Incremental algorithm for acoustoelastic theory of large static pre-deformed fluid-saturated porous media

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Abstract: The incremental acoustoelastic equations for fluid-saturated porous media (FSPM) under the large static pre-deformation are derived in this paper by incremental loading method based on classic acoustoelastic theory of FSPM, which provides quantitative acoustoelastic relation of FSPM with arbitrary constitutive equation. Isotropic FSPM with third-order constitutive equation are taken as an example to give the relation between wave velocity and confining pressure and discuss the effect of loading step on acoustoelastic relations of isotropic FSPM under closed-pore jacketed condition and opened-pore jacketed condition.

Keywords: acoustoelasticity; large static pre-deformation; fluid-saturated porous media; FSPM; incremental algorithm.

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1 Introduction

Acoustoelasticity of fluid-saturated porous media (FSPM) is a nonlinear acoustical phenomenon that describes the influence of stress or deformation states of FSPM on wave velocities (Murnaghan, 1951; Tian and Hu, 2010). It has an extremely important role on the ultrasonic non-destructive measurement of stresses (Crecraft, 1967; Lu and Roy, 1996) and has wide application in the fields of geotechnical engineering, petroleum geophysics for the estimation of crustal stress (Crampin and Peacock, 2005; Tian and Wang, 2006; Winkler and McGowan, 2004).

Since Biot (1956a, 1956b, 1956c, 1956d) established the poroelastic theory of FSPM, some researches on experimental study, theoretical analysis, and numerical simulation of the acoustoelastic properties of FSPM had been investigated. Tosaya (1982) measured the elastic-wave velocities of fluid-saturated clay-bearing rocks subjected to the confining pressure by the ultrasonic method. Winkler and McGowan (2004) measured the nonlinear acoustoelastic constants of dry and water-saturated rocks. Grinfeld and Norris (1996) derived wave speed formulas in closed-pore jacketed and open-pore jacketed configuration to find a complete set of seven third order elastic moduli for poroelastic medium. Considering the finite strain, Ba et al. (2013) derive the wave propagation equation by substituting the 11-term potential function and the Biot kinetic energy function into Lagrange equation. Wang and Tian (2014) gave the acoustoelastic theory for FSPM in the natural and initial coordinates based on the finite deformation theory of the continuum and poroelastic theory and present an explicit formulation about the influence of the effective stress and the fluid pore-pressure on wave velocities.

At present, the acoustoelastic theory of FSPM is based on the assumption of finite deformation. But, if the FSPM experiences a long-term loading process or has larger initial stress, the assumption of finite deformation in classic acoustoelastic theory is failed. However, the initial stress σ_{ij} must satisfy $(\sigma_{ij}\sigma_{ij})(c_{ijklmn}c_{ijklmn}) << (c_{ijkl}c_{ijkl})^2$ in the classic acoustoelastic theory, where c_{ijkl} and c_{ijklmn} are the second-order and third-order elastic constants (Johnson and Rasolofosaon, 1996). Because the third-order elastic constants are usually much greater than the second-order elastic constants, this means that the initial deformation induced by the initial stress applicable for the classic acoustoelastic theory must be infinitesimal, which has been verified in the relevant experiments (Abiza et al., 2012). Furthermore, the FSPM usually have complex nonlinear constitutive law, rather than the constitutive law of the third-order elastic constants in the classic acoustoelastic theory, especially at the large pre-deformation. Although the higher-order elastic constants may be introduced into constitutive law, this will increase the difficulty greatly (Abiza et al., 2012).

In order to solve the above-mentioned problem, we introduce the incremental algorithm of acoustoelastic theory for large static pre-deformed FSPM with the complex constitutive relation by incremental loading method, and then discuss the effect of loading steps on acoustoelastic relations of large static pre-deformed isotropic FSPM under closed-pore jacketed condition and opened-pore jacketed condition.

2 Incremental algorithm for acoustoelastic theory of large static pre-deformed FSPM in natural coordinate

There are three configurations in pre-deformed FSPM, which include natural configuration, initial configuration, and finial configuration. The positions of a particle in the FSPM at natural, initial, and finial configurations are measured by position vectors $\boldsymbol{\xi}$, $\mathbf{X}$, and $\mathbf{x}$, respectively, all directed from the origin of a common Cartesian coordinate systems. A physical variable in the natural, initial, and finial configurations is designated by superscript labels 0, i, and f, respectively. The components of physical quantities which refer to the natural, initial, and finial configurations are denoted by Greek subscripts, upper case Roman subscripts, and lower case Roman subscripts, respectively.

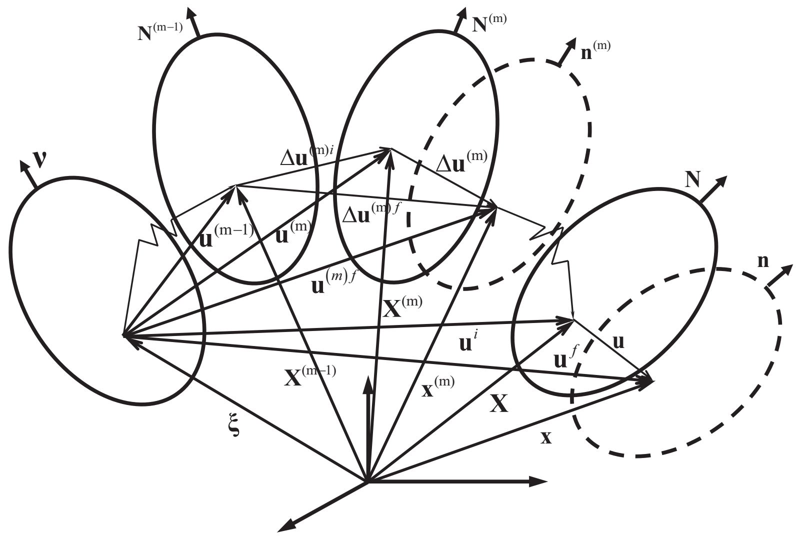
Figure 1 shows the process of statically stepwise loading in initial large pre-deformed FSPM, which loads FSPM at natural state into large pre-deformation state by M steps and the value of loading step depends on the applicable conditions of classic acoustoelastic theory under the infinitesimal pre-deformation. For step m, the position of a solid-skeleton particle at starting state is measured by position vector $\mathbf{X}^{(m-1)}$. FSPM in starting state will turn into terminal state given single-step static loading and the position of a solid-skeleton particle at terminal state is measured by position vector $\mathbf{X}^{(m)}$. Similarly, FSPM in terminal state will turn into ultimate state superposed acoustic dynamic motion and the position of a solid-skeleton particle at ultimate state is measured by position vector $\mathbf{X}^{(m)}$.

There are $m - 1$ loading steps from natural state to starting state in the m^{th} step, and the deformations of the solid-skeleton component and the fluid component in each step is infinitesimal and static. From the natural state to the starting state in the m^{th} step, the natural state to the terminal state in the m^{th} step, and the natural state to the ultimate state in the m^{th} step, the displacements of the solid-skeleton particle is denoted by $\mathbf{u}^{(m-1)}(\xi)$, $\mathbf{u}^{(m)}(\xi)$, and $\mathbf{u}^{(m)f}(\xi)$ in the natural coordinates, and the displacements of the fluid particle is denoted by $\mathbf{U}^{(m-1)}(\xi)$, $\mathbf{U}^{(m)}(\xi)$, and $\mathbf{U}^{(m)f}(\xi)$. The incremental displacements of the solid-skeleton component and the fluid component are denoted by $\Delta\mathbf{u}^{(m)i}$, $\Delta\mathbf{u}^{(m)f}$ and $\Delta\mathbf{U}^{(m)i}$, $\Delta\mathbf{U}^{(m)f}$, respectively. They are related to the vectors by

$$\left.\begin{aligned}\Delta\mathbf{u}^{(m)i}(\xi) &= \mathbf{u}^{(m)}(\xi) - \mathbf{u}^{(m-1)}(\xi)\\ \Delta\mathbf{u}^{(m)f}(\xi) &= \mathbf{u}^{(m)f}(\xi) - \mathbf{u}^{(m-1)}(\xi)\end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned}\Delta\mathbf{U}^{(m)i}(\xi) &= \mathbf{U}^{(m)}(\xi) - \mathbf{U}^{(m-1)}(\xi)\\ \Delta\mathbf{U}^{(m)f}(\xi) &= \mathbf{U}^{(m)f}(\xi) - \mathbf{U}^{(m-1)}(\xi)\end{aligned}\right\} \tag{2}$$

Figure 1 Incremental loading of solid particles for large pre-deformed FSPM



The differences of $\Delta\mathbf{u}^{(m)i}(\xi)$ and $\Delta\mathbf{u}^{(m)f}(\xi)$, $\Delta\mathbf{U}^{(m)i}(\xi)$ and $\Delta\mathbf{U}^{(m)f}(\xi)$ are the dynamic displacements induced by the small acoustic dynamic motion from the terminal state to the ultimate state in the m^{th} step,

$$\left.\begin{aligned}\Delta\mathbf{u}^{(m)}(\xi) &= \Delta\mathbf{u}^{(m)f}(\xi) - \Delta\mathbf{u}^{(m)i}(\xi)\\ \Delta\mathbf{U}^{(m)}(\xi) &= \Delta\mathbf{U}^{(m)f}(\xi) - \Delta\mathbf{U}^{(m)i}(\xi)\end{aligned}\right\} \tag{3}$$

In the global natural coordinates, the solid-skeleton Lagrangian strain tensors in the starting state, terminal state, and ultimate state in the m^{th} step are defined from the squares of the stretch tensor, respectively, as

$$\left.\begin{aligned}E_{\alpha\beta}^{(m-1)} &= \frac{1}{2}\left(\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\beta} + \frac{\partial u_\beta^{(m-1)}}{\partial\xi_\alpha} + \frac{\partial u_\lambda^{(m-1)}}{\partial\xi_\alpha}\frac{\partial u_\lambda^{(m-1)}}{\partial\xi_\beta}\right)\\ E_{\alpha\beta}^{(m)} &= \frac{1}{2}\left(\frac{\partial u_\alpha^{(m)}}{\partial\xi_\beta} + \frac{\partial u_\beta^{(m)}}{\partial\xi_\alpha} + \frac{\partial u_\lambda^{(m)}}{\partial\xi_\alpha}\frac{\partial u_\lambda^{(m)}}{\partial\xi_\beta}\right)\\ E_{\alpha\beta}^{(m)f} &= \frac{1}{2}\left(\frac{\partial u_\alpha^{(m)f}}{\partial\xi_\beta} + \frac{\partial u_\beta^{(m)f}}{\partial\xi_\alpha} + \frac{\partial u_\lambda^{(m)f}}{\partial\xi_\alpha}\frac{\partial u_\lambda^{(m)f}}{\partial\xi_\beta}\right)\end{aligned}\right\} \tag{4}$$

The increments of Lagrangian strain tensors from the starting state to the terminal state and the starting state to the ultimate state in the m^{th} step are denoted by, respectively,

$$\left.\begin{aligned}\Delta E_{\alpha\beta}^{(m)i} &= E_{\alpha\beta}^{(m)} - E_{\alpha\beta}^{(m-1)}\\ &= \frac{1}{2}\left(\begin{aligned}&\frac{\partial\Delta u_\alpha^{(m)i}}{\partial\xi_\beta} + \frac{\partial\Delta u_\beta^{(m)i}}{\partial\xi_\alpha} + \frac{\partial u_\lambda^{(m-1)}}{\partial\xi_\alpha}\frac{\partial\Delta u_\lambda^{(m)i}}{\partial\xi_\beta}\\ &+\frac{\partial\Delta u_\lambda^{(m)i}}{\partial\xi_\alpha}\frac{\partial u_\lambda^{(m-1)}}{\partial\xi_\beta} + \frac{\partial\Delta u_\lambda^{(m)i}}{\partial\xi_\alpha}\frac{\partial\Delta u_\lambda^{(m)i}}{\partial\xi_\beta}\end{aligned}\right)\\ \Delta E_{\alpha\beta}^{(m)f} &= E_{\alpha\beta}^{(m)f} - E_{\alpha\beta}^{(m-1)}\\ &= \frac{1}{2}\left(\begin{aligned}&\frac{\partial\Delta u_\alpha^{(m)f}}{\partial\xi_\beta} + \frac{\partial\Delta u_\beta^{(m)f}}{\partial\xi_\alpha} + \frac{\partial u_\lambda^{(m-1)}}{\partial\xi_\alpha}\frac{\partial\Delta u_\lambda^{(m)f}}{\partial\xi_\beta}\\ &+\frac{\partial\Delta u_\lambda^{m(f)}}{\partial\xi_\alpha}\frac{\partial u_\lambda^{(m-1)}}{\partial\xi_\beta} + \frac{\partial\Delta u_\lambda^{(m)f}}{\partial\xi_\alpha}\frac{\partial\Delta u_\lambda^{(m)f}}{\partial\xi_\beta}\end{aligned}\right)\end{aligned}\right\} \tag{5}$$

Assuming the superposed dynamic motion is small, that is $\|\Delta u_\alpha^{(m)}\| \ll \|\Delta u_\alpha^{(m)i}\|$. The incremental Lagrangian strain tensors from the terminal state to the ultimate state induced by the small-amplitude disturbance are deduced approximately,

$$\begin{aligned}\Delta E_{\alpha\beta}^{(m)} &= \Delta E_{\alpha\beta}^{(m)f} - \Delta E_{\alpha\beta}^{(m)i}\\ &= \frac{1}{2}\left(\frac{\partial\Delta u_\alpha^{(m)}}{\partial\xi_\beta} + \frac{\partial\Delta u_\beta^{(m)}}{\partial\xi_\alpha} + \frac{\partial u_\lambda^{(m)}}{\partial\xi_\alpha}\frac{\partial\Delta u_\lambda^{(m)}}{\partial\xi_\beta} + \frac{\partial\Delta u_\lambda^{(m)}}{\partial\xi_\alpha}\frac{\partial u_\lambda^{(m)}}{\partial\xi_\beta}\right)\end{aligned} \tag{6}$$

The strain in the fluid is defined by the dilatation. The fluid Lagrangian strain tensors from the starting state to the terminal state and the starting state to the ultimate state in the m^{th} step are expressed as, respectively

$$\left.\begin{aligned}\Delta\varepsilon^{(m)i} &= \frac{\partial\Delta U_\alpha^{(m)i}}{\partial\xi_\alpha}\\ \Delta\varepsilon^{(m)f} &= \frac{\partial\Delta U_\alpha^{(m)f}}{\partial\xi_\alpha}\end{aligned}\right\} \tag{7}$$

The incremental Lagrangian strain tensor from the terminal state to the ultimate state induced by the small-amplitude disturbance is expressed as

$$\Delta\varepsilon^{(m)} = \frac{\partial\Delta U_\alpha^{(m)}}{\partial\xi_\alpha}, \tag{8}$$

where $\Delta U_\alpha^{(m)} = \Delta U_\alpha^{(m)f} - \Delta U_\alpha^{(m)i}$.

The total stress $\mathbf{T}$ of the FSPM particle is usually defined as the sum of the solid-skeleton-stress $\bar{\mathbf{T}}$ component and the fluid-stress component $\mathbf{s}$. In natural coordinates, the stresses of a solid-skeleton particle in the FSPM at starting, terminal, and ultimate states in the m^{th} step are measured by Piola-Kirchhoff stress tensor $\mathbf{T}^{(m-1)}(\xi, t)$, $\mathbf{T}^{(m)}(\xi, t)$, and $\mathbf{T}^{(m)f}(\xi, t)$, respectively, and the stresses of a fluid particle are measured by stress tensor $\mathbf{s}^{(m-1)}(\xi, t)$, $\mathbf{s}^{(m)}(\xi, t)$, and $\mathbf{s}^{(m)f}(\xi, t)$. The incremental Kirchhoff stress tensors of a solid-skeleton particle and a fluid particle from the starting state to the terminal state in the m^{th} step are denoted by, respectively,

$$\left.\begin{aligned}\Delta\bar{\mathbf{T}}^{(m)i}(\xi,t)&=\bar{\mathbf{T}}^{(m)}(\xi,t)-\bar{\mathbf{T}}^{(m-1)}(\xi,t)\\ \Delta\mathbf{s}^{(m)i}(\xi,t)&=\mathbf{s}^{(m)}(\xi,t)-\mathbf{s}^{(m-1)}(\xi,t)\end{aligned}\right\}\tag{9}$$

The incremental Kirchhoff stress tensors of a solid-skeleton particle and a fluid particle from the starting state to the ultimate state in the m^{th} step are denoted by, respectively,

$$\left.\begin{aligned}\Delta\bar{\mathbf{T}}^{(m)f}(\xi,t)&=\bar{\mathbf{T}}^{(m)f}(\xi,t)-\bar{\mathbf{T}}^{(m-1)}(\xi,t)\\ \Delta\mathbf{s}^{(m)f}(\xi,t)&=\mathbf{s}^{(m)f}(\xi,t)-\mathbf{s}^{(m-1)}(\xi,t)\end{aligned}\right\}\tag{10}$$

The incremental Kirchhoff stress tensors of a solid-skeleton particle and a fluid particle from the terminal state to the ultimate state induced by the small-amplitude disturbance is given by

$$\left.\begin{aligned}\Delta\bar{\mathbf{T}}^{(m)}(\xi,t)&=\Delta\bar{\mathbf{T}}^{(m)f}(\xi,t)-\Delta\bar{\mathbf{T}}^{(m)i}(\xi,t)\\ \Delta\mathbf{s}^{(m)}(\xi,t)&=\Delta\mathbf{s}^{(m)f}(\xi,t)-\Delta\mathbf{s}^{(m)i}(\xi,t)\end{aligned}\right\}\tag{11}$$

The pre-deformation from natural state to initial state is static, so the particles of solid-skeleton and fluid at the starting state and the terminal state in the m^{th} step must satisfy the equations of equilibrium in the natural coordinates, respectively,

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[\bar{T}_{\beta\gamma}^{(m-1)}\left(\delta_{\alpha\gamma}+\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\gamma}\right)\right]=0\\ &\frac{\partial}{\partial\xi_\beta}\left[s^{(m-1)}\left(\delta_{\alpha\beta}+\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\beta}\right)\right]=0\end{aligned}\right\}\tag{12}$$

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[\bar{T}_{\beta\gamma}^{(m)}\left(\delta_{\alpha\gamma}+\frac{\partial u_\alpha^{(m)}}{\partial\xi_\gamma}\right)\right]=0\\ &\frac{\partial}{\partial\xi_\beta}\left[s^{(m)}\left(\delta_{\alpha\beta}+\frac{\partial u_\alpha^{(m)}}{\partial\xi_\beta}\right)\right]=0\end{aligned}\right\}\tag{13}$$

The equation of motion for solid-skeleton particles and fluid particles at the ultimate state in the m^{th} step can be expressed in the natural coordinates,

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[\bar{T}_{\beta\gamma}^{(m)f}\left(\delta_{\alpha\gamma}+\frac{\partial u_\alpha^{(m)f}}{\partial\xi_\gamma}\right)\right]=\rho_{11}^{o}\frac{\partial^2\Delta u_\alpha^{(m)f}}{\partial t^2}+\rho_{12}^{o}\frac{\partial^2\Delta U_\alpha^{(m)f}}{\partial t^2}\\ &\frac{\partial}{\partial\xi_\beta}\left[s^{(m)f}\left(\delta_{\alpha\beta}+\frac{\partial u_\alpha^{(m)f}}{\partial\xi_\beta}\right)\right]=\rho_{12}^{o}\frac{\partial^2\Delta u_\alpha^{(m)f}}{\partial t^2}+\rho_{22}^{o}\frac{\partial^2\Delta U_\alpha^{(m)f}}{\partial t^2}\end{aligned}\right\}\tag{14}$$

where ρ_{11}^o, ρ_{12}^o, and ρ_{22}^o represent the effective density of the solid moving in the fluid, the coupling density between the solid and the fluid, and the effective density of the fluid flowing through the pores, which are at natural state. Subtracting equation (12) from equations (13) and (14), we obtain the equation of equilibrium and the equation of motion, respectively, for the incremental displacement $\Delta\mathbf{u}^{(m)i}(\xi, t)$ and $\Delta\mathbf{u}^{(m)f}(\xi, t)$ in the m^{th} step at natural coordinates,

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[\Delta\bar{T}_{\alpha\beta}^{(m)i}+\Delta\bar{T}_{\beta\gamma}^{(m)i}\frac{\partial\Delta u_\alpha^{(m)i}}{\partial\xi_\gamma}+\Delta\bar{T}_{\beta\gamma}^{(m)i}\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\gamma}+\bar{T}_{\beta\gamma}^{(m-1)}\frac{\partial\Delta u_\alpha^{(m)i}}{\partial\xi_\gamma}\right]=0\\ &\frac{\partial}{\partial\xi_\beta}\left[\Delta s^{(m)i}\left(\delta_{\alpha\beta}+\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\beta}\right)+s^{(m-1)}\frac{\partial\Delta u_\alpha^{(m)i}}{\partial\xi_\beta}+\Delta s^{(m)i}\frac{\partial\Delta u_\alpha^{(m)i}}{\partial\xi_\beta}\right]=0\end{aligned}\right\}\tag{15}$$

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[\Delta\bar{T}_{\alpha\beta}^{(m)f}+\Delta\bar{T}_{\beta\gamma}^{(m)f}\frac{\partial\Delta u_\alpha^{(m)f}}{\partial\xi_\gamma}+\Delta\bar{T}_{\beta\gamma}^{(m)f}\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\gamma}+\bar{T}_{\beta\gamma}^{(m-1)}\frac{\partial\Delta u_\alpha^{(m)f}}{\partial\xi_\gamma}\right]\\ &=\rho_{11}^{0}\frac{\partial^2\Delta u_\alpha^{(m)f}}{\partial t^2}+\rho_{12}^{0}\frac{\partial^2\Delta U_\alpha^{(m)f}}{\partial t^2}\\ &\frac{\partial}{\partial\xi_\beta}\left[\Delta s^{(m)f}\left(\delta_{\alpha\beta}+\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\beta}\right)+s^{(m-1)}\frac{\partial\Delta u_\alpha^{(m)f}}{\partial\xi_\beta}+\Delta s^{(m)f}\frac{\partial\Delta u_\alpha^{(m)f}}{\partial\xi_\beta}\right]\\ &=\rho_{12}^{0}\frac{\partial^2\Delta u_\alpha^{(m)f}}{\partial t^2}+\rho_{22}^{0}\frac{\partial^2\Delta U_\alpha^{(m)f}}{\partial t^2}\end{aligned}\right\}\tag{16}$$

Similarly, the equation of motion for the incremental displacement $\Delta\mathbf{u}^{(m)}(\xi, t)$ in the m^{th} step can be given by subtracting equation (15) from equation (16),

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[\Delta\bar{T}_{\alpha\beta}^{(m)}+\Delta\bar{T}_{\beta\gamma}^{(m)}\frac{\partial\Delta u_\alpha^{(m)i}}{\partial\xi_\gamma}+\Delta\bar{T}_{\beta\gamma}^{(m)i}\frac{\partial\Delta u_\alpha^{(m)}}{\partial\xi_\gamma}+\Delta\bar{T}_{\beta\gamma}^{(m)}\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\gamma}+\bar{T}_{\beta\gamma}^{(m-1)}\frac{\partial\Delta u_\alpha^{(m)}}{\partial\xi_\gamma}\right]\\ &=\rho_{11}^{0}\frac{\partial^2\Delta u_\alpha^{(m)}}{\partial t^2}+\rho_{12}^{0}\frac{\partial^2\Delta U_\alpha^{(m)}}{\partial t^2}\\ &\frac{\partial}{\partial\xi_\beta}\left[\Delta s^{(m)}\left(\delta_{\alpha\beta}+\frac{\partial u_\alpha^{(m-1)}}{\partial\xi_\beta}\right)+s^{(m-1)}\frac{\partial\Delta u_\alpha^{(m)}}{\partial\xi_\beta}+\Delta s^{(m)i}\frac{\partial\Delta u_\alpha^{(m)}}{\partial\xi_\beta}+\Delta s^{(m)}\frac{\partial\Delta u_\alpha^{(m)i}}{\partial\xi_\beta}\right]\\ &=\rho_{12}^{0}\frac{\partial^2\Delta u_\alpha^{(m)}}{\partial t^2}+\rho_{22}^{0}\frac{\partial^2\Delta U_\alpha^{(m)}}{\partial t^2}\end{aligned}\right\}\tag{17}$$

Assuming the FSPM has a hyperelastic constitutive relation, the internal energy $W(\Delta\mathbf{E}, \Delta\varepsilon)$ can be expanded about the state of zero incremental strain by Taylor series expansion for an isentropic thermodynamic process and considering stress-strain energy relation in hyperelastic FSPM, constitutive equations for the incremental stress tensors

$\Delta T_{\alpha\beta}^{(m)i}$, $\Delta T_{\alpha\beta}^{(m)f}$ and $\Delta \mathbf{s}^{(m)i}$, $\Delta \mathbf{s}^{(m)f}$, in the natural coordinates can be derived, respectively, by neglecting the higher-order terms,

$$\left.\begin{aligned}
\Delta \bar{T}_{\alpha\beta}^{(m)i} &= c_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)i} + M_{\alpha\beta}^{(m-1)} \Delta\varepsilon^{(m)i} \\
&\quad + \frac{1}{2} c_{\alpha\beta\gamma\delta\varepsilon\eta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)i} \Delta E_{\varepsilon\eta}^{(m)i} \\
&\quad + s_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)i} \Delta\varepsilon^{(m)i} + \frac{1}{2} N_{\alpha\beta}^{(m-1)} \left(\Delta\varepsilon^{(m)i}\right)^2 \\
\Delta \bar{T}_{\alpha\beta}^{(m)f} &= c_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)f} + M_{\alpha\beta}^{(m-1)} \Delta\varepsilon^{(m)f} \\
&\quad + \frac{1}{2} c_{\alpha\beta\gamma\delta\varepsilon\eta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)f} \Delta E_{\varepsilon\eta}^{(m)f} \\
&\quad + s_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)f} \Delta\varepsilon^{(m)f} + \frac{1}{2} N_{\alpha\beta}^{(m-1)} \left(\Delta\varepsilon^{(m)f}\right)^2 \\
\Delta s^{(m)i} &= M_{\alpha\beta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)i} + M^{(m-1)} \Delta\varepsilon^{(m)i} \\
&\quad + \frac{1}{2} s_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)i} \Delta E_{\gamma\delta}^{(m)i} \\
&\quad + N_{\alpha\beta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)i} \Delta\varepsilon^{(m)i} + \frac{1}{2} N^{(m-1)} \left(\Delta\varepsilon^{(m)i}\right)^2 \\
\Delta s^{(m)f} &= M_{\alpha\beta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)f} + M^{(m-1)} \Delta\varepsilon^{(m)f} \\
&\quad + \frac{1}{2} s_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)f} \Delta E_{\gamma\delta}^{(m)f} \\
&\quad + N_{\alpha\beta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)f} \Delta\varepsilon^{(m)f} + \frac{1}{2} N^{(m-1)} \left(\Delta\varepsilon^{(m)f}\right)^2
\end{aligned}\right\} \quad (18)$$

where $c_{\alpha\beta\gamma\delta}^{(m-1)}$ and $c_{\alpha\beta\gamma\delta\varepsilon\eta}^{(m-1)}$ are the isentropic second-order and third-order constants of the solid-skeleton, $M_{\alpha\beta}^{(m-1)}$, $s_{\alpha\beta\gamma\delta}^{(m-1)}$, and $N_{\alpha\beta}^{(m-1)}$ are the isentropic coupling constants between the solid and the fluid, $M^{(m-1)}$ and $N^{(m-1)}$ are the isentropic fluid constants at starting state in the m^{th} step. Assuming $\| \Delta E_{\alpha\beta}^{(m)} \| \ll \| \Delta E_{\alpha\beta}^{(m)i} \|$, constitutive equation for the incremental stress tensors $\Delta T_{\alpha\beta}^{(m)}$ and $\Delta s^{(m)}$ can be derived from equation (18),

$$\left.\begin{aligned}
\Delta \bar{T}_{\alpha\beta}^{(m)} &= \left(c_{\alpha\beta\gamma\delta}^{(m-1)} + s_{\alpha\beta\gamma\delta}^{(m-1)} \Delta\varepsilon^{(m)i} \right) \Delta E_{\gamma\delta}^{(m)} \\
&\quad + c_{\alpha\beta\gamma\delta\varepsilon\eta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)i} \Delta E_{\varepsilon\eta}^{(m)} \\
&\quad + \left(M_{\alpha\beta}^{(m-1)} + s_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\gamma\delta}^{(m)i} + N_{\alpha\beta}^{(m-1)} \Delta\varepsilon^{(m)i} \right) \Delta\varepsilon^{(m)} \\
\Delta s^{(m)} &= \left(M_{\alpha\beta}^{(m-1)} + N_{\alpha\beta}^{(m-1)} \Delta\varepsilon^{(m)i} \right) \Delta E_{\alpha\beta}^{(m)} \\
&\quad + s_{\alpha\beta\gamma\delta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)} \Delta E_{\gamma\delta}^{(m)i} \\
&\quad + \left(M^{(m-1)} + N_{\alpha\beta}^{(m-1)} \Delta E_{\alpha\beta}^{(m)i} + N^{(m-1)} \Delta\varepsilon^{(m)i} \right) \Delta\varepsilon^{(m)}
\end{aligned}\right\} \quad (19)$$

Substituting equation (19) into equation (17) yields equation of motion for $\Delta \mathbf{u}^{(m)}(\boldsymbol{\xi}, t)$ and $\Delta \mathbf{U}^{(m)}(\boldsymbol{\xi}, t)$,

$$\left.\begin{aligned}
&\frac{\partial}{\partial \xi_\beta} \left[A_{\alpha\beta\gamma\delta}^{(m)} \frac{\partial \Delta u_\gamma^{(m)}}{\partial \xi_\delta} + \Phi_{\alpha\beta}^{(m)} \frac{\partial \Delta U_k^{(m)}}{\partial \xi_k} \right] \\
&= \rho_{11}^0 \frac{\partial^2 \Delta u_\alpha^{(m)}}{\partial t^2} + \rho_{12}^0 \frac{\partial^2 \Delta U_\alpha^{(m)}}{\partial t^2} \\
&\frac{\partial}{\partial \xi_\beta} \left[B_{\alpha\beta\gamma\delta}^{(m)} \frac{\partial \Delta u_\gamma^{(m)}}{\partial \xi_\delta} + Z_{\alpha\beta}^{(m)} \frac{\partial \Delta U_k^{(m)}}{\partial \xi_k} \right] \\
&= \rho_{12}^0 \frac{\partial^2 \Delta u_\alpha^{(m)}}{\partial t^2} + \rho_{22}^0 \frac{\partial^2 \Delta U_\alpha^{(m)}}{\partial t^2}
\end{aligned}\right\}, \quad (20)$$

where

$$A_{\alpha\beta\gamma\delta}^{(m)} = \bar{T}_{\beta\delta}^{(m-1)} \delta_{\alpha\gamma} + c_{\alpha\beta\gamma\delta}^{(m-1)} + c_{\alpha\beta\lambda\delta}^{(m-1)} \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} + c_{\rho\beta\gamma\delta}^{(m-1)} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} + c_{\rho\beta\lambda\delta}^{(m-1)} \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} + \Delta A_{\alpha\beta\gamma\delta}^{(m)}$$

$$\Phi_{\alpha\beta}^{(m)} = M_{\alpha\beta}^{(m-1)} + M_{\rho\beta}^{(m-1)} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} + \Delta\Phi_{\alpha\beta}^{(m)}$$

$$B_{\alpha\beta\gamma\delta}^{(m)} = s^{(m-1)} \delta_{\alpha\gamma} \delta_{\beta\delta} + M_{\gamma\delta}^{(m-1)} \delta_{\alpha\beta} + M_{\lambda\delta}^{(m-1)} \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} \delta_{\alpha\beta} + M_{\gamma\delta}^{(m-1)} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\beta} + M_{\lambda\delta}^{(m-1)} \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\beta} + \Delta B_{\alpha\beta\gamma\delta}^{(m)}$$

$$Z_{\alpha\beta}^{(m)} = M^{(m-1)} \delta_{\alpha\beta} + M^{(m-1)} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\beta} + \Delta Z_{\alpha\beta}^{(m)}$$

$$\begin{aligned}
\Delta A_{\alpha\beta\gamma\delta}^{(m)} &= \Delta \bar{T}_{\beta\delta}^{(m)i} \delta_{\alpha\gamma} \\
&\quad + \Delta\varepsilon^{(m)i} \left(\begin{aligned} & s_{\alpha\beta\gamma\delta}^{(m-1)} + s_{\alpha\beta\lambda\delta}^{(m-1)} \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} + s_{\rho\beta\gamma\delta}^{(m-1)} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} \\ & + s_{\rho\beta\gamma\delta}^{(m-1)} \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} \end{aligned} \right) \\
&\quad + \left(\begin{aligned} & c_{\rho\beta\gamma\delta}^{(m-1)} \frac{\partial \Delta u_\alpha^{(m)i}}{\partial \xi_\rho} + c_{\alpha\beta\lambda\delta}^{(m-1)} \frac{\partial \Delta u_\gamma^{(m)i}}{\partial \xi_\lambda} \\ & + c_{\rho\beta\lambda\delta}^{(m-1)} \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} \frac{\partial \Delta u_\alpha^{(m)i}}{\partial \xi_\rho} + c_{\rho\beta\lambda\delta}^{(m-1)} \frac{\partial \Delta u_\gamma^{(m)i}}{\partial \xi_\lambda} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} \end{aligned} \right) \\
&\quad + \left(c_{\alpha\beta\gamma\delta\varepsilon\eta}^{(m-1)} + c_{\rho\beta\gamma\delta\varepsilon\eta}^{(m-1)} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} \right) \\
&\quad \left(\frac{\partial \Delta u_\varepsilon^{(m)i}}{\partial \xi_\eta} + \frac{\partial u_\lambda^{(m-1)}}{\partial \xi_\varepsilon} \frac{\partial \Delta u_\lambda^{(m)i}}{\partial \xi_\eta} \right) \\
&\quad + \left(c_{\alpha\beta\lambda\delta\varepsilon\eta}^{(m-1)} + c_{\rho\beta\lambda\delta\varepsilon\eta}^{(m-1)} \frac{\partial u_\alpha^{(m-1)}}{\partial \xi_\rho} \right) \\
&\quad \frac{\partial u_\gamma^{(m-1)}}{\partial \xi_\lambda} \left(\frac{\partial \Delta u_\varepsilon^{(m)i}}{\partial \xi_\eta} + \frac{\partial u_\lambda^{(m-1)}}{\partial \xi_\varepsilon} \frac{\partial \Delta u_\lambda^{(m)i}}{\partial \xi_\eta} \right)
\end{aligned}$$

$$\Delta\Phi^{(m)}_{\alpha\beta}=M^{(m-1)}_{\rho\beta}\frac{\partial\Delta u^{(m)i}_{\alpha}}{\partial\xi_\rho}+s^{(m-1)}_{\alpha\beta\gamma\delta}\frac{\partial\Delta u^{(m)i}_{\gamma}}{\partial\xi_\delta}+N^{(m-1)}_{\alpha\beta}\Delta\varepsilon^{(m)i}+s^{(m-1)}_{\alpha\beta\gamma\delta}\frac{\partial u^{(m-1)}_{\lambda}}{\partial\xi_\gamma}\frac{\partial\Delta u^{(m)i}_{\lambda}}{\partial\xi_\delta}+\left(s^{(m-1)}_{\rho\beta\gamma\delta}\frac{\partial\Delta u^{(m)i}_{\gamma}}{\partial\xi_\delta}+s^{(m-1)}_{\rho\beta\gamma\delta}\frac{\partial u^{(m-1)}_{\lambda}}{\partial\xi_\gamma}\frac{\partial\Delta u^{(m)i}_{\lambda}}{\partial\xi_\delta}+N^{(m-1)}_{\rho\beta}\Delta\varepsilon^{(m)i}\right)\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\rho}$$

$$\begin{aligned}\Delta B^{(m)}_{\alpha\beta\gamma\delta}&=\Delta s^{(m)i}\delta_{\alpha\gamma}\delta_{\beta\delta}+\Delta\varepsilon^{(m)i}\left(\begin{array}{l}N^{(m-1)}_{\gamma\delta}\delta_{\alpha\beta}+N^{(m-1)}_{\lambda\delta}\dfrac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}\delta_{\alpha\beta}\\+N^{(m-1)}_{\gamma\delta}\dfrac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}+M^{(m-1)}_{\lambda\delta}\dfrac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}\dfrac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\end{array}\right)\\&+\left(\begin{array}{l}M^{(m-1)}_{\gamma\delta}\dfrac{\partial\Delta u^{(m)i}_{\alpha}}{\partial\xi_\beta}+M^{(m-1)}_{\lambda\delta}\dfrac{\partial\Delta u^{(m)i}_{\gamma}}{\partial\xi_\lambda}\delta_{\alpha\beta}\\+M^{(m-1)}_{\lambda\delta}\dfrac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}\dfrac{\partial\Delta u^{(m)i}_{\alpha}}{\partial\xi_\beta}\\+M^{(m-1)}_{\lambda\delta}\dfrac{\partial\Delta u^{(m)i}_{\gamma}}{\partial\xi_\lambda}\dfrac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\end{array}\right)\\&+s^{(m-1)}_{\varepsilon\eta\gamma\delta}\left(\delta_{\alpha\beta}+\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\right)\left(\frac{\partial\Delta u^{(m)i}_{\varepsilon}}{\partial\xi_\eta}+\frac{\partial u^{(m-1)}_{\lambda}}{\partial\xi_\varepsilon}\frac{\partial\Delta u^{(m)i}_{\lambda}}{\partial\xi_\eta}\right)\\&+s^{(m-1)}_{\varepsilon\eta\lambda\delta}\left(\delta_{\alpha\beta}+\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\right)\frac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}\left(\frac{\partial\Delta u^{(m)i}_{\varepsilon}}{\partial\xi_\eta}+\frac{\partial u^{(m-1)}_{\lambda}}{\partial\xi_\varepsilon}\frac{\partial\Delta u^{(m)i}_{\lambda}}{\partial\xi_\eta}\right)\end{aligned}$$

$$\Delta Z^{(m)}_{\alpha\beta}=M^{(m-1)}\frac{\partial\Delta u^{(m)i}_{\alpha}}{\partial\xi_\beta}+N^{(m-1)}\Delta\varepsilon^{(m)i}\left(\delta_{\alpha\beta}+\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\right)+N^{(m-1)}_{\gamma\delta}\left(\frac{\partial\Delta u^{(m)i}_{\gamma}}{\partial\xi_\delta}+\frac{\partial u^{(m-1)}_{\lambda}}{\partial\xi_\gamma}\frac{\partial\Delta u^{(m)i}_{\lambda}}{\partial\xi_\delta}\right)\left(\delta_{\alpha\beta}+\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\right)$$

Next, acoustoelastic equation for FSPM in the $(m-1)^{\text{th}}$ step is given as follows. According to equation (17), equation of motion induced by the small-amplitude disturbance in the $(m-1)^{\text{th}}$ step is derived,

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[\Delta\bar{T}^{(m-1)}_{\alpha\beta}+\Delta\bar{T}^{(m-1)}_{\beta\gamma}\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\gamma}+\bar{T}^{(m-1)}_{\beta\gamma}\frac{\partial\Delta u^{(m-1)}_{\alpha}}{\partial\xi_\gamma}\right]=\rho^0_{11}\frac{\partial^2\Delta u^{(m-1)}_{\alpha}}{\partial t^2}+\rho^0_{12}\frac{\partial^2\Delta U^{(m-1)}_{\alpha}}{\partial t^2}\\&\frac{\partial}{\partial\xi_\beta}\left[\Delta s^{(m-1)}\left(\delta_{\alpha\beta}+\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\right)+s^{(m-1)}\frac{\partial\Delta u^{(m-1)}_{\alpha}}{\partial\xi_\beta}\right]=\rho^0_{12}\frac{\partial^2\Delta u^{(m-1)}_{\alpha}}{\partial t^2}+\rho^0_{22}\frac{\partial^2\Delta U^{(m-1)}_{\alpha}}{\partial t^2}\end{aligned}\right\}\quad(21)$$

The incremental stress tensors induced by the small-amplitude disturbance in the $(m-1)^{\text{th}}$ step are denoted as

$$\left.\begin{aligned}\Delta\bar{T}^{(m-1)}_{\alpha\beta}&=c^{(m-1)}_{\alpha\beta\gamma\delta}\frac{\partial\Delta u^{(m-1)}_{\gamma}}{\partial\xi_\delta}+c^{(m-1)}_{\alpha\beta\gamma\delta}\frac{\partial u^{(m-1)}_{\lambda}}{\partial\xi_\gamma}\frac{\partial\Delta u^{(m-1)}_{\lambda}}{\partial\xi_\delta}+M^{(m-1)}_{\alpha\beta}\Delta\varepsilon^{(m-1)}\\\Delta s^{(m-1)}&=M^{(m-1)}_{\alpha\beta}\frac{\partial\Delta u^{(m-1)}_{\alpha}}{\partial\xi_\beta}+M^{(m-1)}_{\alpha\beta}\frac{\partial u^{(m-1)}_{\lambda}}{\partial\xi_\alpha}\frac{\partial\Delta u^{(m-1)}_{\lambda}}{\partial\xi_\beta}+M^{(m-1)}\Delta\varepsilon^{(m-1)}\end{aligned}\right\}\quad(22)$$

Substituting equation (22) into equation (21) yields the equation of motion for $\Delta\mathbf{u}^{(m-1)}(\xi,t)$ and $\Delta\mathbf{U}^{(m-1)}(\xi,t)$,

$$\left.\begin{aligned}&\frac{\partial}{\partial\xi_\beta}\left[A^{(m-1)}_{\alpha\beta\gamma\delta}\frac{\partial\Delta u^{(m-1)}_{\gamma}}{\partial\xi_\delta}+\Phi^{(m-1)}_{\alpha\beta}\frac{\partial\Delta U^{(m-1)}_{k}}{\partial\xi_k}\right]=\rho^0_{11}\frac{\partial^2\Delta u^{(m-1)}_{\alpha}}{\partial t^2}+\rho^0_{12}\frac{\partial^2\Delta U^{(m-1)}_{\alpha}}{\partial t^2}\\&\frac{\partial}{\partial\xi_\beta}\left[B^{(m-1)}_{\alpha\beta\gamma\delta}\frac{\partial\Delta u^{(m-1)}_{\gamma}}{\partial\xi_\delta}+Z^{(m-1)}_{\alpha\beta}\frac{\partial\Delta U^{(m-1)}_{k}}{\partial\xi_k}\right]=\rho^0_{12}\frac{\partial^2\Delta u^{(m-1)}_{\alpha}}{\partial t^2}+\rho^0_{22}\frac{\partial^2\Delta U^{(m-1)}_{\alpha}}{\partial t^2}\end{aligned}\right\}\quad(23)$$

where

$$A^{(m-1)}_{\alpha\beta\gamma\delta}=\bar{T}^{(m-1)}_{\beta\delta}\delta_{\alpha\gamma}+c^{(m-1)}_{\alpha\beta\gamma\delta}+c^{(m-1)}_{\alpha\beta\lambda\delta}\frac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}+c^{(m-1)}_{\rho\beta\gamma\delta}\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\rho}+c^{(m-1)}_{\rho\beta\lambda\delta}\frac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\rho}$$

$$\Phi^{(m-1)}_{\alpha\beta}=M^{(m-1)}_{\alpha\beta}+M^{(m-1)}_{\rho\beta}\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\rho}$$

$$B^{(m-1)}_{\alpha\beta\gamma\delta}=s^{(m-1)}\delta_{\alpha\gamma}\delta_{\beta\delta}+M^{(m-1)}_{\gamma\delta}\delta_{\alpha\beta}+M^{(m-1)}_{\lambda\delta}\frac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}\delta_{\alpha\beta}+M^{(m-1)}_{\gamma\delta}\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}+M^{(m-1)}_{\lambda\delta}\frac{\partial u^{(m-1)}_{\gamma}}{\partial\xi_\lambda}\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}$$

$$Z^{(m-1)}_{\alpha\beta}=M^{(m-1)}\delta_{\alpha\beta}+M^{(m-1)}\frac{\partial u^{(m-1)}_{\alpha}}{\partial\xi_\beta}$$

Assuming that the wave-velocity at terminal state in the $(m-1)^{\text{th}}$ step is same as the wave-velocity at starting state in the $(m-1)^{\text{th}}$ step, the recurrence relation by comparing equation (20) with equation (23) can be deduced as,

$$\left.\begin{aligned}A^{(m)}_{\alpha\beta\gamma\delta}&=A^{(m-1)}_{\alpha\beta\gamma\delta}+\Delta A^{(m)}_{\alpha\beta\gamma\delta}\\\Phi^{(m)}_{\alpha\beta}&=\Phi^{(m-1)}_{\alpha\beta}+\Delta\Phi^{(m)}_{\alpha\beta}\\B^{(m)}_{\alpha\beta\gamma\delta}&=B^{(m-1)}_{\alpha\beta\gamma\delta}+\Delta B^{(m)}_{\alpha\beta\gamma\delta}\\Z^{(m)}_{\alpha\beta}&=Z^{(m-1)}_{\alpha\beta}+\Delta Z^{(m)}_{\alpha}\end{aligned}\right\}\quad(24)$$

Finally, acoustoelastic equations at the terminal state in the m^{th} step (as well as the initial state of large pre-deformation elastic solids) are presented,

$$\left.\begin{aligned}
&\frac{\partial}{\partial \xi_\beta}\left[A^{(M)}_{\alpha\beta\gamma\delta}\frac{\partial \Delta u^{(M)}_\gamma}{\partial \xi_\delta}+\Phi^{(M)}_{\alpha\beta}\frac{\partial \Delta U^{(M)}_k}{\partial \xi_k}\right]\\
&=\rho^0_{11}\frac{\partial^2 \Delta u^{(M)}_\alpha}{\partial t^2}+\rho^0_{12}\frac{\partial^2 \Delta U^{(M)}_\alpha}{\partial t^2}\\
&\frac{\partial}{\partial \xi_\beta}\left[B^{(M)}_{\alpha\gamma\delta}\frac{\partial \Delta u^{(M)}_\gamma}{\partial \xi_\delta}+Z^{(M)}_{\alpha\beta}\frac{\partial \Delta U^{(M)}_k}{\partial \xi_k}\right]\\
&=\rho^0_{12}\frac{\partial^2 \Delta u^{(M)}_\alpha}{\partial t^2}+\rho^0_{22}\frac{\partial^2 \Delta U^{(M)}_\alpha}{\partial t^2}
\end{aligned}\right\} \tag{25}$$

where

$$\left.\begin{aligned}
A^{(M)}_{\alpha\beta\gamma\delta}&=A^{(0)}_{\alpha\beta\gamma\delta}+\sum_{m=1}^{M}\Delta A^{(m)}_{\alpha\beta\gamma\delta}\\
\Phi^{(M)}_{\alpha\beta}&=\Phi^{(0)}_{\alpha\beta}+\sum_{m=1}^{M}\Delta \Phi^{(m)}_{\alpha\beta}\\
B^{(M)}_{\alpha\beta\gamma\delta}&=B^{(0)}_{\alpha\beta\gamma\delta}+\sum_{m=1}^{M}\Delta B^{(m)}_{\alpha\beta\gamma\delta}\\
Z^{(M)}_{\alpha\beta}&=Z^{(0)}_{\alpha\beta}+\sum_{m=1}^{M}\Delta Z^{(m)}_{\alpha\beta}
\end{aligned}\right\}$$

For a homogeneously pre-deformed FSPM, if dynamic disturbance travels as a plane sinusoidal wave form in the direction of the vector **V**, substituting the wave function expression of the solid-skeleton and fluid into equation (25), the characteristic equation for large static pre-deformed elastic solids in global natural coordinates is given as

$$\begin{vmatrix}
\begin{bmatrix} A^{(M)}_{\alpha\beta\gamma\delta}v_\beta v_\delta & \Phi^{(M)}_{\alpha\beta}v_\beta v_\gamma \\ B^{(M)}_{\alpha\beta\gamma\delta}v_\beta v_\delta & Z^{(M)}_{\alpha\beta}v_\beta v_\gamma \end{bmatrix}\\
a-\left(c^{(M)}\right)^2\begin{bmatrix}\rho^0_{11}\delta_{\alpha\gamma} & \rho^0_{12}\delta_{\alpha\gamma}\\ \rho^0_{12}\delta_{\alpha\gamma} & \rho^0_{22}\delta_{\alpha\gamma}\end{bmatrix}
\end{vmatrix}=0 \tag{26}$$

where $c^{(M)}$ is the wave velocity in the natural coordinates.

3 Numerical results and discussions

In this section, the relation between wave-velocity and stress for large static pre-deformed isotropic FSPM subjected to hydrostatic pressure will be derived in natural coordinates, where two cases of loading schemes are considered, including closed-pore jacketed and is open-pore jacketed (Grinfeld and Norris, 1996). Figure 2(a) shows the state of the closed-pore jacketed system corresponds to constancy of the fluid content or mass, and in the geometrically linear approximation this implies that $\varepsilon^i = 0$. Figure 2(b) shows the open porous system corresponds to the case of constant fluid pressure, or $P = 0$.

The closed-pore jacketed condition for the rock sample of confining pressure P_h corresponds to the constant fluid content. Using the linear constitutive relation for a solid skeleton particle and a fluid particle, the solid-skeleton strains in the m^{th} step can be given as

$$\begin{Bmatrix}\Delta E^{(m)i}_{11}\\ \Delta E^{(m)i}_{22}\\ \Delta E^{(m)i}_{33}\end{Bmatrix}=
\begin{bmatrix}
c^{(m-1)}_{1111}+M^{(m-1)}_{11} & c^{(m-1)}_{1122}+M^{(m-1)}_{11} & c^{(m-1)}_{1122}+M^{(m-1)}_{11}\\
c^{(m-1)}_{1122}+M^{(m-1)}_{11} & c^{(m-1)}_{1111}+M^{(m-1)}_{11} & c^{(m-1)}_{1122}+M^{(m-1)}_{11}\\
c^{(m-1)}_{1122}+M^{(m-1)}_{11} & c^{(m-1)}_{1122}+M^{(m-1)}_{11} & c^{(m-1)}_{1111}+M^{(m-1)}_{11}
\end{bmatrix}^{-1}
\begin{Bmatrix}\Delta P_h\\ \Delta P_h\\ \Delta P_h\end{Bmatrix} \tag{27}$$

Figure 2 Different loading schemes in large pre-deformed FSPM

(a) (b)

Furthermore, the solid-skeleton stress and fluid stress are expressed as

$$\left.\begin{aligned}
\Delta s^{(m)i}&=M^{(m-1)}_{\alpha\alpha}\Delta E^{(m)i}_{\alpha\alpha}\\
\Delta \bar{T}^{(m)i}_{\alpha\beta}&=\left(\Delta P_h-\Delta s^{(m)i}\right)\delta_{\alpha\beta}
\end{aligned}\right\}\left(\alpha,\beta=1,2,3\right) \tag{28}$$

If the rock sample which corresponds to the constant initial fluid stress $s^i = -np_f$ (n denotes porosity) is open-pore jacketed and subjected to a hydrostatic pressure P_h, and the fluid pressure is hold on one atmosphere (can be zero in approximation), we have the solid-skeleton stress and fluid stress

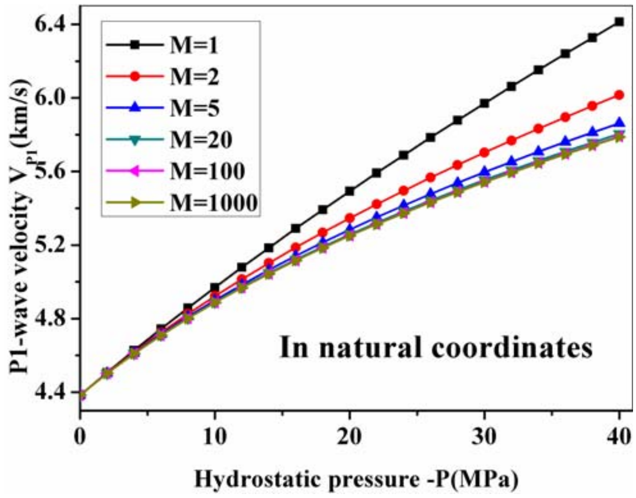
$$\left.\begin{aligned}
\Delta s^{(m)i}&=0\\
\Delta \bar{T}^{(m)i}_{\alpha\beta}&=\Delta P_h\delta_{\alpha\beta}
\end{aligned}\right\}\left(\alpha,\beta=1,2,3\right) \tag{29}$$

In the virtual test, the hydrostatic pressure changes between 0 to 40 MPA. The numerical results for open-pore and close-pore jacketed rock sample under hydrostatic loading are shown in Figures 3 to 5. The material parameters are listed in Table 1 (Nur and Simmons, 1969).

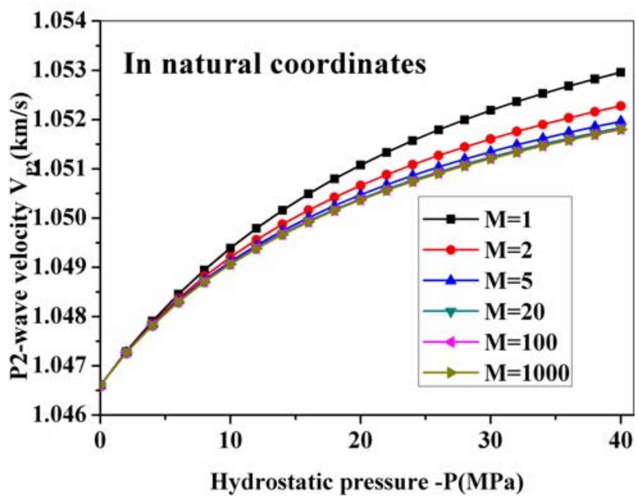
Table 1 Density and elastic constant of isotropic FSPM and the direction of incident wave

ρ^o_{11} *(kg/m³)*	ρ^o_{22} *(kg/m³)*	ρ^o_{12} *(kg/m³)*	λ *(GPa)*	μ *(GPa)*	l *(GPa)*
2,205	650	–350	1.667	18.2	–337
m *(GPa)*	n *(GPa)*	s_{ijkl} *(GPa)*	M *(GPa)*	M_{ij} $(i = j)$ *(GPa)*	M_{ij} $(i \neq j)$ *(GPa)*
–6,742	–6,600	0	0.729	0.396	0
N *(GPa)*	N_{ij} *(GPa)*		$v1$	$v2$	$v3$
0	0		0	0	1

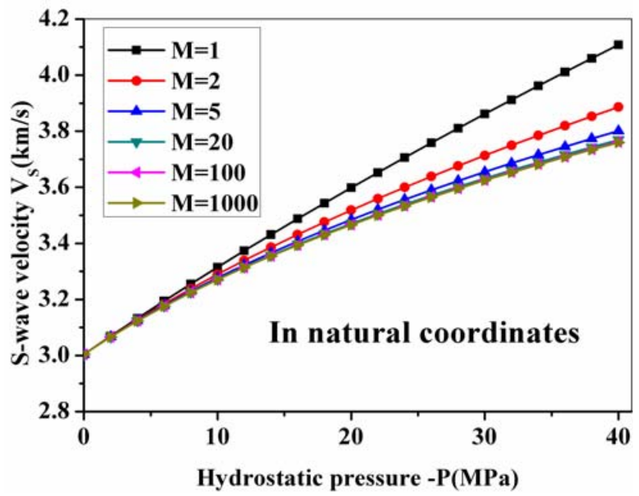
Figure 3 Effects of loading incremental number on wave velocity in natural coordinate under hydrostatic pressure (closed-pore jacketed condition), (a) Vp1-P curve (b) Vp2-P curve (c) Vs-P curve (see online version for colours)



(a)

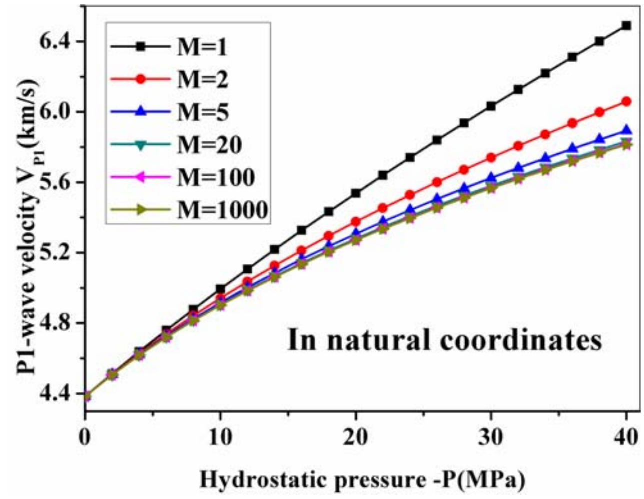


(b)

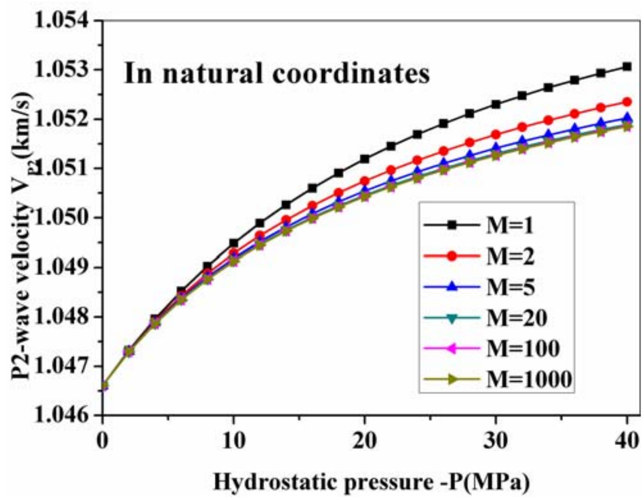


(c)

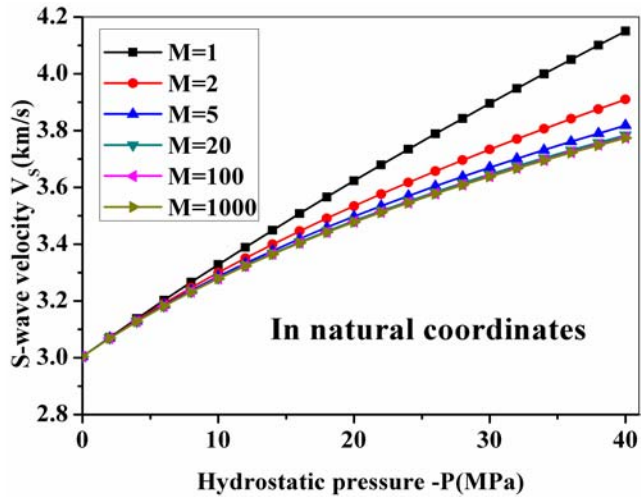
Figure 4 Effects of loading incremental number on wave velocity in natural coordinate under hydrostatic pressure (opened-pore jacketed condition), (a) Vp1-P curve (b) Vp2-P curve (c) Vs-P curve (see online version for colours)



(a)

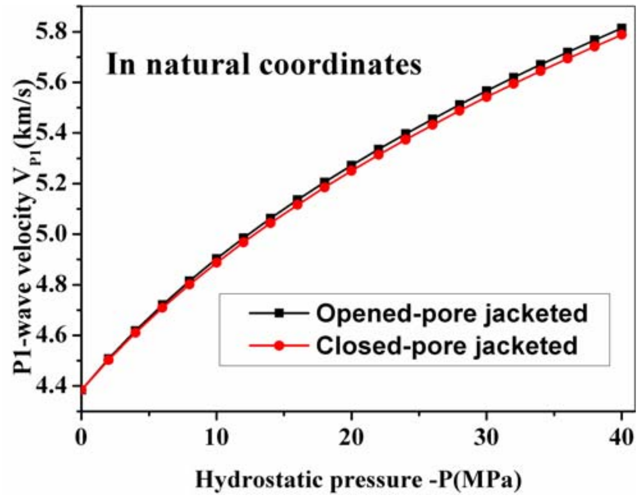


(b)

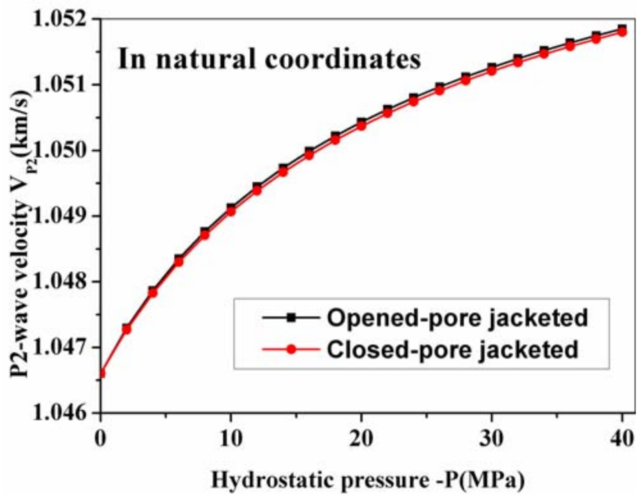


(c)

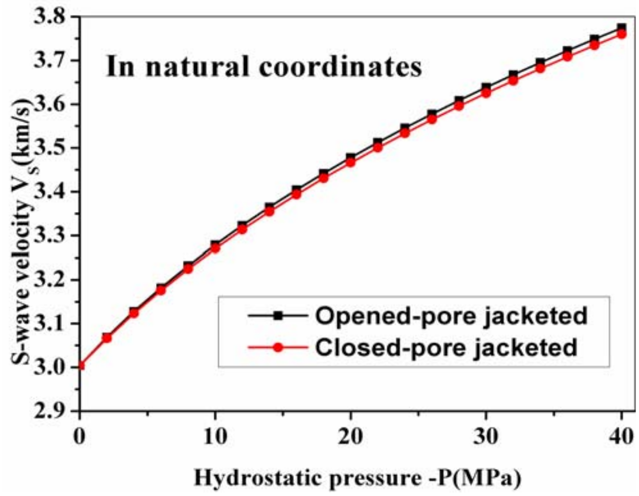
Figure 5 Effects of loading scheme on wave velocity in natural coordinate, (a) Vp1-P curve (b) Vp2-P curve (c) Vs-P curve (see online version for colours)



(a)



(b)



(c)

Figures 3 and 4 show the effects of the loading step number on the elastic wave-velocity in the closed-pore and open-pore jacketed rock sample under hydrostatic loading. There are three types of body waves in the FSPM, which are fast longitudinal wave, slow longitudinal wave, and transverse wave. For a fixed initial stress, the variations of stress-dependent wave-velocity are given with different loading step (M = 1, M = 2, M = 5, M = 20, M = 100, M = 1,000). The velocity-pressure curve with M = 1 corresponds to the results of classic acoustoelastic theory. The velocity-pressure curves for longitudinal and transverse waves show the good convergence of the incremental algorithm with the increase of loading step number in natural coordinates. Wave-velocities for the fast and slow longitudinal wave and transverse wave increase with the increase of hydrostatic pressure. When the hydrostatic pressure is below 10 MPa, the results of the incremental algorithm are basically consistent with the results of classic acoustoelastic theory. When the hydrostatic pressure is greater than 10 MPa, their differences increase with the increase of hydrostatic pressure.

The longitudinal wave-velocity and transverse wave-velocity in the open-pore jacketed condition are larger comparing with the closed-pore jacketed condition. Figure 5 shows the effect of loading scheme on wave velocity in natural coordinates. In the open-pore jacketed condition, external loading is mainly supported by the solid skeleton, and the nonlinear elastic property of solid plays a leading role. In the closed-pore jacketed condition, the external loading is partly supported by the fluid component, so the acoustoelasticity of fluid need to be considered in the experiment. Because the three order elastic constant of fluid is far smaller than that of solid, the wave velocity in the open-pore jacketed condition is higher than that in the closed-pore jacketed condition at the same hydrostatic pressure.

4 Conclusions

The incremental acoustoelastic equation of large pre-deformed FSPM with arbitrary constitutive law has been presented by incremental loading method based on the classic acoustoelastic theory. The examples for isotropic FSPM of third-order constitutive equation under closed-pore jacketed condition and opened-pore jacketed condition verify the good convergence of the incremental algorithm with the increase of loading step number.

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