157209	112.359557
4.6	1.429479	2.038064	4.935377	2.187979	114.471261
5.0	1.444242	2.066528	5.042014	2.217982	116.527496
5.4	1.458642	2.094305	5.146156	2.247246	118.530443

Table 3 Breakdown rate of the main server vs. E_{system} – exponential service

τ	*EXPS*				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
0.2	1.387374	1.840607	3.835346	1.958794	87.779613
0.6	1.415465	1.887056	3.983858	2.007192	90.607041
1.0	1.442798	1.932263	4.128756	2.054285	93.348438
1.4	1.469404	1.976272	4.270111	2.100119	96.007739
1.8	1.495311	2.019126	4.408000	2.144738	98.588638
2.2	1.520546	2.060867	4.542507	2.188187	101.094604
2.6	1.545134	2.101533	4.673720	2.230506	103.528901
3.0	1.569099	2.141164	4.801728	2.271735	105.894599
3.4	1.592464	2.179796	4.926622	2.311913	108.194595
3.8	1.615250	2.217463	5.048493	2.351077	110.431620
4.2	1.637478	2.254200	5.167430	2.389263	112.608256
4.6	1.659168	2.290039	5.283521	2.426504	114.726939
5.0	1.680339	2.325010	5.396855	2.462834	116.789977
5.4	1.701008	2.359143	5.507516	2.498283	118.799555

Table 4 Breakdown rate of the main server vs. E_{system} – hyper-exponential service

τ	*HYP-EXPS*				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
0.2	2.456263	2.958756	5.282026	3.065762	88.931713
0.6	2.538514	3.059664	5.487963	3.168622	91.815290
1.0	2.618850	3.158109	5.688712	3.268922	94.611539
1.4	2.697309	3.254153	5.884408	3.366729	97.324411
1.8	2.773933	3.347859	6.075189	3.462112	99.957615
2.2	2.848765	3.439291	6.261196	3.555140	102.514643
2.6	2.921850	3.528512	6.442571	3.645883	104.998782
3.0	2.993232	3.615587	6.619453	3.734408	107.413129
3.4	3.062956	3.700577	6.791981	3.820783	109.760609
3.8	3.131069	3.783547	6.960291	3.905074	112.043981
4.2	3.197615	3.864555	7.124515	3.987346	114.265854
4.6	3.262637	3.943662	7.284785	4.067661	116.428697
5.0	3.326181	4.020926	7.441225	4.146082	118.534845
5.4	3.388287	4.096403	7.593958	4.222666	120.586514

The observation from Tables 2, 3 and 4 as follows:

- The observation from the effect of an increase in the breakdown rate of the main server which leads to the E_{system} increases for the precise combinations of arrival and service times.
- While an increase in breakdown rate of the main server that is breakdown occurs more often, it gives a detailed account of whenever the main server struck by the breakdown immediately the standby server would interrupt and take over the service process at a slower service rate which leads to the E_{system} increases respectively.
- From the viewpoint of arrival times, MAP-PC increases tremendously and the ERLA increases slowly for both the E_{system}. Similarly, the observation from the viewpoint of service times, the HPY-EXPS increases fastly and the ERLS increases slowly.

Illustrative Example 9.2

To observe the outcome of the main server's vacation rate (η) versus the total cost of the system (TC). We choose $\lambda = 1$; $\delta_1 = 5$; $\delta_2 = 5$; $\delta_3 = 3$; $\theta_1 = 0.8$; $\theta_2 = 0.8$; $\theta_3 = 0.7$; $\sigma = 2$; $\Psi = 9$; $\tau = 1$; $\zeta = 3$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.6$; $d_2 = 0.4$.

The overall observation from Tables 5, 6 and 7 are given below:

- When an increase the vacation rate of the main server then the TC decreases for the precise combinations of service and arrival times.
- While enhancing the vacation rate of the main server, it will convey the detailed account of the main server could go for a vacation when the service completion epoch in that vacation times, the waiting customers in the system may renege from the system which communicates the TC decreases.

- Observe the results of arrival times, MAP-PC largely decreases and ERLA decreases at a minimum level. However, consider the service times, the ERLS decreases at minimum level and the HYP-EXPS decreases much greater level than the other service times.

Table 5 Vacation rate of the main server vs. TC – Erlang service

η	*ERLS*				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
2	80.448591	81.370471	84.977615	81.550156	487.247239
3	79.667979	80.706331	84.574041	80.939441	486.711176
4	79.198514	80.318698	84.330171	80.597285	486.403479
5	78.883436	80.063185	84.164664	80.378508	486.205534
6	78.656874	79.881524	84.044155	80.226648	486.068006
7	78.485981	79.745494	83.952125	80.115130	485.967088
8	78.352444	79.639699	83.879365	80.029796	485.889965
9	78.245215	79.555000	83.820296	79.962412	485.829152
10	78.157220	79.485622	83.771331	79.907867	485.779994
11	78.083714	79.427731	83.730046	79.862817	485.739446
12	78.021396	79.378677	83.694745	79.824987	485.705437
13	77.967897	79.336572	83.664199	79.792774	485.676509
14	77.921472	79.300030	83.637499	79.765014	485.651605
15	77.880808	79.268013	83.613955	79.740847	485.629942

Table 6 Vacation rate of the main server vs. TC – exponential service

η	*EXPS*				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
2	80.787137	81.896856	86.024836	81.962848	487.736869
3	80.020331	81.256603	85.657285	81.366831	487.205481
4	79.556469	80.880247	85.430477	81.032134	486.900266
5	79.244204	80.630955	85.274397	80.817893	486.703917
6	79.019312	80.453081	85.159618	80.669098	486.567523
7	78.849551	80.319519	85.071305	80.559798	486.467461
8	78.716860	80.215413	85.001073	80.476148	486.391012
9	78.610306	80.131918	84.943788	80.410087	486.330745
10	78.522875	80.063423	84.896116	80.356611	486.282039
11	78.449854	80.006196	84.855790	80.312442	486.241872
12	78.387962	79.957653	84.821213	80.275353	486.208190
13	78.334842	79.915946	84.791222	80.243769	486.179543
14	78.288758	79.879720	84.764953	80.216553	486.154886
15	78.248403	79.847956	84.741746	80.192859	486.133441

Table 7 Vacation rate of the main server vs. TC – hyper-exponential service

η	*HYP-EXPS*				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
2	83.713786	85.309313	91.590150	85.065686	491.155637
3	83.098817	84.833300	91.444015	84.562095	490.661927
4	82.703138	84.535864	91.324691	84.269593	490.374044
5	82.426874	84.330833	91.229269	84.078862	490.187433
6	82.223269	84.180333	91.152106	83.944878	490.057228
7	82.067207	84.064891	91.088688	83.845712	489.961437
8	81.943928	83.973399	91.035739	83.769414	489.888114
9	81.844184	83.899030	90.990903	83.708927	489.830235
10	81.761890	83.837347	90.952464	83.659819	489.783414
11	81.692876	83.785333	90.919150	83.619167	489.744776
12	81.634197	83.740863	90.890003	83.584968	489.712357
13	81.583713	83.702397	90.864288	83.555804	489.684775
14	81.539832	83.668787	90.841433	83.530643	489.661027
15	81.501348	83.639165	90.820985	83.508716	489.640367

Illustrative Example 9.3

To examine the impact of start-up rate of main server (σ) versus the Total cost of the system (TC). We fix $\lambda = 1$; $\delta_1 = 5$; $\delta_2 = 5$; $\delta_3 = 4$; $\theta_1 = 0.6$; $\theta_2 = 0.6$; $\theta_3 = 0.5$; $\eta = 4$; $\Psi = 10$; $\tau = 2$; $\zeta = 5$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.4$; $d_2 = 0.6$.

A quick observation from Figure 2 as follows:

While increasing the start-up rate which is handled by the main server then the TC also increases. Whenever the main server return from vacation who starts the start-up process and when the start-up completion epoch, the main server will start service if anyone in the system otherwise would be in the idle state up to customers arrival. In this investigation, the combination of service times with different arrival times explicit the HYP-EXPS increases fastly and the ERLS increases slowly. Likewise, in the same manner, arrangements of arrival with service times the ERLA increases slowly and the MAP-PC increases tremendously.

Illustrative Example 9.4

To test the fundamental arrival rate of customers (λ) versus the expected waiting time ($E(W)$). To choose $\delta_1 = 20$; $\delta_2 = 20$; $\delta_3 = 19$; $\theta_1 = 0.7$; $\theta_2 = 0.7$; $\theta_3 = 0.6$; $\eta = 5$; $\Psi = 10$; $\tau = 1$; $\zeta = 6$; $\sigma = 4$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.7$; $d_2 = 0.3$.

The observation from Figure 3 is given.

If the fundamental arrival rate increases which deliberate the $E(W)$ also increases for the different combinations of arrival and service times. The queueing line increases due to customer's arrival increases which leads to the customer have to wait more time to receive service from the server. However, consider the arrival times, the expected waiting time increases highly in HYP-EXPA and slowly in ERLA except for the

MAP-PC. In the same manner, consider the service times ERLS increases slowly and HYP-EXPS increases much faster compared to other service times except the MAP-PC. Now consider the MAP-PC arrival, the expected waiting time increases but all service times collide in the same line.

Figure 2 Start-up rate vs. total cost of the system, (a) ERLA (b) EXPA (c) HYP-EXPA (d) MAP-NC (e) MAP-PC (see online version for colours)

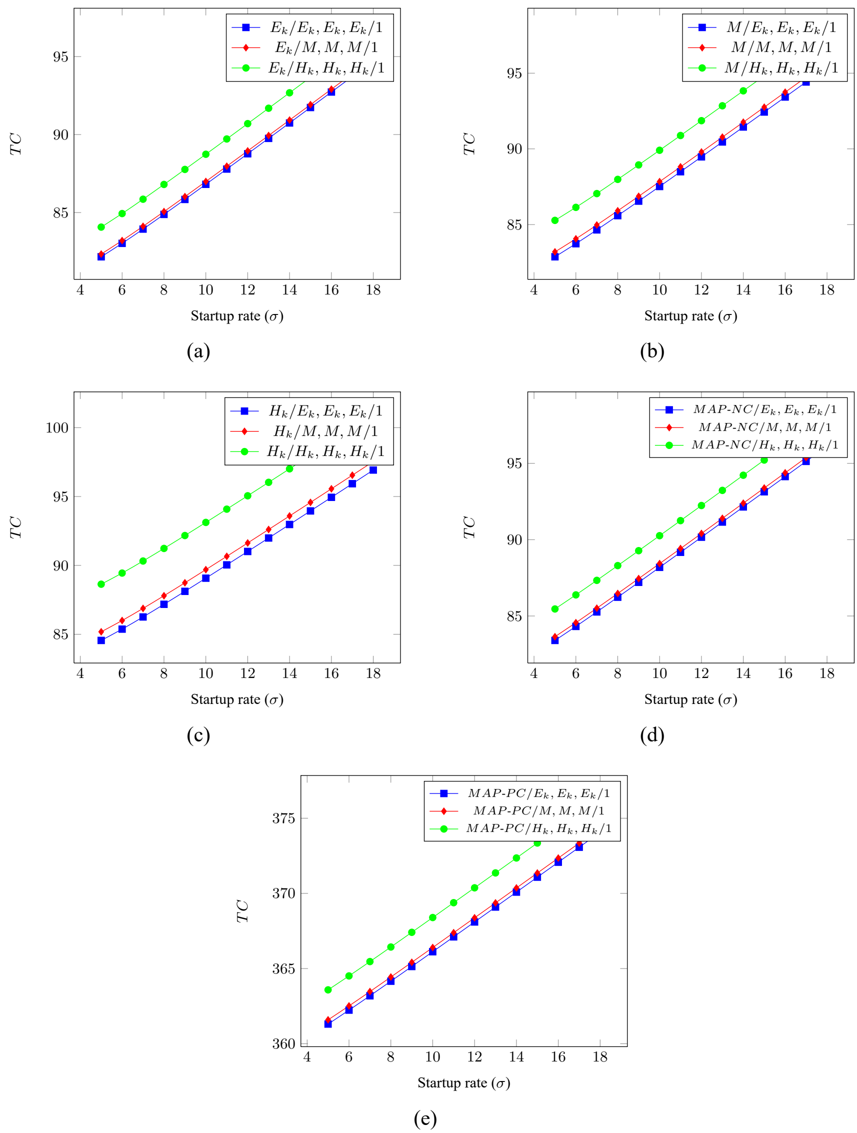
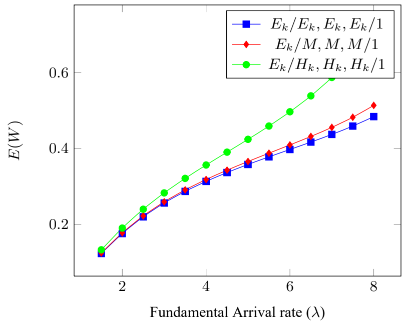
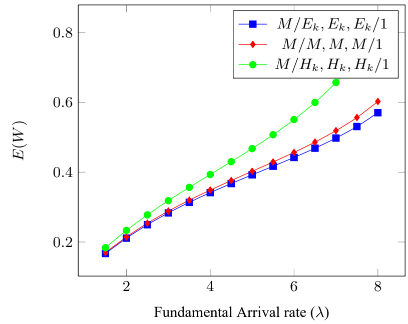


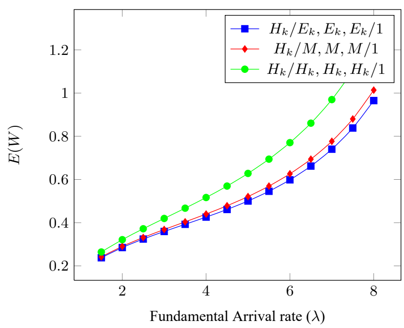
Figure 3 Fundamental arrival rate vs. expected waiting time, (a) ERLA (b) EXPA (c) HYP-EXPA (d) MAP-NC (e) MAP-PC (see online version for colours)



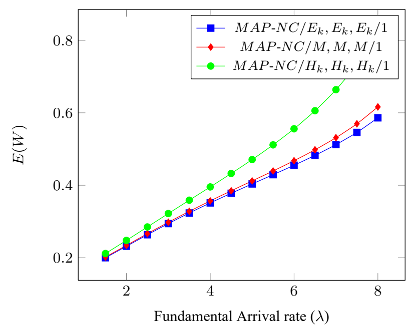
(a) ERLA



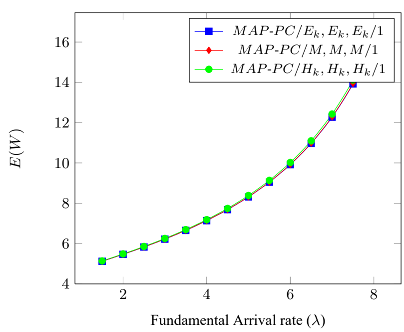
(b) EXPA



(c) HYP-EXPA



(d) MAP-NC



(e) MAP-PC

Illustrative Example 9.5

To see the features of the main server's vacation rate (η) and additional service rate (δ_3) versus the expected number of customers in the system (E_{system}). We prefer $\lambda = 1$; $\delta_1 = 4$; $\delta_2 = 4$; $\theta_1 = 0.8$; $\theta_2 = 0.8$; $\theta_3 = 0.7$; $\Psi = 7$; $\zeta = 7$; $\tau = 2$; $\sigma = 3$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.6$; $d_2 = 0.4$.

The overall observation from Figures 4, 5 and 6.

We increase the values of both the main server's additional service rate and vacation rate, the E_{system} decreases with various groupings of arrival and service times. Whenever the main server completes the additional service to the customer, the main server checks if there is any customer in the system suppose if there is no customer in the system the main server immediately goes on vacation. Due to the main server increase the additional service rate the E_{system} decreases likewise increase the vacation rate of the main server in that situations reneging may happen so that is also the reason for the decrease in the E_{system}. Let look at the service times, the HYP-EXPS decreases quickly and the ERLS slowly decreases. Similarly, in the arrival times, the ERLA decreases slowly and MAP-PC decreases fastly.

Illustrative Example 9.6

We determine the comparison of the mode I service rate of main server (δ_1) and start-up rate of main server(σ) versus the expected waiting time ($E(W)$). We fix $\lambda = 1$; $\delta_2 = 7$; $\delta_3 = 6$; $\theta_1 = 0.7$; $\theta_2 = 0.7$; $\theta_3 = 0.6$; $\Psi = 10$; $\eta = 8$; $\zeta = 6$; $\tau = 2$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.6$; $d_2 = 0.4$.

The overall observation from Figures 7, 8 and 9.

While maximising the values of both the main server's start-up rate and mode I service rate simultaneously then the expected waiting time decreases with the distinct combinations of service and arrival times. When the vacation completion epoch, the main server does the start-up process and then offers any one of the modes of service to the customers which deliberates the expected waiting time decreases. From the viewpoint of arrival times, MAP-PC decreases tremendously and the ERLA decreases slowly likewise consider the service times, the ERLS decreases slowly compared to the other service times and the HYP-EXPS highly decreases. An increase in the mode I service rate and start-up rate which is handled by the main server leads to a decrease in the expected waiting time.

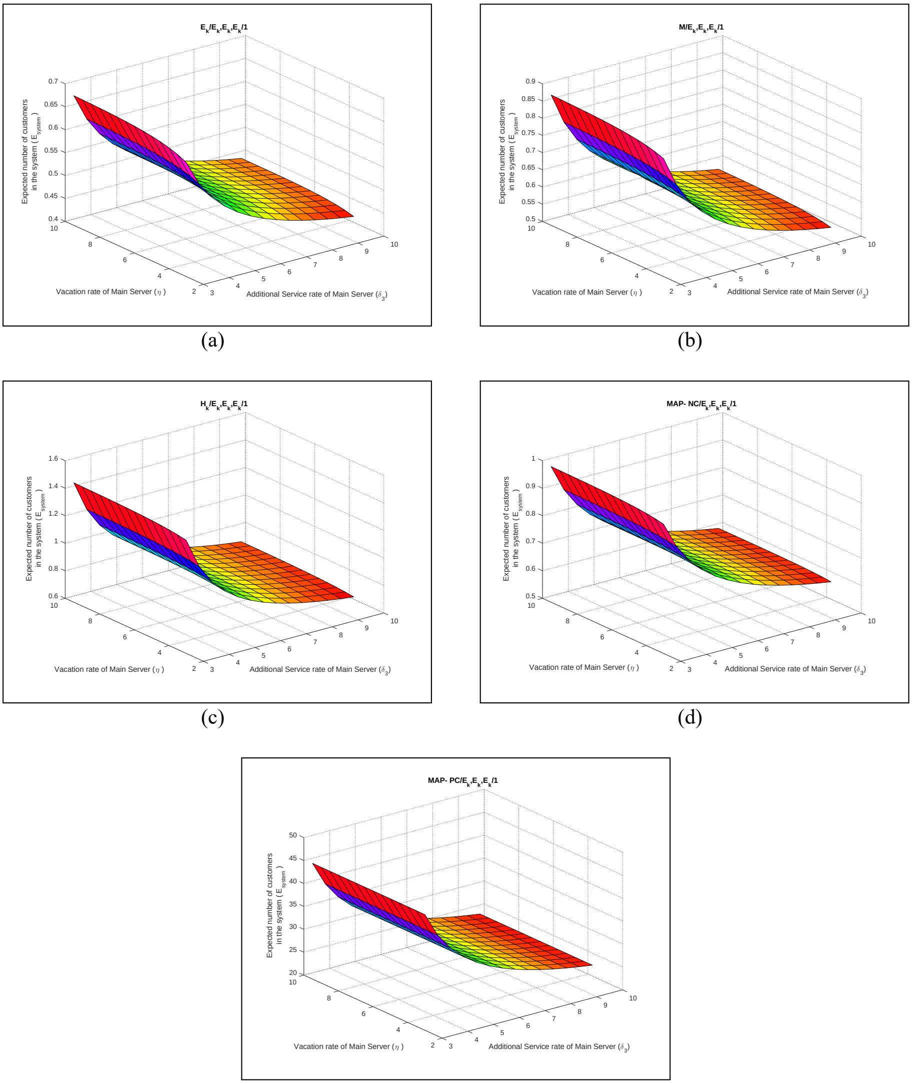
Illustrative Example 9.7

We probe the consequence of the mode II service rate of standby server ($\theta_2\delta_2$) and repair rate of the main server (Ψ) versus the Total cost of the system (TC). To choose $\lambda = 1$; $\delta_1 = 10$; $\delta_2 = 10$; $\delta_3 = 9$; $\theta_1 = 0.6$; $\theta_3 = 0.5$; $\eta = 2$; $\sigma = 3$; $\zeta = 6$; $\tau = 1$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.5$; $d_2 = 0.5$.

The observation from Figures 10, 11 and 12.

While maximising both the values of the main server's repair rate and the standby server's mode II service rate which leads to the increase in the TC for the different groupings of service and arrival times. When the repair completion epoch the main server would interrupt the standby server and carry over the service process whatever the service it is. But here, when the standby does mode II service to the customers at that epoch the main server completes the repair then immediately service process switchover to the main server such that it deliberates the TC also increases. Let us consider the service times, the ERLS increases slowly and HYP-EXPS fastly increases. However, now consider the arrival times, the MAP-PC increases tremendously and the EXPA increases slowly in comparison to all other arrival times.

Figure 4 Additional service rate and the vacation rate of the main server vs. E_{system} – Erlang service (see online version for colours)



(a) (b) (c) (d) (e)

Figure 5 Additional service rate and the vacation rate of the main server vs. E_{system} – exponential service (see online version for colours)

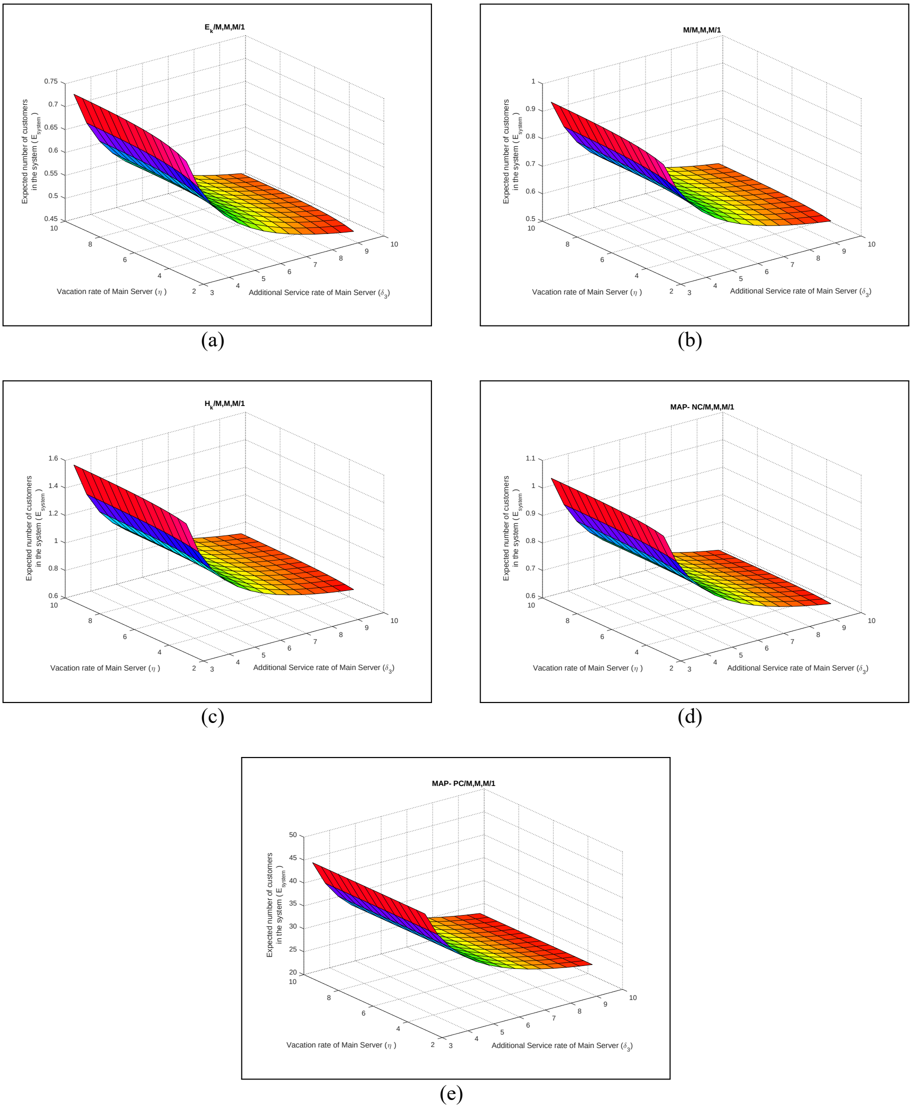
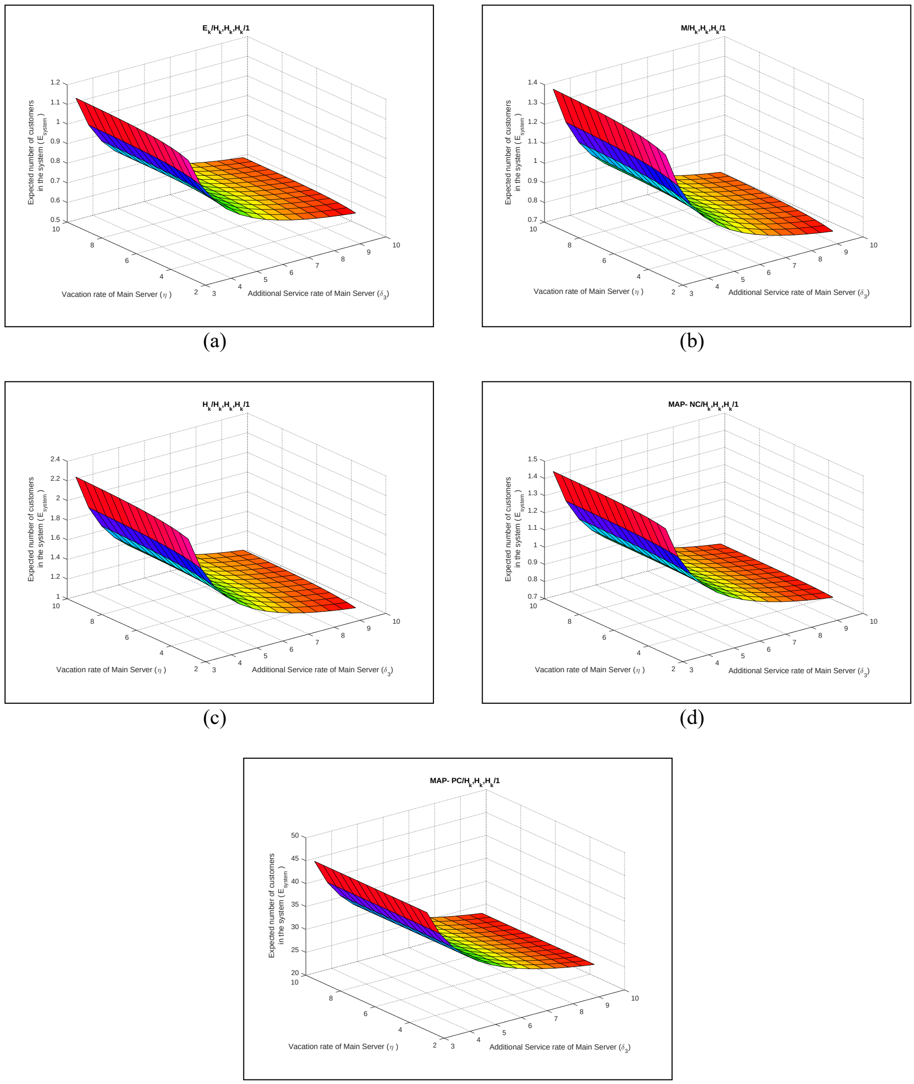


Figure 6 Additional service rate and the vacation rate of the main server vs. E_{system} – hyper-exponential service (see online version for colours)



(a) (b) (c) (d) (e)

Figure 7 Mode I service rate and the start-up rate of the main server vs. $E(W)$ – Erlang service (see online version for colours)

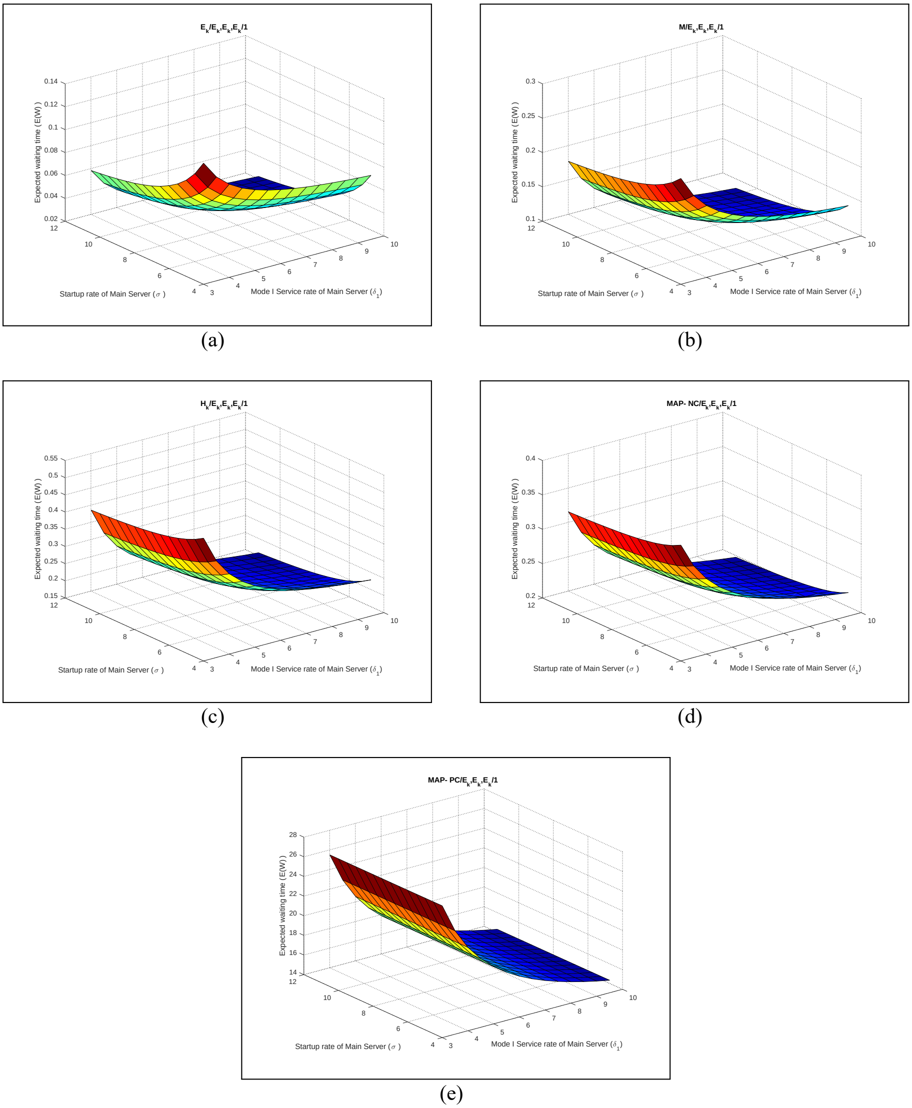


Figure 8 Mode I service rate and the start-up rate of the main server vs. $E(W)$ – exponential service (see online version for colours)

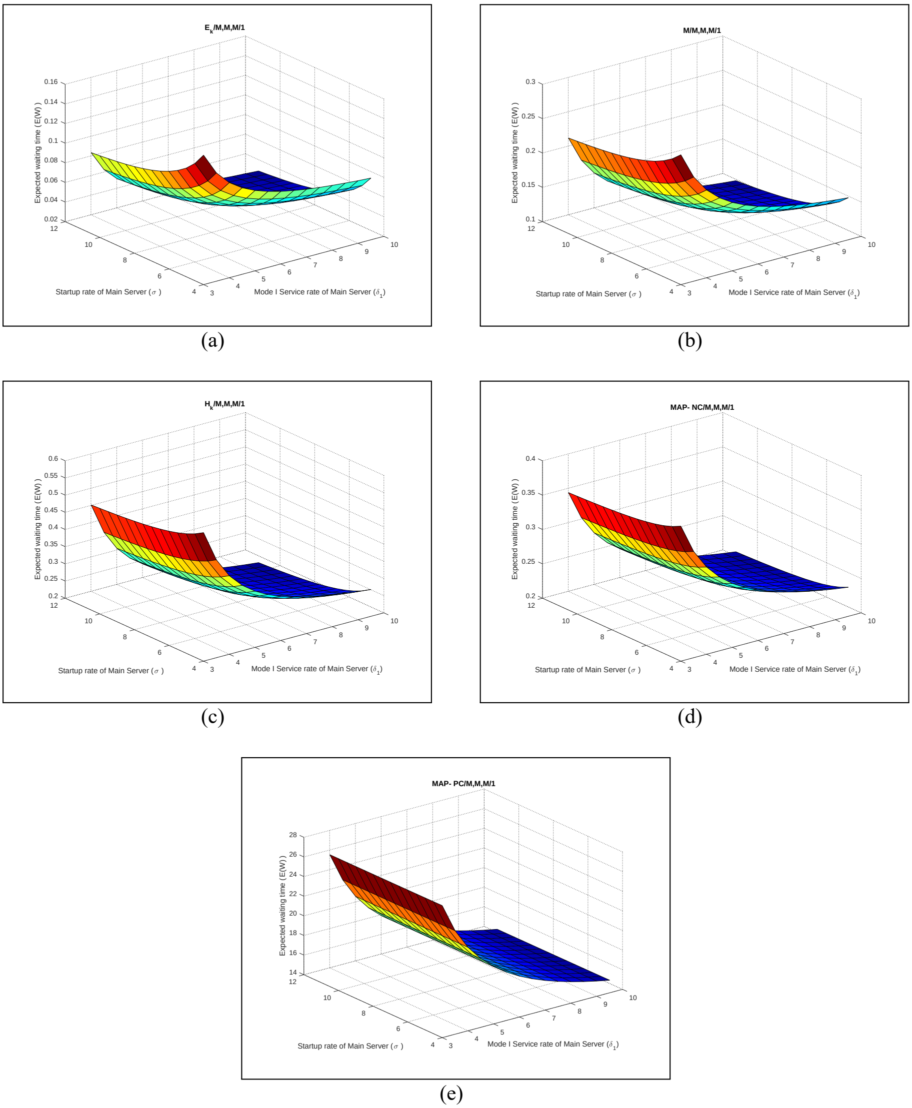
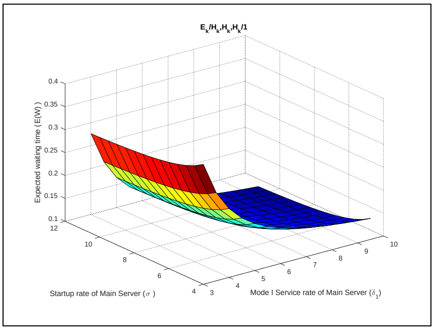
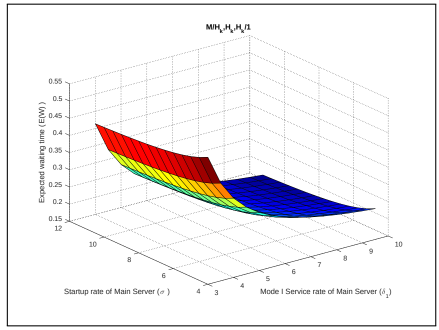


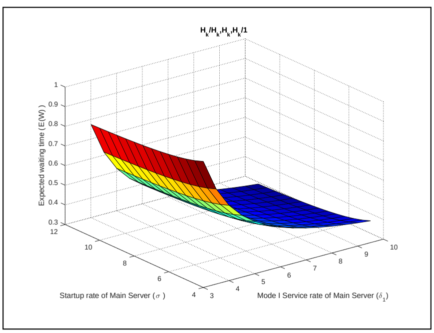
Figure 9 Mode I service rate and the start-up rate of the main server vs. $E(W)$ – hyper-exponential service (see online version for colours)



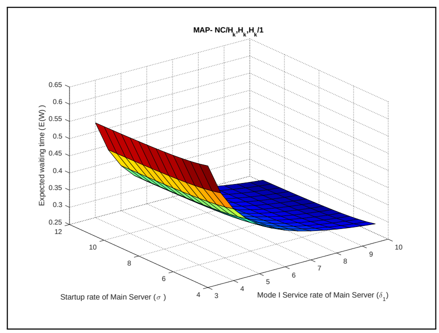
(a)



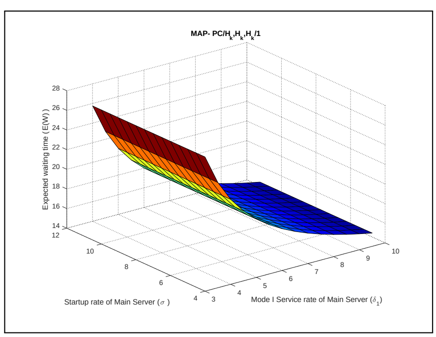
(b)



(c)

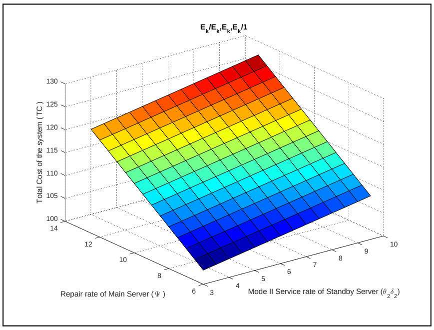


(d)

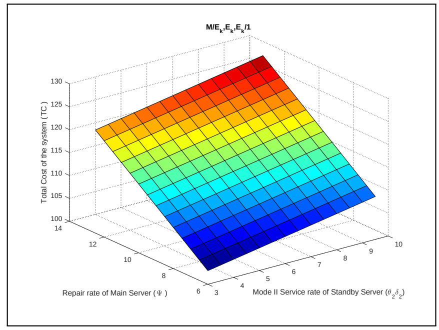


(e)

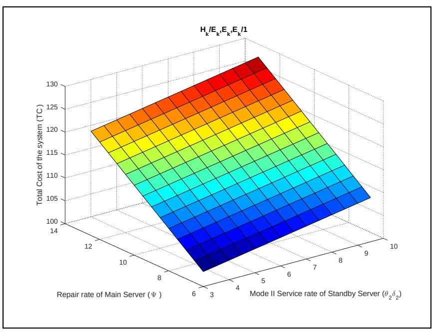
Figure 10 Mode II service rate of the standby server and the repair rate of the main server vs. TC – Erlang service (see online version for colours)



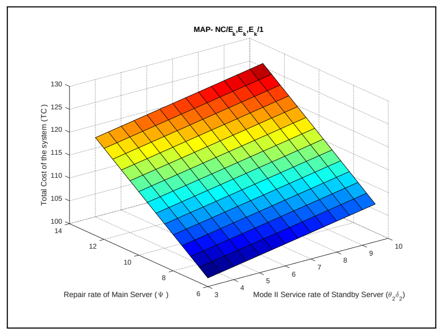
(a)



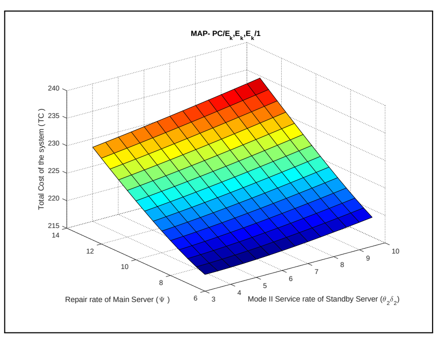
(b)



(c)



(d)



(e)

Figure 11 Mode II service rate of the standby server and the repair rate of the main server vs. TC – exponential service (see online version for colours)

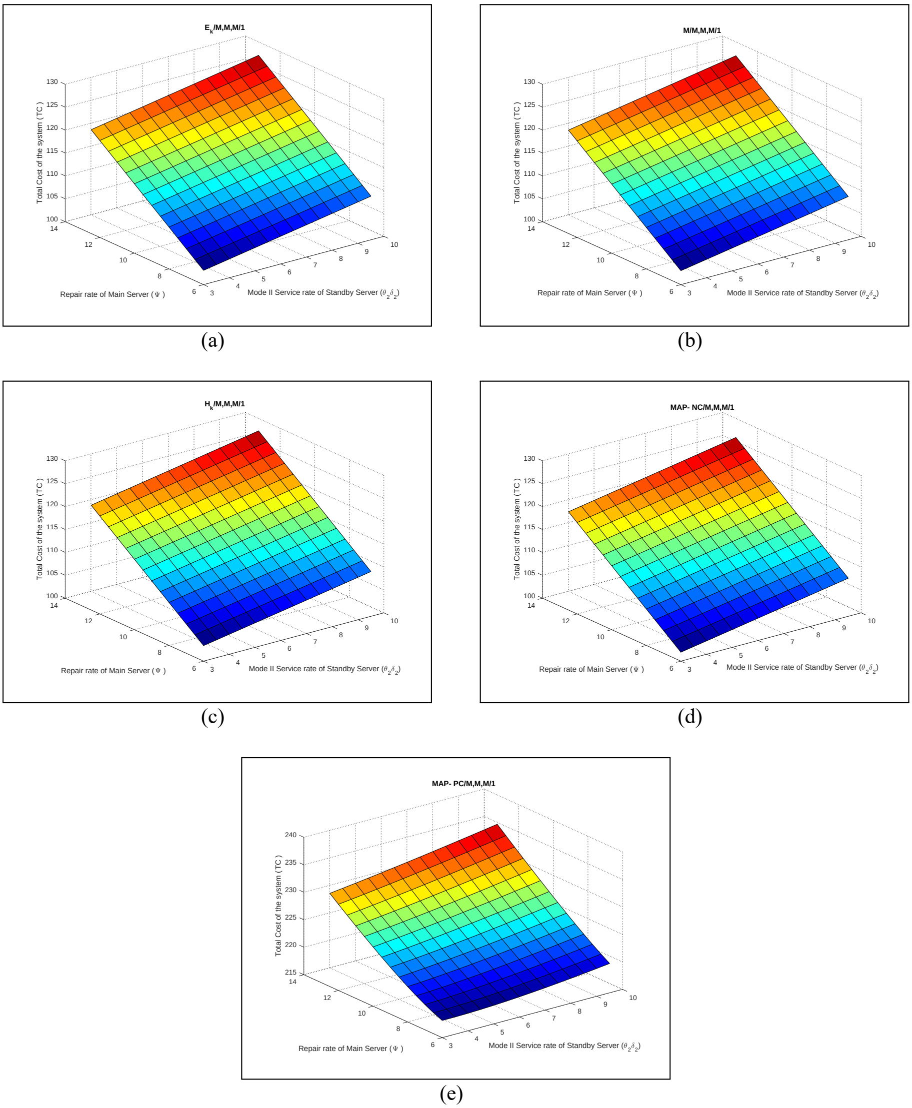
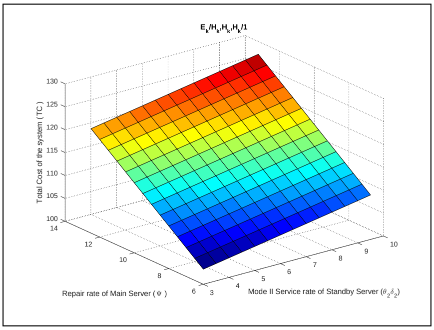
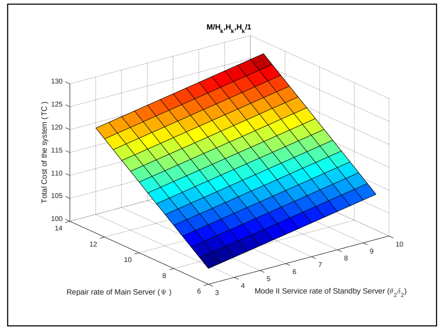


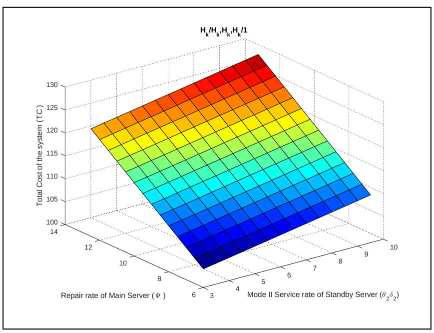
Figure 12 Mode II service rate of the standby server and the repair rate of the main server vs. TC – hyper-exponential service (see online version for colours)



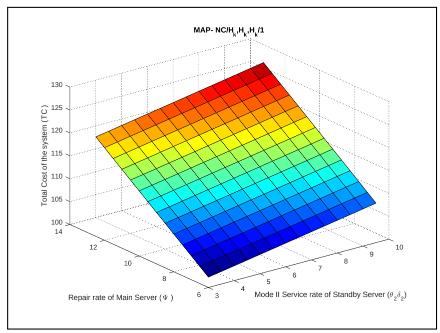
(a)



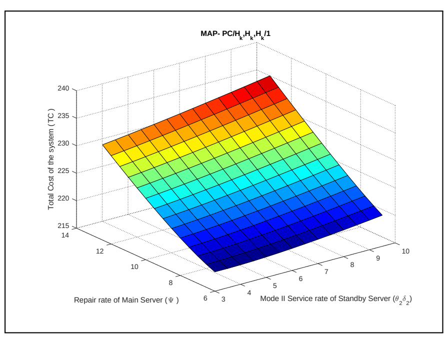
(b)



(c)



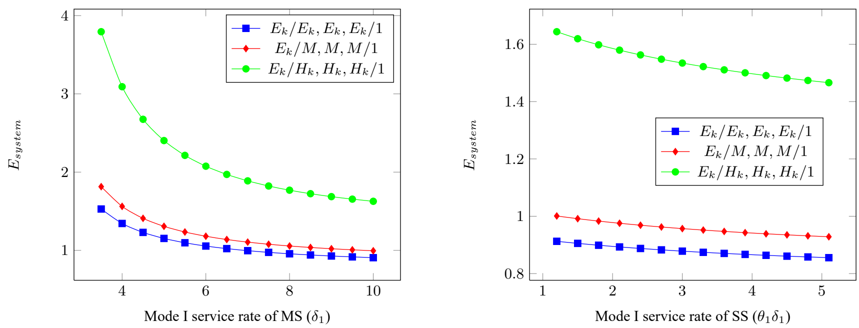
(d)



(e)

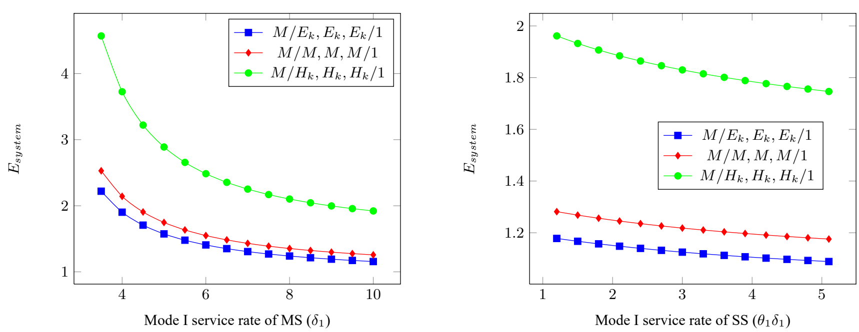
Comparing the mode I service scenarios of main server and standby server

Figure 13 E_{system} vs. mode I service rate of main server and standby server – ERLA (see online version for colours)



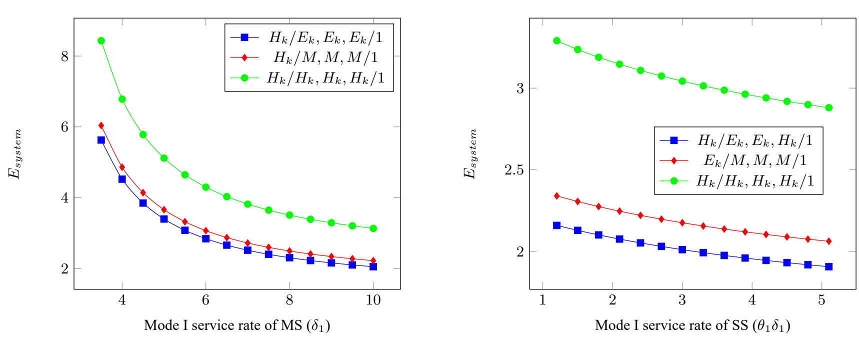
(a) (b)

Figure 14 E_{system} vs. mode I service rate of main server and standby server of EXPA (see online version for colours)



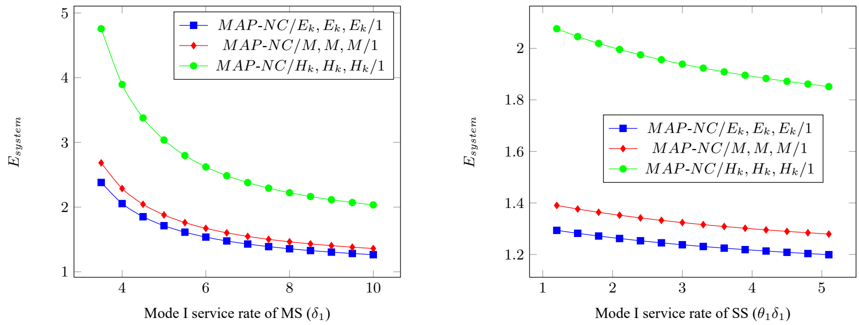
(a) (b)

Figure 15 E_{system} vs. mode I service rate of main server and standby server of HYP-EXPA (see online version for colours)



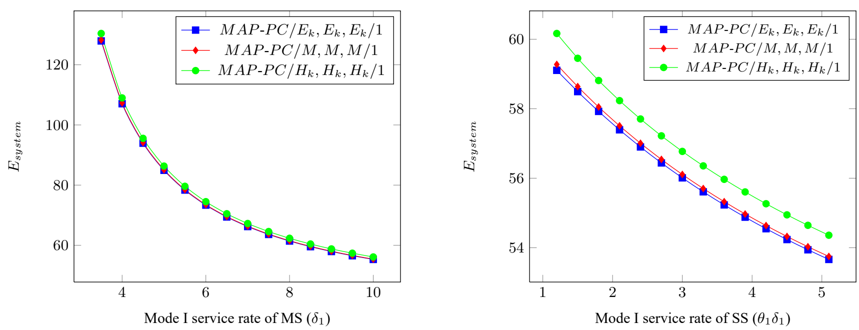
(a) (b)

Figure 16 E_{system} vs. mode I service rate of main server and standby server of MAP-NC (see online version for colours)



(a) (b)

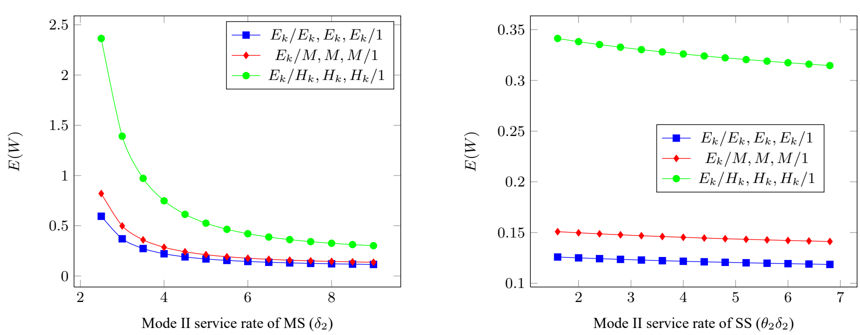
Figure 17 E_{system} vs. mode I service rate of main server and standby server of MAP-PC (see online version for colours)



(a) (b)

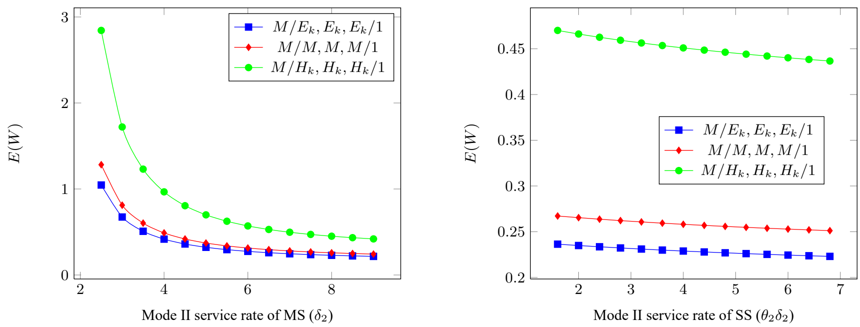
Comparing the mode II service scenarios of main server and standby server

Figure 18 $E(W)$ vs. mode II service rate of main server and standby server of ERLA (see online version for colours)



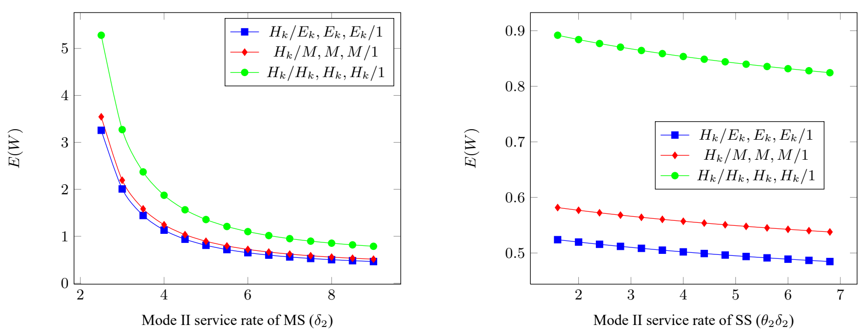
(a) (b)

Figure 19 $E(W)$ vs. mode II service rate of main server and standby server of EXPA (see online version for colours)



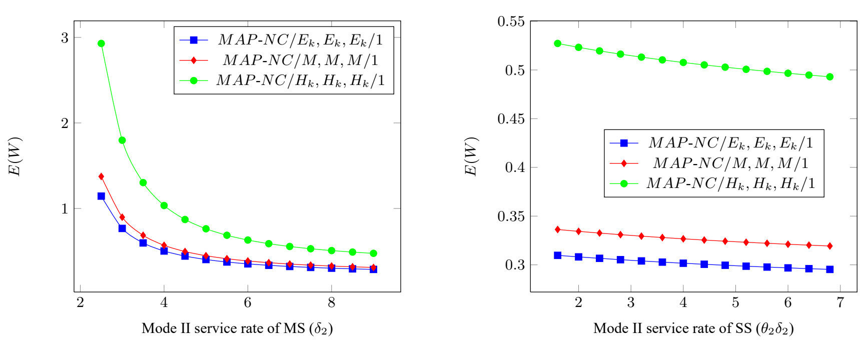
(a) (b)

Figure 20 $E(W)$ vs. mode II service rate of main server and standby server of HYP-ERLA (see online version for colours)



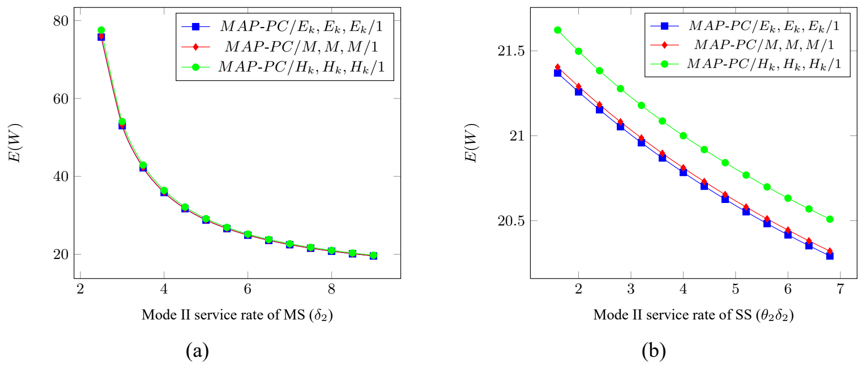
(a) (b)

Figure 21 $E(W)$ vs. mode II service rate of main server and standby server of MAP-NC (see online version for colours)



(a) (b)

Figure 22 $E(W)$ vs. mode II service rate of main server and standby server of MAP-PC (see online version for colours)



For comparing the mode I service scenarios of the MS and the SS. For the main server's mode I service, we fix $\lambda = 2$; $\delta_2 = 5$; $\delta_3 = 4$; $\theta_1 = 0.7$; $\theta_2 = 0.7$; $\theta_3 = 0.6$; $\Psi = 6$; $\eta = 3$; $\zeta = 8$; $\tau = 1$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.4$; $d_2 = 0.6$; $\sigma = 4$. For the standby server's mode I service, we prefer $\lambda = 2$; $\delta_1 = 6$; $\delta_2 = 6$; $\delta_3 = 5$; $\theta_2 = 0.7$; $\theta_3 = 0.6$; $\Psi = 5$; $\eta = 3$; $\zeta = 8$; $\tau = 1$; $c_1 = 0.5$; $d_1 = 0.5$; $\sigma = 4$; $c_2 = 0.4$; $d_2 = 0.6$. From Figures 13–17, we observe that the E_{system} decreases while increasing the main server and standby server service rate of mode I service. Consider the distinct arrangements of arrival and service times, the expected number of customers in the system decreases fastly during main server service and slowly during standby server service except for MAP-PC. In the case of MAP-PC, E_{system} decreases rapidly and the results of all service times converge during main server service and decrease slowly during standby server service compared to the main server service. Next, comparing the mode II service scenarios of the MS and the SS. For the main server's mode II service, we consider $\lambda = 2$; $\delta_1 = 8$; $\delta_3 = 7$; $\theta_1 = 0.5$; $\theta_2 = 0.5$; $\theta_3 = 0.4$; $\Psi = 12$; $\eta = 9$; $\zeta = 7$; $\tau = 2$; $\sigma = 8$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.6$; $d_2 = 0.4$. For the standby server's mode II service, we fix $\lambda = 2$; $\delta_1 = 8$; $\delta_2 = 8$; $\delta_3 = 7$; $\theta_1 = 0.5$; $\theta_3 = 0.4$; $\Psi = 12$; $\eta = 9$; $\zeta = 7$; $\tau = 2$; $\sigma = 8$; $c_1 = 0.5$; $d_1 = 0.5$; $c_2 = 0.6$; $d_2 = 0.4$. From Figures 18–22, we observe that the $E(W)$ decreases when increasing the main server and standby server service rate of mode II service with the different groupings of service and arrival times. In Figure 22, consider the service times, the expected waiting time decrease rapidly during main server service and it also converges but while service offering by standby server, the expected waiting time decreases slower than the main server service. In Figures 18–21, the $E(W)$ decreases gradually during the main server service than the standby server service. Therefore, while the MS or SS rendering service to the customers, the E_{system} and $E(W)$ decreases due to server availability in the system.

10 Conclusions

We consider the customers whose arrival based on the Markovian arrival process with two types of heterogeneous service, additional service in which service follows a phase-type distribution with start-up time, standby server, impatient behaviour of customers, server vacation, breakdown and phase-type repair are considered in this manuscript. We have analysed the cost analysis for optimisation of our model, waiting time distribution and the busy period of the system for our model. By using the numerical values of arrival and service times, we obtained tabulation of numerical values and examined the pictorial representations of 2D and 3D graphs. However, it precisely and clearly communicated the consequence of distinct parameters related to our model on the performance characteristics of the system measures and optimisation of cost analysis. The overall motivation of our model is that we have studied the situations faced by the customers who choose their day-to-day lifestyle in the banking sector as well as the internet banking system who are wishing to deposit or withdrawal their transactions and the someone seek additional service. It is an interesting viewpoint and much helpful to understand the phase-type service from the involvement of customers either go to bank directing for using banking sector or use internet banking wherever they are at the time of transaction service whether the main server or standby server is available in the system.

References

Ayyappan, G. and Karpagam, S. (2019) 'An $M^{[X]}/G(a,b)/1$ queueing system with server breakdown and repair, stand-by server and single vacation', *Internationnal Journal of Mathematics in Operational Research*, Vol. 14, No. 2, pp.221–235, DOI: 10.1504/IJMOR.2019.097756.

Ayyappan, G. and Udayageetha, J. (2020) 'Transient analysis of $M^{[X1]}, M^{[X2]}/G_1, G_2/1$ retrial queueing system with priority services, working breakdown, start up/close down time, Bernoulli vacation, reneging and balking', *Pakistan Journal of Statistics and Operation Research*, Vol. 16, No. 1, pp.203–216 [online] http://dx.doi.org/10.18187/pjsor.v16i1.2181.

Bagyam, J.E.A. and Udayachandrika, K. (2011) 'Non-persistent retrial queueing system with two types of heterogeneous service', *Global Journal of Theoretical and Applied Mathematics Sciences*, Vol. 1, No. 2, pp.157–164, ISSN: 2248-9916.

Baruah, M., Madan, K.C. and Eldabi, T. (2012) 'Balking and re-service in a vacation queue with batch arrival and two types of heterogeneous service', *Journal of Mathematics Research*, Vol. 4, No. 4, pp.114–124 [online] http://dx.doi.org/10.5539/jmr.v4n4p114.

Chakravarthy, S.R. (2010) 'Markovian arrival process', *Wiley Encyclopaedia of Operation Research and Management Science* [online] https://doi.org/10.1002/9780470400531.eorms0499.

Chakravarthy, S.R. and Agarwal, A. (2003) 'Analysis of a machine repair problem with an unreliable server and phase type repairs and services', *Wiley Periodicals, Inc. Naval Research Logistics*, Vol. 50, pp.462–480, DOI: 10.1002/nav.10069.

Choudhury, G. and Kalita, C.R. (2017) 'An $M/G/1$ queue with two types of general heterogeneous service and optional repeated service subject to server's breakdown and delayed repair', *Quality Technology & Quantitative Management*, pp.1–33 [online] https://doi.org/10.1080/16843703.2017.1331499.

Dudin, A.N. and Nishimura, S. (1999) 'Optimal control for a $BMAP/G/1$ queue with two service modes', *Mathematical Problems in Engineering*, Vol. 5, No. 3, pp.255–273.

Jain, M., Rakhee and Maheshwari, S. (2004) 'N-policy for a machine repair system with spares and reneging', *Applied Mathematical Modelling*, Vol. 28, pp.513–531 [online] https://doi.org/10.1016/j.apm.2003.10.013.

Kalita, P. and Choudhury, G. (2021) 'A single server queue under random vacation policy', *RAIRO-Operations Research*, Vol. 55, pp.225–251.

Kannadasan, G. and Sathiyamoorthi, N. (2021) 'The fuzzy queue with single working vacation serving at the slower rate during the start-up period', *Advances and Applications in Mathematical Sciences*, Vol. 20, No. 5, pp.761–773.

Ke, J-C. (2001) 'The control policy of an $M^{[X]}/G/1$ queueing system with server startup and two vacation types', *Mathematical Methods of Operations Research*, Vol. 54, pp.471–490 [online] https://doi.org/10.1007/s001860100167.

Ke, J-C. (2003) 'The optimal control of an M/G/1 queueing system with servervacations, startup and breakdowns', *Computers & Industrial Engineering*, Vol. 44, pp.567–579 [online] https://doi.org/10.1016/S0360-8352(02)00235-8.

Kemeny, J.G. and Snell, J.L. (1960) *Finite Markov Chains, The University Series in Undergraduate Mathematics*, Springer-Verlag, New York; Berlin, Heidelberg; Tokyo, Van Nostrand, Priceton, NJ, USA.

Kumar, M.S. and Arumuganathan, R. (2008) 'On the single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service', *Quality Technology & Quantitative Management*, Vol. 5, No. 2, pp.145–160, DOI: 10.1080/16843703.2008.11673393.

Kumar, R. and Sharma, S. (2019) 'Transient analysis of an M/M/c queuing system with retention of reneging customers', *International Journal of Operational Research*, Vol. 36, No. 1, pp.78–91.

Latouche, G. and Ramaswami, V. (1993) 'A logarithmic reduction algorithm for quasi-birth-death processes', *Journal of Applied Probability*, Vol. 30, No. 3, pp.650–674 [online] https://doi.org/10.2307/3214773.

Neuts, M.F. (1979) 'A versatile Markovian point process', *Journal of Applied Probability*, Vol. 16, No. 4, pp.764–779.

Nobel, R.D. and Tijms, H.C. (1999) 'Optimal control for an $M^X/G/1$ queue with two service modes', *European Journal of Operational Research*, Vol. 113, pp.610–619, DOI: 10.1016/s0377-2217(98)0008.

Singh, C.J., Kaur, S. and Jain, M. (2020) 'Unreliable server retrial G-queue with bulk arrival, optional additional service and delayed repair', *International Journal of Operational Research*, Vol. 38, No. 1, pp.82–111.

Sudhesh, R. and Azhagappan, A. (2019) 'Analysis of an M/M/c queue with heterogeneous servers, balking and reneging', *International Journal of Operational Research*, Vol. 36, No. 3, pp.293–309.

Wang, K-H., Wang, T-Y. and Pearn, WL. (2007) 'Optimal control of the N policy M/G/1 queueing system with server breakdowns and general startup times', *Applied Mathematical Modelling*, Vol. 31, pp.2199–2212.